Evolution of a quantum system during measurement and decoherence

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Recibido el 16 de febrero de 1999; aceptado el 8 de abril de 1999

The process of measurement and decoherence is studied with a model consisting of a measuring apparatus in interaction with both an oscillator and the environment. In the present approach, the Weisskopf-Wigner approximation is used in order to calculate the evolution operator of the complete system, and to construct directly the density operator. The quantum fluctuations of the zero-point field make an important contribution of the decoherence; this process is described in an explicit form.

Keywords: Quantum measurement; decoherence

Se estudia el proceso de medición y decoherencia con un modelo que consiste de un aparato de medición en interacción con un oscilador y el entorno. Se utiliza la aproximación de Weisskopf-Wigner para calcular el operador de evolución del sistema completo y construir directamente el operador de densidad. Se describe explícitamente cómo las fluctuaciones cuánticas del campo de punto cero contribuyen en forma importante a la decoherencia.

Descriptores: Medición cuántica; decoherencia

PACS: 03.65.bZ; 03.65.-w

1. Introduction

Several aspects of the quantum theory of measurement have been studied in recent years [1–6]. An essential feature of quantum measurement is that the interaction with a measuring apparatus reduces the initially pure state of a measured system to a mixture of states. Additionally, the apparatus is also in interaction with its environment, and the overall effect of a measurement is the loss of quantum coherence. Decoherence is an important effect which may occur in a time of the order of the atomic time scale; it has been observed experimentally [7], and must be taken into account in high precision measurements of quantum states [8].

The process of a quantum measurement can be conveniently described with the formalism of the density operator developed by von Neumann [9]. In the approach taken by many authors, the density operator of a quantum system, in interaction with both a measuring apparatus and a thermal bath, is obtained from a master equation based on the Markoff approximation. In the present paper, we follow an approach used previously by Glauber [10], and Glauber and Man'ko [11], which is based on the Weisskopf-Wigner approximation. In this way, we obtain a Langevin type equation for the motion of the measuring 'pointer', together with the unitary operator describing the evolution of the complete system. The results are similar to those obtained with the Markoff approximation, but the present analysis is more general in certain aspects since the evolution of the states can be described directly, and the density operator is obtained directly from the analytic expressions of the states. In particular, we give an explicit description of the decoherence produced by the vacuum fluctuations of the zero-point field.

The model to be used in the present paper is essentially the one proposed by Walls, Collet and Milburn [4], with a slightly more general form of the interaction between the measuring apparatus and the measured system. Our aim is to describe explicitly the evolution of the states of a quantum system and a meter interacting with a thermal bath. All these systems are modeled by harmonic oscillators. In order to analyze each part of the measuring process, the interaction between the measured system and the apparatus is considered first in Sect. 2. The full quantum measurement process is analyzed in Sect. 3, using the Weisskopf-Wigner approximation; it is shown that the motion of the pointer is described by a Langevin equation, and explicit expressions are obtained for the density operator of the full system and its reduced forms. A basic result is that decoherence is produced both by the thermal bath and the zero-point field. The limit of zero temperature is considered in Sect. 4, where the complete evolution of the states is described analytically.

2. System and measuring apparatus

In order to gain some insight into the physical processes involved, let us consider first the case of a quantum system in interaction with a measuring apparatus, without, for the moment, taking into account the effects of the environment. The system and apparatus will be modeled by a pair of harmonic oscillators. The apparatus consists of a pointer in direct interaction with the system, and the pointer is in a coherent state. The quantum system is taken to be an oscillator in a certain energy state, and the particles number of this state is the quantity to be measured.

The Hamiltonian to be used is

$$H_{SM} = \hbar\omega_s N + \hbar\omega_0 a^\dagger a + \hbar N(f^* a + f a^\dagger), \tag{1}$$

where N is the particles number operator of the system, a and $a^\dagger$ are the annihilation and creation operators of the measuring apparatus, ω_s and ω_0 are the frequencies of the system and apparatus respectively, and f is a certain function of time which controls the interaction. The number operator commutes with the boson operators of the pointer. This Hamiltonian describes a typical non demolition measurement, since the interaction term commutes with the Hamiltonian of the measured system. As a consequence, the energy of the system is conserved and the number operator is a constant of motion: $dN/dt = 0$.

The Heisenberg equation of motion for the meter takes the form

$$\frac{da}{dt} = -i\omega_0 a - i f N, \tag{2}$$

and its solution is

$$a(t) = e^{-i\omega_0 t}\left[a(0) - iN\int_0^t e^{i\omega_0 t'} f(t')\, dt'\right]. \tag{3}$$

The classical version of this equation describes a harmonic oscillator with an external force proportional to $\dot{f}(t)N$.

A particularly important case is that of a periodic external force, such that

$$f(t) = f_0 e^{-i\omega_0 t + i\epsilon t}, \tag{4}$$

where the case $\epsilon = 0$ corresponds to a resonance between the external force and the characteristic frequency of the pointer. This is the most important case, since it corresponds to an ideal measurement producing maximum motion of the pointer; in fact, if $\epsilon = 0$, the pointer will move with constant velocity. It is also worth noting that the case studied previously by Mancini and Tombesi [12], and Bose, Jacobs, and Knight [13] corresponds to a constant value of the function $f(t)$; in other words, it is a special case of Eq. (4) with $\omega_0 = \epsilon$.

We now look for the unitary operator U_{SM} associated to the Hamiltonian (1). The equation

$$i\hbar\frac{\partial U_{SM}}{\partial t} = H_{SM} U_{SM} \tag{5}$$

admits a solution of the form

$$U_{SM} = e^{-i\omega_s t N + i v N^2 - \frac{1}{2}\Gamma\Gamma^* N^2} e^{-\Gamma^* N a^\dagger} e^{\Gamma N a} e^{-i\omega_0 t a^\dagger a}, \tag{6}$$

where, in general,

$$\Gamma(t) = -i\int_0^t f^*(t') e^{i\omega_0(t-t')}\, dt' \tag{7}$$

and

$$v(t) = \frac{1}{2}\int_0^t [\Gamma(t') f(t') + c.\,c.\,]\, dt'. \tag{8}$$

In the particular case of an external force described by Eq. (4), it follows that

$$\Gamma = \frac{f_0^*}{\epsilon} e^{i\omega_0 t}(e^{-i\epsilon t} - 1) \tag{9}$$

and

$$v = \frac{|f_0|^2}{\epsilon}\left[t - \frac{1}{\epsilon}\sin(\epsilon t)\right]. \tag{10}$$

Given the unitary operator in its general form (6), it is straightforward to find the evolution of any state of the system and the measuring apparatus. Suppose, then, that the system is initially in the state $|N\rangle_S$ with well defined energy, and that the pointer is in the coherent state $|\alpha\rangle_M$. If the interaction between the system and the pointer acts during a time interval, t, starting at $t = 0$, then the initial state $|\psi(0)\rangle \equiv |N\rangle_S|\alpha\rangle_M$ evolves into the state

$$|\psi(t)\rangle = e^{-i\Phi N + i v N^2}|N\rangle_S|\alpha e^{-i\omega_0 t} - \Gamma^* N\rangle_M, \tag{11}$$

where

$$\Phi = \omega_s t + \frac{i}{2}(\Gamma\alpha e^{-i\omega_0 t} - \Gamma^*\alpha^* e^{i\omega_0 t}) \tag{12}$$

is a phase, and v and Γ are given in general by Eqs. (7) and (8), or in the particular case of a resonant external force, by Eqs. (9) and (10). To obtain equation (11), we have used the definition of a coherent state $a|\alpha\rangle = \alpha|\alpha\rangle$, together with the formula

$$e^{\beta a^\dagger} e^{i\phi a^\dagger a}|\alpha\rangle = e^{\frac{1}{2}(|e^{i\phi}\alpha + \beta|^2 - |\alpha|^2)}|e^{i\phi}\alpha + \beta\rangle, \tag{13}$$

valid for any coherent state. We see from Eq. (11) that, after the measurement, the pointer has moved to a new coherent state which depends on the quantum observable N, and that the motion also depends implicitly, through the function Γ, on the form of the interaction between observed system and apparatus. Clearly, the ideal measurement corresponds to the resonant case $\epsilon = 0$, since this produces the maximum motion of the pointer.

Suppose now that the system is not initially in a state with well defined energy, but rather in a superposition of energy eigenstates; that is

$$|\psi(0)\rangle = \sum_{N=0}^{\infty} C_N |N\rangle_S|\alpha\rangle_M. \tag{14}$$

After a time t, the new state is given by a superposition of states which can be determined using the unitary operator obtained above. Then, the full density operator $|\psi(t)\rangle\langle\psi(t)|$ can be directly traced over the meter states using the relation

$$\mathrm{Tr}\,|\beta\rangle\langle\alpha| = \langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^*\beta}, \tag{15}$$

and it follows that the reduced density operator of the measured system evolves according to

$$\rho_S = \sum_{NM} C_N C_M^* \times e^{-i\omega_s t(N-M) + i v(N^2 - M^2) - \frac{1}{2}(N-M)^2|\Gamma|^2}|N\rangle\langle M|. \tag{16}$$

Notice the important fact that the exponential in the above equation contains negative definite terms for the non diagonal elements. For instance, near a resonance, $\epsilon \ll \omega_0$, the function $|\Gamma|$ increases linearly with time. This implies that one can improve the precision of the measurement taking the characteristic frequency of the interaction term to be in resonance with the pointer, but the precision is obtained at the expense of suppressing all non diagonal terms of the density operator.

3. Decoherence

After the preliminary analysis of the previous section, we are in a position to study a complete system consisting of a harmonic oscillator (describing the quantum system to be measured), a measuring apparatus (also described by a harmonic oscillator) and a thermal bath in interaction with the apparatus. The Hamiltonian of the model is [4]

$$H = H_{SM} + \sum_n \omega_n b_n^\dagger b_n + \hbar \sum_n (\kappa_n a^\dagger b_n + \kappa_n^* b_n^\dagger a), \quad (17)$$

where b_n and $b_n^\dagger$ are the boson operators of the thermal bath corresponding to modes of frequency ω_n, and κ_n are coupling constants between the pointer and the reservoir. As a first step, we note that the Heisenberg equations of motion take the form

$$\frac{da}{dt} = -i(\omega_0 a + \sum_n \kappa_n b_n + fN) \quad (18)$$

and

$$\frac{db_n}{dt} = -i(\kappa_n^* a + \omega_n b_n), \quad (19)$$

and, of course, $dN/dt = 0$. Eq. (19) admits the formal solution

$$b_n(t) = e^{-i\omega_n t} b_n(0) - i\kappa_n^* \int_0^t a(t') e^{i\omega_n(t'-t)}\, dt'; \quad (20)$$

and therefore, defining $a_0(t) \equiv a(t)e^{i\omega_0 t}$, Eq. (18) takes the form

$$\frac{da_0(t)}{dt} = -\sum_n |\kappa_n|^2 \int_0^t a_0(t') e^{i(\omega_n-\omega_0)(t'-t)}\, dt' + G(t), \quad (21)$$

where

$$G(t) = -ie^{i\omega_0 t}\Big[\sum_n \kappa_n e^{-i\omega_n t} b_n(0) + Nf(t)\Big]. \quad (22)$$

As shown by Glauber [10] (see also Ref. 11), Eq. (21) can be solved using standard techniques of Laplace transforms. Defining

$$\tilde{a}_0(s) = \int_0^\infty e^{-st} a_0(t)\, dt, \quad (23)$$

it follows from Eq. (21) that:

$$\tilde{a}_0(s) = \frac{a(0) + \tilde{G}(s)}{s + \sum_n \dfrac{|\kappa_n|^2}{s + i(\omega_n - \omega_0)}}, \quad (24)$$

where $\tilde{G}(s)$ is the Laplace transform of $G(t)$.

To proceed further, we use the Weisskopf-Wigner approximation. This approximation consists in taking the pole near $s = 0$ in Eq. (24) as the leading contribution to the inverse Laplace transform, and then looking for first order corrections in $|\kappa|^2$ to this pole. This amounts to the replacement (see, *e.g.* Ref. 14, Chap. 7)

$$-i\sum_n \frac{|\kappa_n|^2}{\omega_n - \omega_0 - is} \to -i\int \frac{g(\omega_n)|\kappa(\omega_n)|^2 d\omega_n}{\omega_n - \omega_0 - is} \to \frac{\gamma}{2} + i\Delta\omega, \quad (25)$$

where $g(\omega_n)$ is the density of modes,

$$\gamma = 2\pi g(\omega_0)|\kappa(\omega_0)|^2, \quad (26)$$

and

$$\Delta\omega = -\int \frac{g(\omega_n)|\kappa(\omega_n)|^2}{\omega_n - \omega_0}\, d\omega_n. \quad (27)$$

As it will become clear below, the term γ is the damping coefficient, and $\Delta\omega$ is the energy shift of the meter due to its interaction with all the modes of the background radiation field.

The outcome of this approximation is exactly the same as if one had changed Eq. (21) to the equation

$$\frac{da(t)}{dt} + \left[\frac{\gamma}{2} + i(\omega_0 + \Delta\omega)\right] a(t) = -i\sum_n \kappa_n b_n(0) e^{-i\omega_n t} - iNf(t). \quad (28)$$

This is a Langevin equation and its solution can be written in the form

$$a(t) = \langle a(t)\rangle - i\sum_n Q_n(t) b_n(0), \quad (29)$$

where the average $\langle a\rangle$ of the operator a is

$$\langle a(t)\rangle = a(0)e^{-[\gamma/2 + i(\omega_0+\Delta\omega)]t} - iN\int_0^t e^{[(\gamma/2)+i(\omega_0+\Delta\omega)](t'-t)} f(t')\, dt', \quad (30)$$

and the fluctuating part is given in terms of the functions

$$Q_n(t) = \kappa_n \frac{e^{-i\omega_n t} - e^{-[(\gamma/2)+i(\omega_0+\Delta\omega)]t}}{(\gamma/2) + i(\omega_0 + \Delta\omega - \omega_n)}. \quad (31)$$

3.1. Thermal average

To proceed further, we recall that the average of any operator O over a thermal bath at temperature T is defined as

$$\langle O\rangle = \frac{\mathrm{Tr}(Oe^{-H_B/k_BT})}{\mathrm{Tr}(e^{-H_B/k_BT})}, \tag{32}$$

where, in accordance with the Hamiltonian appearing in Eq. (17),

$$H_B = \sum_n \omega_n b_n^\dagger b_n \tag{33}$$

in the Schrödinger representation. It follows, in particular, that

$$\langle b_n^\dagger(0)\rangle = \langle b_n(0)\rangle = 0 \tag{34}$$

and

$$\langle b_n^\dagger(0)b_m(0)\rangle = \delta_{nm}\mathcal{N}(\omega_n), \tag{35}$$

where

$$\mathcal{N}(\omega) = \frac{1}{e^{\hbar\omega/k_BT}-1}. \tag{36}$$

It further follows that

$$\langle a^\dagger(t)a(t)\rangle - \langle a^\dagger(t)\rangle\langle a(t)\rangle = \sum_n \mathcal{N}(\omega_n)|Q_n(t)|^2, \tag{37}$$

and since $\sum_n |Q_n(t)|^2$ is strongly peaked around $\omega_n = \omega_0 + \Delta\omega \equiv \omega_0'$, we have approximately (see Appendix A)

$$\langle a^\dagger(t)a(t)\rangle - \langle a^\dagger(t)\rangle\langle a(t)\rangle \approx (1-e^{-\gamma t})\mathcal{N}(\omega_0'); \tag{38}$$

This last formula shows that the average energy of the pointer tends to a thermal equilibrium according to its characteristic frequency of oscillation.

Notice also that, in general, the averaged operators satisfy the commutation relation

$$\Big[\langle a(t)\rangle, \langle a^\dagger(t)\rangle\Big] = e^{-\gamma t}, \tag{39}$$

but the complete operators do not violate the Heisenberg uncertainty relations. Indeed, the condition $[a, a^\dagger] = 1$ implies that

$$\sum_n |Q_n(t)||^2 = 1 - e^{-\gamma t}; \tag{40}$$

this relation can be proved independently, as shown in Appendix A.

3.2. System operator

Let us now consider an operator S which depends only on the variables of the system to be measured. If the Hamiltonian (17) describes the complete system, the Heisenberg equation of motion reduces to

$$\frac{dS}{dt} = -i(\omega_s + f^* a + f a^\dagger)[S, N]. \tag{41}$$

Clearly, the natural basis for the operator S is the number basis; accordingly, we set $S_{MN} = \langle M|S|N\rangle$, and the solution of Eq. (60) takes the form

$$S_{MN}(t) = e^{i(M-N)W(t)} S_{MN}(0), \tag{42}$$

where the function W is

$$W = \omega_s t + \int_0^t (f^* a + f a^\dagger)\, dt'. \tag{43}$$

It is convenient to write this function in a form exhibiting explicitly the averaged and the fluctuating parts:

$$W(t) = \langle W(t)\rangle - i\sum_n K_n(t) b_k(0) + i\sum_n K_n^*(t) b_n^\dagger(0); \tag{44}$$

here, the functions K_n are defined through $dK_n/dt = f^* Q_n$.

The thermal average of the evolution operator appearing in Eq. (42) can be calculated using Eq. (B.6). The result is

$$\langle e^{i(M-N)W}\rangle = e^{i(M-N)(\omega_s t + \langle W\rangle)} \times \exp\Big\{ -(M-N)^2 \sum_n |K_n(t)|^2 \Big[\frac{1}{2} + \mathcal{N}(\omega_n)\Big]\Big\}, \tag{45}$$

from where the thermal average of any operator which depends on the system variables can be calculated.

Consider the particular case of a periodic external force described by the function f given in Eq. (4). It follows that

$$K_n(t) = -i\frac{f_0^*\kappa_n}{(\gamma/2) + i(\omega_0 + \Delta\omega - \omega_n)} \times \Big[\frac{e^{i(\omega_0-\omega_n-\epsilon)t} - 1}{i(\omega_0 - \omega_n - \epsilon)} + \frac{e^{-[(\gamma/2)+i(\epsilon+\Delta\omega)]t} - 1}{(\gamma/2) + i(\epsilon + \Delta\omega)}\Big], \tag{46}$$

and the expression appearing in Eq. (45) is given by

$$\sum_n |K_n|^2 \Big[\frac{1}{2} + \mathcal{N}(\omega_n)\Big] \approx |f_0|^2 \frac{\gamma t}{(\gamma/2)^2 + (\Delta\omega + \epsilon)} \Big[\frac{1}{2} + \mathcal{N}(\omega_0)\Big]. \tag{47}$$

for large values of t (see Appendix C).

In particular, if the system is initially in the state

$$|\Psi\rangle = \sum_N C_N |N\rangle \tag{48}$$

(in the Schrödinger representation), then the elements of the density operator ρ of the system are $C_M C_N^*$, in the number representation, and furthermore, according to Eqs. (42)–(45), the evolution of this operator is given by

$$\rho(t) = \sum_{MN} e^{i(M-N)\left[\omega_s t + \langle W(t)\rangle\right]} C_M C_N^* |M\rangle\langle N| \exp\left\{ -(M-N)^2 |f_0|^2 \frac{\gamma t}{(\gamma/2)^2 + (\Delta\omega + \epsilon)} \left[\frac{1}{2} + \mathcal{N}(\omega_0)\right]\right\}. \quad (49)$$

We see that the non diagonal terms of this operator decay exponentially. The decay coefficient depends on the interaction time t between the meter and the system, the strength of the measuring interaction, $|f|$, and also the temperature of the environment. It is a noteworthy feature of the above result that the decoherence is not suppressed at zero temperature, since there is always a contribution of the zero point field, given by the term $1/2$ which adds to the Planck distribution; this term is due to the quantum fluctuations in vacuum.

4. Zero temperature limit

Since decoherence is not suppressed at zero temperature, it is worth analyzing the evolution of a complete system including a background field without thermal radiation but only vacuum fluctuations. Starting again with the complete Hamiltonian (17), we calculate the unitary operator U which is associated to it. Since we have already solved the problem for the system coupled with the measuring apparatus, it is convenient to define an operator V such that $U = U_{SM}V$; then, this new operator satisfies the equation

$$i\hbar\frac{\partial V}{\partial t} = H'V, \quad (50)$$

where $H' = U_{SM}^{-1}(H - H_{SM})U_{SM}$ is an effective Hamiltonian.

Now, for the operator U_{SM} given by Eq (6), we notice that any function $F(a, a^\dagger)$ of the operators a and $a^\dagger$ satisfies the relation

$$U_{SM}^{-1}F(a,a^\dagger)U_{SM} = F(ae^{-i\omega_0 t} - N\Gamma^*, \quad a^\dagger e^{i\omega_0 t - N\Gamma}), \quad (51)$$

from where a straightforward algebra leads to the result:

$$H' = \hbar\sum_n \omega_n b_n^\dagger b_n + \hbar\sum_n(\kappa_n e^{i\omega_0 t}a^\dagger b_n + \kappa_n^* e^{-i\omega_0 t}b_n^\dagger a) - \hbar N\sum_n(\kappa_n\Gamma b_n + \kappa_n^*\Gamma^* b_n^\dagger). \quad (52)$$

In order to solve Eq. (50) with the above Hamiltonian, the method of normal ordered operators will be used (Ref. 14, Chap. 3). Define

$$e^G \equiv V(\alpha, \beta_n, \alpha^*, \beta_n^*) = \langle\alpha, \beta_n|V^{(n)}|\alpha, \beta_n\rangle \quad (53)$$

where $V^{(n)}$ is the operator V in normal ordered form, and $|\alpha, \beta_n\rangle$ is the coherent state of the meter and the background. Then, Eq. (50) together with the Hamiltonian (52) is equivalent to

$$i\frac{\partial G}{\partial t} = \sum_n \Big[\omega_n\beta_n^*\Big(\beta_n + \frac{\partial G}{\partial\beta_n^*}\Big) + \kappa_n e^{i\omega_0 t}\alpha^*\Big(\beta_n + \frac{\partial G}{\partial\beta_n^*}\Big) + \kappa_n^* e^{-i\omega_0 t}\beta_n^*\Big(\alpha + \frac{\partial G}{\partial\alpha^*}\Big) - N\Gamma\kappa_n\Big(\beta_n + \frac{\partial G}{\partial\beta_n^*}\Big) - N\Gamma^*\kappa_n^*\beta_n^*\Big]. \quad (54)$$

Setting now the ansatz

$$G = A\alpha^*\alpha + \sum_{mn} B_{mn}\beta_m^*\beta_n + \sum_n C_n\alpha^*\beta_n + \sum_n D\alpha\beta_n^* + N\big[\lambda\alpha + \mu\alpha^* + \sum_n \nu_n\beta_n + \sum_n \rho_n\beta_n^*\big], \quad (55)$$

where A, B_{rs}, C_r, D, λ, μ, ν_r, and ρ_n are all functions of time t (vanishing at $t = 0$), and substituting it in Eq. (54), a closed set of differential equations for all these functions can be obtained.

For our present purpose, we only need the functions μ and ρ_n, so that we restrict our calculations to these functions. Consider then the equations

$$\frac{d\mu}{dt} = -ie^{i\omega_0 t}\sum_n \kappa_n\rho_n \quad (56)$$

and

$$\frac{d\rho_n}{dt} = -i\omega_n\rho_n - i\kappa_n^*(e^{-i\omega_0 t}\mu - \Gamma^*), \quad (57)$$

that follow from Eqs. (54) and (55). At this stage, it is convenient to define

$$\zeta = \mu - e^{i\omega_0 t}\Gamma^*, \quad (58)$$

from where it follows that

$$\rho_n = -i\kappa_n^* e^{-i\omega_n t}\int_0^t e^{i(\omega_n - \omega_0)t'}\zeta(t')dt'. \quad (59)$$

Using now the Weisskopf-Wigner approximation just as in Sect. 3, we find that the solution of the above equations is given by

$$\zeta(t) = -i \int_0^t e^{[(\gamma/2)+i\Delta\omega](t'-t)} e^{i\omega_0 t'} f(t')\, dt'. \qquad (60)$$

Furthermore, in the particular case of a periodic external force f, given by Eq. (4), we find

$$\zeta = -i f_0 \frac{e^{i\epsilon t} - e^{-[(\gamma/2)+i\Delta\omega]t}}{(\gamma/2) + i(\Delta\omega + \epsilon)} \qquad (61)$$

and the functions ρ_n follow directly from Eq. (59).

We are now in a position to study the time evolution of the complete state $|N\rangle_S|0\rangle_M|\mathbf{0}\rangle_R$. For simplicity, it is assumed that the meter is initially in the ground state. The complete evolution operator is a product of the operator U_{SM} given by Eq. (6) and the operator V given by Eqs. (53) and (54); therefore

$$U|N\rangle_S|0\rangle_M|\mathbf{0}\rangle_R = e^{-i\omega_s tN + i\Phi N^2}|N\rangle_S|Ne^{-i\omega_0 t}\zeta\rangle_M|N\rho_n\rangle_R, \qquad (62)$$

where the phase Φ is given by

$$\Phi = v - \frac{i}{2}(\zeta\Gamma e^{-i\omega_0 t} - c.c.). \qquad (63)$$

It clearly follows that, due to the measurement, the background field is excited to an infinite-mode coherent state; similarly, the pointer state moves to a coherent state with an amplitude proportional both to the strength of the interaction and to the energy of the state to be measured.

In general, the state of the system is a superposition of energy states, and the complete initial state is given by

$$|\Psi(0)\rangle = \Big(\sum_N C_N|N\rangle_S\Big)|0\rangle_M|\mathbf{0}\rangle_R. \qquad (64)$$

Therefore, according to Eq. (62), the full density operator

$$\rho_{SMR}(t) = U(t)|\Psi(0)\rangle\langle\Psi(0)|U^{-1}(t) \qquad (65)$$

reduces, after tracing over the background states, to the density operator for the system and apparatus

$$\rho_{SM} = \sum_{NM} C_N C_M^* \exp\Big\{ - i\omega_s t(N-M) + i\Phi(N^2 - M^2) - \frac{1}{2}(N-M)^2 \sum_n |\rho_n|^2 \Big\} \times |N\rangle_S|Ne^{-i\omega_0 t}\zeta\rangle_{M\,M}\langle Me^{-i\omega_0 t}\zeta|_s\langle M|, \qquad (66)$$

where we have traced over the reservoir states using the fact, just as in Sect. 2, that we are dealing with coherent states.

We can further trace the operator ρ_{SM} over the meter states. Then the final result turns out to be

$$\rho_s = \sum_{NM} \exp\Big\{ - i\omega_s t(N-M) + i\Phi(N^2 - M^2) - \frac{1}{2}(N-M)^2\big(\sum_n |\rho_n|^2 + |\zeta|^2\big)\Big\} C_N C_M^* |N\rangle_{SS}\langle M|. \qquad (67)$$

The term $\sum_n |\rho_n|^2 + |\zeta|^2$ appearing in the above formula is evaluated in Appendix B. It is sufficient to notice that for the oscillating force given by Eq. (4), and for $\gamma t \gg 1$, this term is approximately

$$|\rho_n|^2 + |\zeta|^2 \approx \frac{|f_0|^2 \gamma t}{(\gamma/2)^2 + (\Delta\omega + \epsilon)^2}, \qquad (68)$$

which is completely consistent with Eq. (49). Once more we find that the non diagonal terms of the density operator vanish exponentially. As for the phase given by Eq. (63), it is simply

$$\Phi \approx \frac{|f_0|^2}{\epsilon} t, \qquad (69)$$

in this particular case.

At this point, it is interesting to compare the density matrix given by Eq. (67), together with Eq. (D.4), with the one obtained using the Markoff approximation. The result obtained by Walls, Collet,and Milburne [4] coincides exactly with our result in the limiting case where both the energy shift $\Delta\omega$ and the resonance displacement ϵ vanish. Thus, we may conclude that the Markoff approximation is valid as long as the external force is close to resonance with the meter, and the frequency displacement of the meter due the background radiation field is negligible.

5. Conclusions

To sum up, we have developed an approach to the problem of quantum decoherence based on the Weisskopf-Wigner approximation. An advantage of these formulation is that the evolution operator can be obtained in a closed form. This, in turns, permits to calculate the evolution of the states associated with a quantum system, a measuring apparatus and the environment. From our results, it is explicitly seen how the pointer, modeled as a coherent state, moves when an interaction with the measured system is switched on, and it is also clear what the effects of this measurement are on the background field. The emphasis of this work has been on the interaction between a quantum system and the environment mediated by a measuring apparatus. A direct interaction between system and environment can also be studied with the present formalism; this will be the subject of future work.

It is well known that the zero-point field is at the origin of many quantum phenomena, such as the Casimir effect [15], and it also plays a basic role in the process of decoherence leading from quantum to classical world (see, *e.g.*, Ref. 16). This is shown explicitly in our formule (68).

A possible realization of the present model could be achieved with a photoelectron counter. The thermal bath and the detector represent the usual interaction producing a linear loss mechanism, and the meter measures the number of quanta per unit time. This particular realization is discussed in Ref. 4.

Acknowledgments

The author is grateful to R. Jáuregui and L. de la Peña for many useful comments and discussions

Appendix A

Consider a sum of the form $\sum_n |Q_n|^2$, where Q_n is given by Eq. (31). This sum can be approximated by an integral of the form

$$I \equiv \int_{-\omega_0-\Delta\omega}^{\infty} \frac{|\kappa(\omega_n)|^2 g(\omega_n)}{(\gamma/2)^2 + x^2} \times \left[e^{-\gamma t} + 1 - 2e^{-(\gamma/2)t}\cos(xt)\right] dx, \quad \text{(A.1)}$$

where $x = \omega_n - \omega_0 - \Delta\omega$. Since the integrand is strongly peaked around $x = 0$ (that is, around $\omega_n = \omega_0 + \Delta\omega \approx \omega_0$) the slowly varying term $|\kappa(\omega_n)|^2 g(\omega_n)$ can be taken outside the integration as $|\kappa(\omega_0)|^2 g(\omega_0)$, and, furthermore, the lower range of the integral can be extended to $-\infty$. Then, using

$$\int_{-\infty}^{\infty} \frac{dx}{(\gamma/2)^2 + x^2} = \frac{2\pi}{\gamma} \quad \text{(A.2)}$$

and

$$\int_{-\infty}^{\infty} \frac{\cos(xt)dx}{(\gamma/2)^2 + x^2} = \frac{2\pi}{\gamma} e^{-\gamma t/2}, \quad \text{(A.3)}$$

together with the definition (26) of γ, it follows that

$$I = 1 - e^{-\gamma t}, \quad \text{(A.4)}$$

in full agreement with Eq. (40).

Appendix B

The thermal average of any operator can be conveniently calculated using the following theorem (Ref. 14, p. 161): For any function $F(b, b^\dagger)$ of the boson operators b and $b^\dagger$, its thermal average is given by

$$\langle F(b,b^\dagger)\rangle \equiv (1 - e^{-\lambda})\mathrm{Tr}\left(F(b,b^\dagger)e^{-\lambda b^\dagger b}\right) =_{c,b} \langle 0,0|F\left(\sqrt{1+\mathcal{N}}b + \sqrt{\mathcal{N}}c^\dagger, \sqrt{1+\mathcal{N}}b^\dagger + \sqrt{\mathcal{N}}c\right)|0,0\rangle_{b,c}, \quad \text{(B.1)}$$

where

$$\mathcal{N} = \frac{1}{e^\lambda - 1}, \quad \text{(B.2)}$$

and c, $c^\dagger$ are dummy boson operators which commute with b and $b^\dagger$, and have an associated vacuum state $|0\rangle_c$. From

$$e^{y(\sqrt{1+\mathcal{N}}b + \sqrt{\mathcal{N}}c^\dagger)}|0,0\rangle_{b,c} = e^{\frac{1}{2}|y|^2\mathcal{N}}|0\rangle_b|y\sqrt{\mathcal{N}}\rangle_c, \quad \text{(B.3)}$$

it follows that

$$\langle e^{xb^\dagger}e^{yb}\rangle = e_c^{\frac{1}{2}(|x|^2+|y|^2)\mathcal{N}}\langle x^*\sqrt{\mathcal{N}}|y\sqrt{\mathcal{N}}\rangle_c, \quad \text{(B.4)}$$

where $|\ldots\rangle_c$ are the coherent states associated to the dummy operators. The expectation values in Eq. (B.4) can be easily calculated using the formula for the scalar products of coherent states, and it finally follows that

$$\langle e^{xb^\dagger}e^{yb}\rangle = e^{xy\mathcal{N}}. \quad \text{(B.5)}$$

From this last expression, a general formula valid for an infinite number of modes can be obtained:

$$\langle \exp\left\{\sum_n (x_n b_n^\dagger + y_n b_n)\right\}\rangle = \times \exp\left\{\frac{1}{2}\sum_n x_n y_n\right\}\exp\left\{\sum_n x_n y_n \mathcal{N}(\omega_n)\right\}, \quad \text{(B.6)}$$

where now the thermal average is as defined in Eq. (32), and use has been made of the Baker-Campbell-Hausdorff formula:

$$e^{xb^\dagger + yb} = e^{xb^\dagger}e^{yb}e^{\frac{1}{2}xy}. \quad \text{(B.7)}$$

Appendix C

If the external force f is given by Eq. (4), the functions K_n, defined by Eq. (44), take the explicit form given in Eq.(46). It then follows that

$$\sum_n |K_n|^2 \approx |f_0|^2|\kappa(\omega_0)|^2 g(\omega_0)\mathcal{I}(t), \quad \text{(C.1)}$$

where we have defined the integral

$$\mathcal{I}(t) = \int_{-\infty}^{\infty} \frac{dx}{(\gamma/2)^2 + x^2} \Big| \frac{e^{-i(x+u)t} - 1}{x + u} - i\chi \Big|^2, \quad \text{(C.2)}$$

with $x = \omega_n - \omega_0 - \Delta\omega$, $u = \Delta\omega + \epsilon$,

$$\chi = \frac{e^{-(\gamma/2)t - iut} - 1}{(\gamma/2) + iu}, \quad \text{(C.3)}$$

and the following approximations have been made: it is assumed that $|\kappa|^2 g$ is a slowly varying function and that the integrand is strongly peaked around $x = 0$ (which also implies that the lower limit of integration can be taken as $-\infty$). The integral $\mathcal{I}$ can be given in exact form using successive integration with respect of t of the integral given by Eq. (A.2). Instead of going into a textbook exercise in integration, we simply note that the integral $\mathcal{I}$ contains several oscillating terms and one dominant term of order γt, from where it follows that for large time t,

$$\mathcal{I} \approx \frac{2\pi t}{(\gamma/2)^2 + u^2}. \quad \text{(C.4)}$$

We finally obtain Eq. (47) as a good approximation valid for $\gamma t \gg 1$, with the additional assumption that the Planck factor in the integrand can also be taken outside the integration using again the fact that the whole integrand is strongly peaked around $\omega_n = \omega_0 + \Delta\omega$.

Appendix D

Since, from the definition (58),

$$\frac{d\zeta}{dt} = -ie^{i\omega_0 t}\Big(\sum_n \kappa_n \rho_n + f\Big) \quad \text{(D.1)}$$

and also from Eq. (59),

$$\frac{d}{dt}(e^{i\omega_0 t}\rho_n) = -i\kappa_n^* e^{i(\omega_n - \omega_0)t}\zeta, \quad \text{(D.2)}$$

it follows by direct substitution that

$$\frac{d}{dt}\Big(\sum_n |\rho_n|^2 + |\zeta|^2\Big) = -ie^{i\omega_0 t} f\zeta^* + ie^{-i\omega_0 t} f^*\zeta. \quad \text{(D.3)}$$

This is a general formula valid for any function f. In the particular case that the external force f is given by Eq. (4), it follows by direct integration that

$$\sum_n |\rho_n|^2 + |\zeta|^2 = |f_0|^2 \frac{\gamma t}{(\gamma/2)^2 + (\Delta\omega + \epsilon)} + |f_0|^2 \Big[\frac{e^{-(\gamma/2)t + i(\epsilon + \Delta\omega)t} - 1}{[(\gamma/2) - i(\epsilon + \Delta\omega)^2]^2} + c.\,c.\Big]. \quad \text{(D.4)}$$

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