Noether's theorem and the "extended", conservative approach to time driven systems

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We discuss Kaplan's conservative approach to time driven dynamical systems in relation with Hamilton's Principle and the symmetries of the associated action integral. We show how the symmetries of Kaplan's "extended", conservative, dynamical system "conspire" so as to give rise to the symmetries of the original, time driven system. These considerations provide, via Noether's Theorem, an alternative path between the conservation laws of both the "extended" and the driven system.

Keywords: Dynamical systems; symmetries; Noether's theorem

Estudiamos el tratamiento conservativo de sistemas dinámicos forzados en relación con el Principio de Hamilton y las simetrías de la integral de acción asociada. Mostramos cómo las simetrías del sistema dinámico conservativo "extendido" de Kaplan se combinan dando origen a las simetrías del sistema forzado original. Estas consideraciones constituyen una forma diferente de vincular, vía el Teorema de Noether, las leyes de conservación de los sistemas "extendido" y forzado.

Descriptores: Sistemas dinámicos; simetrías; teorema de Noether

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1. Introduction

Recently Kaplan has introduced an interesting approach to time driven systems [1]. Given a dynamical system characterized by a time dependent Lagrangian (or Hamiltonian), Kaplan's proposal is based on the introduction of an ad-hoc "extended", conservative, dynamical system that incorporates (at least) one additional degree of freedom to those of the original system. In the limit when the "inertia" associated with these new variables tends to infinity, the equations of motion of the original system are recovered within Kaplan's formalism. The explicit time dependence of the original Lagrangian arises from the "motion" associated to the *new* dynamical variables, whose behaviour becomes "decoupled" from that of the "old" ones, that pertain to the original system.

Actually, Kaplan's approach paves a quite natural road towards understanding time driven systems. It is possible to argue that most time driven systems encountered in nature (if not all of them) admit a description à la Kaplan, in terms of an extended conservative system. After all, physical systems are generally assumed to be governed, at a fundamental level, by a time independent Lagrangian (or Hamiltonian). The associated law of energy conservation is the most important general principle of Physics. When a dynamical system requires a non conservative description, this is so because the system of interest is interacting with "something else".

One of the appealing features of Kaplan's procedure lies in the fact that it provides a physical understanding of the origin of the integrals of motion of time driven systems [1, 2]. These time dependent integrals of motion are shown to arise as appropriate linear combinations of the integrals of motion of the extended conservative system. On the other hand, it is well known that many important conservation laws of Lagrangian dynamical systems are related, via Noether's theorem, to the symmetries of the concomitant action integral [3–6]. The purpose of the present note is to investigate the relationship between Kaplan's approach and Noether's procedure. Such a study would pave the way for establishing useful connexions between Kaplan's beautiful approach and some of the main ideas of contemporary physics, that stress the relevance of symmetries in understanding physical processes. The pedagogical value of Kaplan's approach should in such a manner be greatly increased. To this end, we study how the symmetries of the action integral characterizing the original time driven system can be obtained from the symmetries of the action integral associated with the extended system. The important example of Jacobi's Integral of motion for rotating potentials [7, 8] is considered along these lines. The application of Kaplan's approach to a particle moving in a plane-wavelike potential is also studied.

2. General formalism

Let us consider a Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}, \lambda), \tag{1}$$

which depends on a small parameter λ, $\mathbf{q}$ and $\dot{\mathbf{q}}$ standing for the sets of generalized coordinates and velocities, respectively. The Lagrangian governing Kaplan's "extended" dynamical system can be cast in this form, the small parameter λ being proportional to the inverse of the "inertia" characterizing the new degrees of freedom [2].

The action integral, given by

$$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, \lambda) dt, \tag{2}$$

contains all that relevant information about the system which is independent of its precise dynamical state at a given time. Through the Lagrangian L, the action is a functional of the paths $\mathbf{q}(t, \lambda)$. Notice that once a path is given, the velocities $\dot{\mathbf{q}}(t, \lambda)$ are immediately obtained.

If S is invariant (to first order in the small parameter ε) under the infinitesimal symmetry transformation given by

$$\begin{aligned} \mathbf{q}' &= \mathbf{q} + \varepsilon \mathbf{K}(\mathbf{q}, \mathbf{q}', t), \\ t' &= t + \varepsilon \Gamma(\mathbf{q}, \mathbf{q}', t), \end{aligned} \tag{3}$$

then, according to Noether's theorem [4], the system admits the following integral of motion:

$$\mathbf{K} \cdot \mathbf{p} - \Gamma E, \tag{4}$$

where $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$ stands for the generalized moments associated to the generalized coordinates $\mathbf{q}$, and E denotes the energy $E = \sum \mathbf{p} \cdot \dot{\mathbf{q}} - L$. It is useful, for later discussions, to observe explicitly that, if the action integral S remains invariant under the transformation (3), for every λ value, the partial derivatives $\partial^n S / \partial \lambda^n$ remain invariant as well. Notice that these derivatives correspond to a given "fixed" trajectory $\mathbf{q}(t)$. This means that they only take into account the explicit λ dependence in (2).

According to Hamilton's principle, the actual motion of the system is such that the variation of the action for arbitrary $\delta \mathbf{q}$ is zero,

$$\delta S = \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, \lambda) dt = 0. \tag{5}$$

Let $\mathbf{q}(t, \lambda)$ be a "real" path of the system, that is to say, a trajectory for which the action is stationary. On expanding $\mathbf{q}(t, \lambda)$ in powers of the small parameter λ,

$$\mathbf{q}(t, \lambda) = \mathbf{q_0}(t) + \lambda \mathbf{q_1}(t) + \lambda^2 \mathbf{q_2}(t) + \dots, \tag{6}$$

the equations of motion for $\mathbf{q_0}(t)$, $\mathbf{q_1}(t)$, ..., are obtained by recourse to a power expansion (in λ) of the action integral S

$$S = S_0 + \lambda S_1 + \lambda^2 S_2 + \dots, \tag{7}$$

and requiring afterwards that

$$\delta S_i = 0 (i = 0, 1, 2, \dots), \tag{8}$$

for arbitrary infinitesimal variations

$$\delta \mathbf{q}_j = 0 (j = 0, 1, 2, \dots). \tag{9}$$

Therefore, Eq. (8) constitutes an expression of Hamilton's principle whenever power series expansions about a small parameter are utilized.

We consider now the particular instance of a Lagrangian given by

$$L = L_a(\mathbf{q}_a, \dot{\mathbf{q}}_a) + \lambda L_b(\mathbf{q}_a, \dot{\mathbf{q}}_a, \mathbf{q}_b, \dot{\mathbf{q}}_b), \tag{10}$$

where L_a depends only on a subset $\mathbf{q}_a, \dot{\mathbf{q}}_a$ of the complete set $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_a, \dot{\mathbf{q}}_a, \mathbf{q}_b, \dot{\mathbf{q}}_b)$ of generalized coordinates and velocities. Here the pertinent Kaplan's extended system is described by a Lagrangian of the form (10), where $\mathbf{q}_a$ denotes the "new" generalized coordinates, while $\mathbf{q}_b$ stands for the "old" coordinates of the original time driven system. The action integral corresponding to the Lagrangian (10) reads

$$\begin{aligned} S &= S_a + \lambda S_b \\ &= S_0 + \lambda S_1 + \dots, \end{aligned} \tag{11}$$

where

$$\begin{aligned} S_a &= \int_{t_1}^{t_2} L_a(\mathbf{q}_a, \dot{\mathbf{q}}_a) dt, \\ S_b &= \int_{t_1}^{t_2} L_b(\mathbf{q}_a, \dot{\mathbf{q}}_a, \mathbf{q}_b, \dot{\mathbf{q}}_b) dt, \\ S_0 &= \int_{t_1}^{t_2} L_a(\mathbf{q}_{a0}, \dot{\mathbf{q}}_{a0}) dt, \end{aligned} \tag{12}$$

and

$$\begin{aligned} S_1 = \int_{t_1}^{t_2} \left[\left(\frac{\partial L_a}{\partial \mathbf{q}_a} \right)_0 \cdot \mathbf{q}_{a1} + \left(\frac{\partial L_a}{\partial \dot{\mathbf{q}}_a} \right)_0 \cdot \dot{\mathbf{q}}_{a1} \right] dt \\ + \int_{t_1}^{t_2} L_b(\mathbf{q}_{a0}, \dot{\mathbf{q}}_{a0}, \mathbf{q}_{b0}, \dot{\mathbf{q}}_{b0}) dt. \end{aligned} \tag{13}$$

Hamilton's principle requires that δS_0 vanish identically for any infinitesimal variation $\delta \mathbf{q}_{a0}$. This requirement yields the equations of motion for $\mathbf{q}_{a0}$, which are easily seen to coincide with the ones corresponding to the Lagrangian L_a. Notice also that, $\delta S_1 = 0$ for arbitrary variations $\delta \mathbf{q}_{a0}$, $\delta \mathbf{q}_{a1}$, and $\delta \mathbf{q}_{b0}$, which should provide the equations of motion for $\mathbf{q}_{a1}$ and $\mathbf{q}_{b0}$. It is interesting to point out that the Euler-Lagrange equations for $\mathbf{q}_{a1}$, associated with the variational requirement $\delta S_1 = 0$, are

$$\frac{d}{dt} \left(\frac{\partial L_c}{\partial \dot{\mathbf{q}}_{a1}} \right) = \frac{\partial L_c}{\partial \mathbf{q}_{a1}}, \tag{14}$$

where

$$L_c = \left[\left(\frac{\partial L_a}{\partial \mathbf{q}_a} \right)_0 \cdot \mathbf{q}_{a1} + \left(\frac{\partial L_a}{\partial \dot{\mathbf{q}}_a} \right)_0 \cdot \dot{\mathbf{q}}_{a1} \right], \tag{15}$$

which are just the Euler-Lagrange equations for $\mathbf{q}_{a0}$ derived from $\delta S_0 = 0$. On the other hand, the Euler-Lagrange equations deduced from $\delta S_1 = 0$ when considering arbitrary variations of $\mathbf{q}_{a0}$, actually yield the set of differential equations for $\mathbf{q}_{a1}$. Interestingly enough, this series-expansion-procedure generates the differential evolution equations for

$\mathbf{q}_{a0}, \mathbf{q}_{b0}, \mathbf{q}_{a1}, \cdots$. It should be noticed that, on imposing the condition $\delta S_1 = 0$, one is led to the equations of motion for $\mathbf{q}_{b0}$ that would have been obtained from the Lagrangian $L_b(\mathbf{q}_{a0}, \dot{\mathbf{q}}_{a0}, \mathbf{q}_{b0}, \dot{\mathbf{q}}_{b0})$ considered just as a function of both $\mathbf{q}_{b0}$ and $\dot{\mathbf{q}}_{b0}$, when $\mathbf{q}_{a0}(t)$ has been replaced by a solution of $\delta S_0 = 0$. This is a time driven system, as the Lagrangian explicitly depends on time.

3. Rotating potentials

In Ref. 2 we have shown how Kaplan's approach illuminates the discussion of a problem which has important applications in dynamical Astronomy: the motion of a particle of mass m in a uniformly rotating potential $mV(r, \theta - \omega t)$. Here V denotes the potential per unit mass, (r, θ) are the polar coordinates of the particle, and ω the rotation angular velocity. The concomitant Lagragian reads

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mV(r, \theta - \omega t). \tag{16}$$

In order to obtain Kaplan's "companion" conservative system to the original time driven one, we have introduced a plane rigid rotator of moment of inertia I. The position of the rotator, which has a fixed point and lies on the same plane where the particle moves, is given by the angular coordinate ϕ. Hence, the configuration space (r, θ, ϕ) of our extended dynamical system is of a three-dimensional character. We assume that the interaction between our rigid rotator and the particle is given by the potential $mV(r, \theta - \phi)$. Thus, the concomitant Lagrangian of our model is

$$L = \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mV(r, \theta - \phi), \tag{17}$$

or

$$L = \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}I\lambda(\dot{r}^2 + r^2\dot{\theta}^2) - I\lambda V(r, \theta - \phi), \tag{18}$$

where we have introduced the quantity λ (that should serve as a small expansion parameter), defined as the ratio

$$\lambda = \frac{m}{I}, \tag{19}$$

between the mass of the particle and the moment of inertia of the rotator. On comparing the Lagrangian (18) with Eq. (10), we make the identification

$$L_a = \frac{1}{2}I\dot{\phi}^2 \tag{20}$$

and

$$L_b = \frac{1}{2}I(\dot{r}^2 + r^2\dot{\theta}^2) - IV(r, \theta - \phi). \tag{21}$$

We recall now that the Lagrangian (18) is invariant (and so is the concomitant action S) under the infinitesimal transformations

$$t' = t + \varepsilon, \tag{22}$$

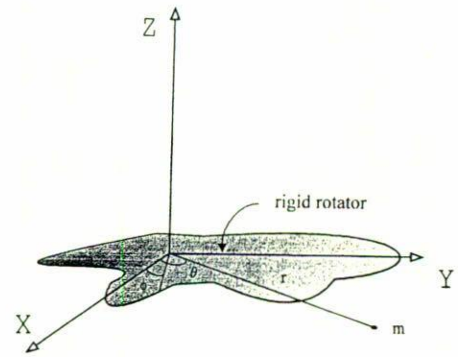

FIGURE 1. A plane rigid rotator and a particle both contained in the XY-plane. The position of the rigid rotator is characterized by the angular coordinate ϕ and that of the particle by the polar coordinates r and θ.

associated with the conservation of energy, and

$$\theta' = \theta + \varepsilon,$$
$$\phi' = \phi + \varepsilon, \tag{23}$$

related to the conservation of the z-axis-component of angular momentum L_z.

Since the above referred to invariance holds for arbitrary values of λ, both L_a and L_b, individually, share the invariance property, a fact readily verified by simple inspection of Eqs. (20) and (21). From the stationary condition $\delta S_0 = 0$, the appropriate Euler-Lagrange equations yield

$$\dot{\phi}_0 = \omega = \text{constant}, \tag{24}$$

so that

$$\phi_0(t) = \phi_0(0) + \omega t. \tag{25}$$

Moreover, since we also have $\delta S_1 = 0$, it follows that the action

$$S_0^* = S_b(\phi_0, r_0, \theta_0) = \int_{t_1}^{t_2} \left[\frac{1}{2}I(\dot{r}_0^2 + r_0^2\dot{\theta}_0^2) - IV(r_0, \theta_0 - \phi_0)\right] dt, \tag{26}$$

with $\phi_0(t)$ given by Eq. (25), must verify

$$\delta S_0^* = 0, \tag{27}$$

for arbitrary infinitesimal variations δr_0 and $\delta\theta_0$. (Notice that $S_0^* \neq S_0$.) The condition (27) furnishes the equations of motion for $r_0(t)$ and $\theta_0(t)$. These are precisely the equations of motion for a particle moving in a uniformly rotating potential whose Lagrangian is (16).

The action S_0^*, which can be cast in the form

$$S_0^* = \int_{t_1}^{t_2} L_b\left[\phi_0(t), r_0(t), \theta_0(t)\right] dt, \tag{28}$$

remains invariant under the infinitesimal transformations (22) and (23) when performed upon ϕ_0, r_0 and θ_0. However, whenever $\phi_0(t)$ is fixed, the invariance of S_0^* under (22) and (23) is lost. Let us now analize just how *each* one of the previous transformations contributes to the variation of S_0^*, when they are applied upon $\phi_0(t)$. The result of applying (22) to $\phi_0(t)$, to first order in ϵ, is

$$\Delta S_0^* = \int_{t_1}^{t_2} L_b\left[\phi_0(t+\varepsilon), r_0(t), \theta_0(t)\right] dt - \int_{t_1}^{t_2} L_b\left[\phi_0(t), r_0(t), \theta_0(t)\right] dt = \int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial \phi}\right)_0 \dot{\phi}_0 \varepsilon dt, \tag{29}$$

where $(\partial L_b/\partial\phi)_0$ stands for the partial derivative with respect to ϕ, evaluated on $\phi_0(t), r_0(t)$, and $\theta_0(t)$. Thus we have

$$\Delta S_0^* = \varepsilon\omega \int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial \phi}\right)_0 dt. \tag{30}$$

On the other hand, when $\phi_0(t)$ is subjected to the transformation (23) we obtain

$$\Delta S_0^* = \int_{t_1}^{t_2} L_b\left[\phi_0(t)+\varepsilon, r_0(t), \theta_0(t)\right] dt - \int_{t_1}^{t_2} L_b\left[\phi_0(t), r_0(t), \theta_0(t)\right] dt, \tag{31}$$

so that

$$\Delta S_0^* = \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial \phi}\right)_0 dt. \tag{32}$$

Therefore, on applying both

$$t' = t + \varepsilon \tag{33}$$

and

$$\phi_0' = \phi_0 + \varepsilon\omega \tag{34}$$

onto $\phi_0(t)$, one is led to

$$\Delta S_0^* = 0. \tag{35}$$

Now, since S_0^* remains invariant either when the transformations

$$t' = t + \varepsilon, \tag{36}$$

$$\theta' = \theta + \varepsilon\omega, \tag{37}$$

and

$$\phi' = \phi + \varepsilon\omega, \tag{38}$$

are performed upon the whole set $\phi_0(t), r_0(t), \theta_0(t)$, or when both (36) and (38) are applied only onto $\phi_0(t)$, it is clear that S_0^* also remains invariant when the transformations (36) and (37) are applied upon $r_0(t)$ and $\theta_0(t)$. Therefore, Noether's invariant, given by Eq. (4), provides us with Jacobi's Integral of motion [7, 8]

$$C_J = E - \omega L_z, \tag{39}$$

where L_z is the z-component of the angular momentum.

4. Plane-wavelike potentials

We now consider the motion of a particle of mass m in a plane-wavelike external potential

$$mV(x - vt), \tag{40}$$

v being the uniform wave propagation velocity. Following Kaplan's procedure, we introduce a particle of mass M, its position denoted by y, and assume that the interaction between both particles is given by the potential $mV(x-y)$. The concomitant "extended" Lagrangian reads

$$L = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{x}^2 - mV(x-y). \tag{41}$$

On introducing the quantity λ, defined as the ratio of the masses

$$\lambda = \frac{m}{M}, \tag{42}$$

the Lagrangian (41) can be recasted in the form

$$L = L_a + \lambda L_b, \tag{43}$$

where

$$L_a = \frac{1}{2}M\dot{y}^2, \tag{44}$$

and

$$L_b = \frac{1}{2}M\dot{x}^2 - MV(x-y). \tag{45}$$

It is easily seen that the infinitesimal transformations

$$t' = t + \varepsilon, \tag{46}$$

and

$$x' = x + \varepsilon,$$
$$y' = y + \varepsilon, \tag{47}$$

leave the Lagrangian (41) invariant, which entails that both L_a and L_b are, individually, invariant too. From the stationary requirement $\delta S_0 = 0$, the Euler-Lagrange equations provide us with

$$\dot{y}_0 = \text{constant} \tag{48}$$

so that

$$y_0(t) = y_0(0) + \dot{y}_0 t. \tag{49}$$

Additionally, from the condition $\delta S_1 = 0$, we conclude that the action

$$S_0^* = S_b(y_0, x_0)$$
$$= \int_{t_1}^{t_2} \left[\frac{1}{2} M\dot{x}_0^2 - MV(x_0 - y_0)\right] dt, \tag{50}$$

with $y_0(t)$ given by (49), must be of a stationary character. On applying (46) upon $y_0(t)$ we obtain, to first order in ϵ,

$$\Delta S_0^* = \int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial y}\right)_0 \dot{y}_0\, \varepsilon\, dt$$
$$= \dot{y}_0\, \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial y}\right)_0 dt, \tag{51}$$

where $(\partial L_b/\partial y)_0$ stands for the partial derivative with respect to y, evaluated on $y_0(t)$ and $x_0(t)$. On the other hand, when the traslation (47) acts upon $y_0(t)$ one finds

$$\Delta S_0^* = \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L_b}{\partial y}\right)_0 dt, \tag{52}$$

which entails that, when both transformations

$$t' = t + \varepsilon \tag{53}$$

and

$$y' = y + \varepsilon v \tag{54}$$

(where $v = \dot{y}_0$) are applied to $y_0(t)$, one is led to

$$\Delta S_0^* = 0. \tag{55}$$

Therefore, since S_0^* remains invariant either when the transformations

$$t' = t + \varepsilon \tag{56}$$

$$x' = x + \varepsilon v \tag{57}$$

$$y' = y + \varepsilon v \tag{58}$$

are applied to the whole set $y_0(t), x_0(t)$, or when (56) and (58) are performed just onto $y_0(t)$, it is clear that S_0^* also remains invariant when the transformations (56) and (57) are applied only to $x_0(t)$. Then, according to Noether's theorem (see Eqn. (4)), the symmetry transformations (53) and (54) provide us with the invariant

$$E - vp, \tag{59}$$

where $p = m\dot{x}$ is the conjugate momentum associated to x. Notice that the variational problem $\delta S_0^* = 0$ yields, for $x_0(t)$, the same solutions as those corresponding to a particle moving in the wavelike potential (40).

5. Conclusions

We considered Kaplan's approach to time driven systems in connection with: *i)* Hamilton's principle and *ii)* the symmetries of the associated action integral. The symmetries of Kaplan's "extended" conservative system were shown to arise as an appropriate linear combination of the symmetries of the original time driven system. As an example, Jacobi's integral of motion for rotating potentials was considered in this regard. The application of Kaplan's ideas to the motion of a particle in a plane-wavelike potential was also discussed.

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