Curvilinear coordinates for the evaluation of the optical transfer function

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In the present work a new coordinate system of curvilinear coordinates is proposed, in order to evaluate the optical transfer function, OTF, by the autocorrelation method. By using these curvilinear coordinates the elements of the exit pupil grid are defined by curves instead of straight lines as is the case when using rectangular coordinates. So that, these curvilinear coordinates divide the domain of integration in segments that follow the shape of the exit pupil. The curvilinear coordinates allow a better approximation in the calculation of the OTF with less elements in the grid and, therefore, a saving in computational time.

Keywords: Optical transfer function; autocorrelation; curvilinear coordinates

En el presente trabajo se propone un nuevo sistema de coordenadas, llamadas coordenadas curvilíneas, para evaluar la función de transferencia óptica, OTF, por el método de la autocorrelación. Usando estas coordenadas curvilíneas los elementos de la grilla en la pupila de salida están definidos por líneas curvas y no por líneas rectas como se obtiene al usar coordenadas rectangulares. De tal manera que dichas coordenadas curvilíneas dividen el dominio de integración en segmentos que siguen la forma de la pupila de salida. Las coordenadas curvilíneas permiten una mejor aproximación en el cálculo del OTF usando menos elementos de la grilla y, por lo tanto, se disminuye el tiempo de cómputo.

Descriptores: Función de transferencia óptica; autocorrelación; coordenadas curvilíneas

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1. Introduction

In optical design the optical transfer function, OTF, is used as a means of assessing and predicting the image quality of optical systems but it can also be used as a merit function during the optimization process. There are two basic methods for evaluating the OTF: (1) evaluating the autocorrelation of the exit pupil function, or (2) taking the inverse Fourier transform of the intensity point-spread function. When evaluating the OTF by the autocorrelation method the domain of integration is bounded by two arcs of circles of unit radius, that is, two symmetrical circular segments joined at their straight sides. In order to perform the numerical integration, the domain may be divided by a rectangular mesh into elements of equal area. This, however, gives an error in the evaluation of OTF due to the zig-zag boundaries in the domain. This problem has been discussed by J. MacDonald [1] who proposed a modified polar mesh. We introduce an alternative solution, curvilinear coordinates, for dividing the domain of integration into a number of elementary areas.

2. The OTF autocorrelation formulae

It is well known that the "normalized image contrast", $T(s)$, is defined as the rat io of the OTF under the presence of aberration, $t(s)$, to the OTF for an aberration-free system $t_0(s)$, that is,

$$T(s) = \frac{t(s)}{t_0(s)}. \tag{1}$$

H.M Hopkins [2] has shown that for the reduced spatial frequency s, the normalized image contrast, $T(s)$, is given by

$$T(s) = \frac{1}{A} \iint\limits_S \exp\{iksW(x,y;s)\}\, dx\, dy, \tag{2}$$

where $k = 2\pi/\lambda$, A is the area of the pupil, $W(x,y;s)$ is the aberration function which is a polynomial in x and y, and S is the area common to two pupil areas centred on the points $(\pm s/2, 0)$.

3. Curvilinear coordinates

In reference to Fig. 1, the domain of integration, S, denoted by the dash lines, is defined by shearing the pupil an amount proportional to the spatial frecuency. r_0 is the radius of the exit pupil which is equal to one, that is, $r_0 = 1$. Θ is the angle that subtends the line that joins the centre of curvature of the exit pupil, C, to the point where the two shearing pupils intersect, P. Finally, H is the distance between points C and O.

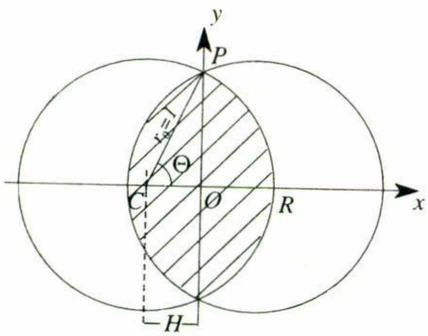

FIGURE 1. The domain of integration, S, which is denoted by the dash area, is defined by shearing the pupil an amount proportional to the spatial frecuency

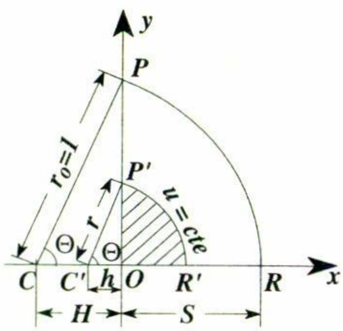

FIGURE 2. The normalized area within the arc u = const. is the value of the u-coordinate.

In order to solve the problem in a completely general way we establish a system of curvilinear coordinates u, v on the plane of the sheared pupil such that each elementary area is bounded by two pairs of lines

$$u_l = l\Delta u, \qquad u_{l+1} = (l+1)\Delta u, \tag{3}$$

$$v_m = m\Delta v, \qquad v_{m+1} = (m+1)\Delta v, \tag{4}$$

where l, m are non-negative integers and Δu Δv are the sides of the elementary areas. Inside the domain of integration such pairs of lines will divide the region into elements of equal area and will conform exactly to the shape of the boundaries. These coordinates will be normalized to extreme values of one:

$$0 \leq u \leq 1, \qquad 0 \leq v \leq 1. \tag{5}$$

Consider now the region OPR (Fig. 2) the upper half of a segment of radius $r_0 = 1$. The chord is in coincidence with the y-axis and the centre of the circle is point C on the x-axis at a distance H from the origin of coordinates O. The area of the half-segment, that is, the area of the region OPR, is

$$A = \frac{1}{2}(\Theta - H\sin\Theta), \tag{6}$$

where

$$\Theta = \cos^{-1} H. \tag{7}$$

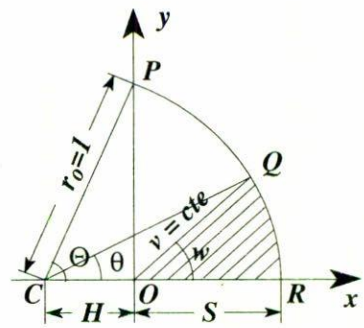

FIGURE 3. The normalized area within the arc v = const. is the value of the v-coordinate.

On the x-y plane the coordinate u = cte defines the arc of circle P'R' of radius r (see Fig. 2) with centre at point C' on the x-axis at a distance h from the origin O such that

$$h = r\cos\Theta = rH. \tag{8}$$

The value of the coordinate, u, is the ratio

$$u = \frac{\text{area(OP'R'P)}}{A} = \frac{\frac{1}{2}r^2(\Theta - \cos\Theta\sin\Theta)}{\frac{1}{2}(\Theta - \cos\Theta\sin\Theta)} = r^2, \tag{9}$$

hence,

$$r = \sqrt{u}, \qquad \frac{dr}{du} = \frac{1}{2}r. \tag{10}$$

The equation of the arc u = cte is

$$(x+h)^2 + y^2 = r^2,$$

or

$$(x+Hr)^2 + y^2 - r^2 = 0,$$

or

$$x^2 + 2Hrx + (H^2-1)r^2 + y^2 = 0. \tag{11}$$

Also, on the x-y plane the coordinate v = cte (see Fig. 3) defines the line OQ through the origin of coordinates forming an angle w with the x-axis.

The value of the coordinate v is the ratio

$$\begin{aligned} v &= \frac{1}{A}\text{area(OQR)} \\ &= \frac{1}{A}[\text{area(CQR)} - \text{area(CQO)}] \\ &= \frac{1}{2A}(\theta - H\sin\theta). \end{aligned} \tag{12}$$

For a given value of v, a first approximation to θ is

$$\theta^{(1)} = v\Theta, \tag{13}$$

successive better approximations are obtained with the Newton-Raphson method

$$v^{(n)} = \frac{1}{2A}\left[\theta^{(n)} - H\sin\theta^{(n)}\right], \tag{14}$$

$$\theta^{(n+1)} = \theta^{(n)} - \frac{v^{(n)}}{dv/d\theta}, \tag{15}$$

where

$$\frac{dv}{d\theta} = \frac{1}{2A}\left[1 - H\cos\theta\right]. \tag{16}$$

The equation of the line $v = $ cte is

$$y = x\tan w, \tag{17}$$

hence

$$\frac{\partial y}{\partial x} = \tan w, \tag{18}$$

and

$$\tan w = \frac{\sin\theta}{\cos\theta - H}. \tag{19}$$

Eliminating y from Eqs. (11) and (18) we obtain an equation of second degree in x:

$$F = (1 + \tan^2 w)x^2 + 2Hrx + (H^2 - 1)r^2 = 0. \tag{20}$$

Given the pair of curvilinear coordinates (u, v) of a point, the corresponding pair of Cartesian coordinates (x, y) is found as follows:

(1) With the value of v and the Newton-Raphson method Eqs. (14) and (15), obtain θ and then $\tan w$ with Eq. (19).

(2) With the value of u compute r with Eq. (10).

(3) Solve for x the equation of second degree (20).

(4) Compute y with Eq. (17).

The partial derivatives of x and y with respect to the coordinate u can be obtained directly from Eqs. (20), (10) and (18) [3]:

$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial r}\frac{dr}{du} = -\frac{\partial \mathrm{F}/\partial r}{\partial \mathrm{F}/\partial x}\frac{dr}{du}$$
$$= -\frac{1}{2r}\frac{Hx + (H^2 - 1)r}{(1 + \tan^2 w)x + Hr} \tag{21}$$

$$\frac{\partial y}{\partial u} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial u} = -\frac{\tan w}{2r}\frac{Hx + (H^2 - 1)r}{(1 + \tan^2 w)x + Hr}. \tag{22}$$

We shall now derive expressions for the partial derivatives of x and y with respect to the coordinate v. First, we write

$$\frac{\partial x}{\partial v} = \frac{dx}{ds}\frac{ds}{d\theta}\frac{d\theta}{dw}\frac{dw}{dv}, \tag{23}$$

$$\frac{\partial y}{\partial v} = \frac{dy}{ds}\frac{ds}{d\theta}\frac{d\theta}{dw}\frac{dw}{dv}, \tag{24}$$

where ds is the element of arc along a line $u = $ cte. From the geometry of Fig. 4,

$$\frac{dx}{ds} = -\sin\theta, \qquad \frac{dy}{ds} = \cos\theta, \tag{25}$$

and

$$\frac{ds}{d\theta} = r. \tag{26}$$

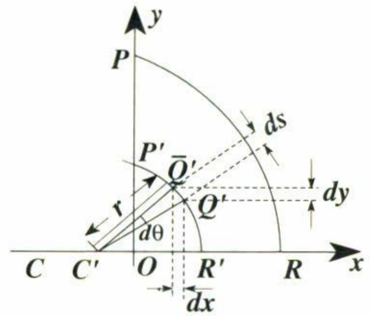

FIGURE 4. Geometry for the derivation of dx/ds, dy/ds and $ds/d\theta$.

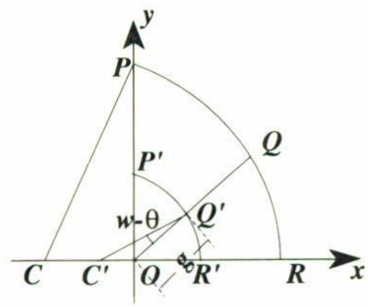

FIGURE 5. Geometry for the derivation of $d\theta/dw$.

In triangle COQ of Fig. 5 by the law of sine's

$$\frac{h}{\sin(w - \theta)} = \frac{r}{\sin w} \tag{27}$$

hence, we may write

$$F = r\sin(w - \theta) - h\sin w = 0 \tag{28}$$

then

$$\frac{d\theta}{dw} = -\frac{\partial F/\partial w}{\partial F/\partial \theta} = -\frac{r\cos(w - \theta) - h\cos w}{r\cos(w - \theta)}$$
$$= \frac{g}{r\cos(w - \theta)} \tag{29}$$

where

$$g = \overline{\mathrm{OQ}'}. \tag{30}$$

Consider the infinitesimal triangle $\mathrm{OQ\bar{Q}}$ of Fig. 6,

$$dv = \frac{\mathrm{area}(\mathrm{OQ\bar{Q}})}{A}, \tag{31}$$

where $\bar{\mathrm{Q}}$ is a point on arc $u = 1$, infinitely close to Q. It is easy to show that

$$dv = \frac{G^2}{2A}dw \tag{32}$$

where $\mathrm{G} = \overline{\mathrm{OQ}}$, then

$$\frac{dw}{dv} = \frac{2A}{G^2}. \tag{33}$$

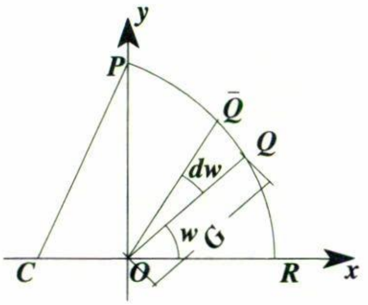

FIGURE 6. Geometry for the derivation of dw/dv.

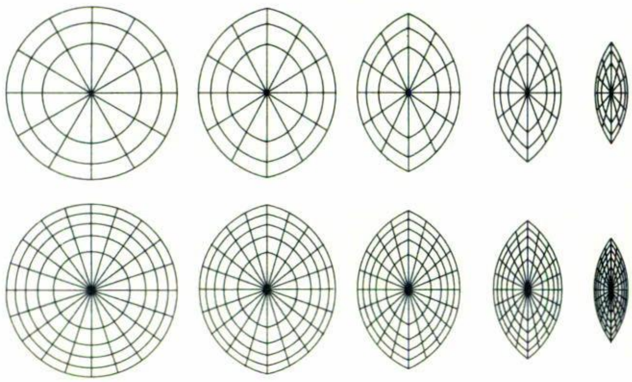

FIGURE 7. Domain of integration divided into 3 rings and 12 sectors (upper row) and 5 rings and 20 sectors (lower row) for normalized frequencies of 0%, 20%, 40%, 60% and 80% from left to right.

We now substitute Eqs. (25), (26), (28) and (33) into Eqs. (23) and (24) to obtain the following expressions:

$$\frac{\partial x}{\partial v} = -\frac{2Ag\sin\theta}{G^2\cos(w-\theta)}, \tag{34}$$

$$\frac{\partial y}{\partial v} = -\frac{2Ag\cos\theta}{G^2\cos(w-\theta)}, \tag{35}$$

By noting that,

$$h\cos\theta + g\cos(w-\theta) = r \quad \therefore \quad g = \frac{r-h\cos\theta}{\cos(w-\theta)} \tag{36}$$

$$H\cos\theta + G\cos(w-\theta) = 1 \quad \therefore \quad G = \frac{1-H\cos\theta}{\cos(w-\theta)} \tag{37}$$

so that,

$$\frac{g}{G^2\cos(w-\theta)} = \frac{(r-h\cos\theta)}{(1-H\cos\theta)^2}. \tag{38}$$

Finally, substituting in Eqs. (34) an (35)

$$\frac{\partial x}{\partial v} = -2A\frac{(r-h\cos\theta)}{(1-H\cos\theta)^2}\sin\theta. \tag{39}$$

$$\frac{\partial y}{\partial v} = 2A\frac{(r-h\cos\theta)}{(1-H\cos\theta)^2}\cos\theta. \tag{40}$$

Figure 7 shows the domain of integration.

TABLE I. Values of the MTF for an aberration-free system in focus. Note that the normalized frequency values for the case 6×6 elements of rectangular coordinates is different due to the limitation of the smallest step-size in the normalized frequency being equal to the smallest element size in the sample grid (for 6×6 squares, the smallest normalized frequency step is $1/6$).

Normalized frecuency ν/ν_0	Ideal MTF Eq. A.1	Curvilinear coordinates 3×12	Rectangular coordinates 20×20
0.	1.0	1.0	1.0
0.1	0.87291	0.87289	0.87342
0.2	0.74708	0.74706	0.74684
0.3	0.62386	0.62384	0.62025
0.4	0.50465	0.50463	0.50633
0.5	0.39101	0.391	0.39241
0.6	0.28477	0.28476	0.29114
0.7	0.18813	0.18812	0.18987
0.8	0.10409	0.10409	0.10127
0.9	0.03739	0.03739	0.03797
1.0	0.	0.	0.

Normalized frecuency ν/ν_0	Rectangular coordinates 6×6
0.	1.0
0.16667	0.8125
0.33333	0.6250
0.5	0.4375
0.66667	0.2500
0.83333	0.1250
1.0	0.0

4. Results

In order to show the approximation of using the canonical coordinates for the evaluation of the MTF for systems with circular apertures as proposed in this paper, we have evaluated the modulation transfer function for an aberration-free system in focus, in three different ways: the first one is to evaluate the area common to two pupils according to Eq. (A.1) in appendix A, the second one is by using rectangular coordinates and the third one is by using the canonical coordinates.

The results are presented in Table I, In reference to Table I, the first column shows the normalized frequency, v/v_0. The second column shows the MTF values as a function of the normalized frequency when Eq. (A.1) is used. It is important to note that the values of the MTF given by Eq. (A.1) are exact, in the sense that it gives the value of the area common to the aperture and the aperture displaced, for this reason the data in this column is named "ideal" MTF. The third column shows the data for the evaluation of the MTF when using curvilinear coordinates. The domain of integration was divided into 3 rings and 12 sectors as shown in the upper row

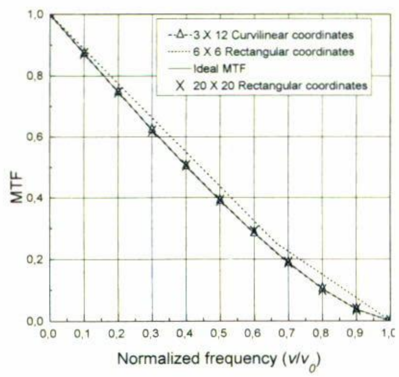

FIGURE 8. Modulation transfer function *versus* normalized frecuency, v/v_0, for an aberration-free system in focus.

of Fig. 7. The equations presented in Sect. 3 where used to perform the evaluation of the integral given by,

$$\mathrm{MTF}(v) = \frac{1}{4\pi} \iint\limits_{\substack{\text{curvilinear}\\ \text{element}}} h_u h_v \, du \, dv \tag{41}$$

where

$$h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2}$$

and

$$h_v = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2} \tag{42}$$

In Fig. 8 the data are plotted in order to show that these curvilinear coordinates allow the "ideal" MTF. The fourth column shows the data of the MTF when the pupil is divided into 20 by 20 rectangular elements. The fifth column shows the normalized frequency values, v/v_0, for the case when the grid is divided into 6 by 6 rectangular elements. Note that the difference between the in the first column and the fifth column is due to the limitation of the smallest step-size in the normalized frequency being equal to the smallest element size in the sample grid. The sixth column shows the MTF values as a function of the normalized frequency when the pupil is divided in 6 by 6 rectangular elements of equal area. If the center of a rectangular element is inside the common area of the two pupils then this element contributes to the MTF. In Fig. 8, we can appreciate the error in the evaluation of the MTF when the rectangular grid is used. To improve the approximation, it would be necessary to divide the region in 20 by 20 rectangular elements, that is, it would be necessary to perform four hundred raytraces to reach the "ideal"value of the MTF as shown in Fig. 8, instead of 36 as is the case when using curvilinear coordinates.

5. Summary and conclusions

When the optical transfer function is evaluated by the autocorrelation method, the domain of integration is defined by the shearing pupil, that is, the domain of integration is bounded by two arcs of circles of unit radius. In the present work, we have proposed a system of curvilinear coordinates such that the domain of integration is divided into elements of equal area and conform exactly the shape of the boundaries of the domain. These curvilinear coordinates u, v have been written in terms of the Cartesian coordinates x, y as well as their partial derivatives, so that the evaluation of the optical transfer function can be carried out. In the results, we show that when using curvilinear coordinates the values of the MTF are exact for an aberration-free system of circular apertures. Currently, we are working on the algorithm to implement these functions on aberrated systems of circular apertures.

Appendix A

When no aberrations are present, the MTF as a function of frequency, v, is given by the common area between two shearing pupils as [4],

$$\mathrm{MTF}(v) = \frac{2}{\pi}(\phi - \cos\phi \sin\phi) \tag{A.1}$$

where

$$\phi = \cos^{-1}\left(\frac{v}{v_0}\right) \tag{A.2}$$

v is the cutoff frequency in cycles per millimeter and v_0 is the cutoff frecuency given by

$$v_0 = \frac{2NA}{\lambda} \tag{A.3}$$

where NA is the numerical aperture and λ is the wavelength in millimeters. Note that Eq. (41) applies to circular apertures.

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