Transient heat conduction in a plate with variable thermal conductivity and non-linear boundary conditions by network simulation method

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The thermal response of a plate is predicted using an analysis based on the network simulation method. One side of the plate is exposed to a time-dependent radiative heat flux on the surface, simulating solar incident radiation flux, and polynomial temperature dependent thermal conductivity is assumed. A general network model is proposed for this process, whatever the polynomial degree and including initial and boundary conditions. With this network model and using the electrical circuit simulation program, PSPICE, time-dependent temperature and heat flow profiles can be obtained at any location

Keywords: Network simulation method; digital simulation; nonlinear heat conduction

Se obtiene la respuesta térmica de una placa, cuya conductividad depende polinómicamente de la temperatura, mediante el método de simulación por redes; una de sus caras se somete a radiación armónica incidente, simulando la exposición solar; condiciones de convección y re-radiación son añadidas a ambos lados de la placa. Se diseña un modelo de circuito correspondiente a este proceso, independiente del grado del polinomio, que incluye las condiciones de contorno e iniciales. Su simulación, mediante el programa de resolución de circuitos PSPICE, permite obtener la totalidad del campo térmico transitorio y los flujos de calor en cada sección.

Descriptores: Método de simulación por redes; simulación digital; conducción de calor no lineal

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1. Introduction

The evaluation of time-dependent responses in solids possessing variable thermal properties is of great importance for many engineering applications. Transient non-linear thermal analysis of structures and materials subjected to harsh thermal environments is important because non-linear and/or non-uniform heatiing may have a significant effect on their performance characteristics. It is of particular importance in the development of advanced materials like ceramics and glass, or in the study of component behavior and structural design in furnace technology, high temperature engines or energy storage systems. Solar panel plates, space and aircraft frames, and the metal supports of buildings are just some of the cases to which the technique may be applied. Several approximate and numerical methods have been suggested for overcoming the difficulties inherent in this problem [1–6]. In this work we are interested in investigating conduction in a plate of polynomial variable thermal conductivity, the front surface of wich is exposed to a time-dependent radiative heat flux on the surface, simulating solar incident radiation flux. Initially the plate is isothermal and at the same temperature as its environment, which has a temporal sinusoidal-variation, a case which is similar to that of a solar collector. The boundary condition on the front face is radiative heating countered by natural convective cooling and radiant exchange with the background radiation sink. Energy is also lost to the environment on the rear face of the plate by natural convection and radiation. Variations in convection coefficient with temperature difference between surface and surrounding air are assumed.

This investigation uses the network simulation method, which is well suited to treating transport processes [7–9] and which allows the description of any such process in terms of electrical networks. The spatial variable in the heat conduction equation is discretized as in finite-difference schemes although the time variable remains continuous. This permits the mathematical model to be described by a network model, the partial differential equations involved in the conduction problem being transformed into a set of coupled ordinary differential equations (equations of the network). Highly developed methods of circuits analysis may then be used to observe the dynamic behavior of systems obviating the need to deal explicitly with what are usually very complex differential equations describing the process. The simulation program PSPICE[1]® [10, 11] was found to be very useful for this purpose.

The assumed conditions, which make the problem highly non-linear as regards temperature, are easily implemented in the network model. This model is then simulated using the circuit simulation program, PSPICE, and the thermal behavior of the plate, either during transient or stationary time intervals, can thus be obtained.

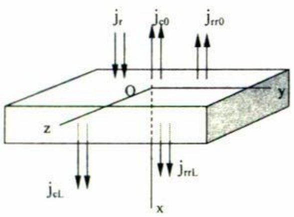

FIGURE 1. Slab of thickness L.

2. Network model of an absorber plate

The transient one-dimensional heat conduction differential equation in a single plate may be expressed as

$$\rho c_e \left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{k\partial T}{\partial x}\right); \quad t > 0, \quad 0 < x < L, \quad (1)$$

where x and t are the spatial coordinate and time, respectively, ρ is the density, c_e, the specific heat, k, the thermal conductivity, and L, the thickness of the plate. The front surface of the plate is assumed to be receiving heat radiation and to be losing heat by natural convection and radiation by exchange with its surrounding (Fig. 1); in this case, the thermal flux at $x = 0$, can be given by

$$\begin{aligned} j_x = 0 &= -k\left(\frac{\partial T}{\partial x}\right)_{(0,t)} \\ &= h(T_{x=0} - T_a) + j_r + \varepsilon\sigma(T^4_{x=0} - T^4_\infty), \quad (2)\end{aligned}$$

where h is the heat-transfer coefficient, T_a and T_∞ are, respectively, the ambient fluid and radiation sink temperatures, ε is the radiative emissivity of surface, σ, the Stefan-Boltzman constant, and j_r, the radiant heat flux.

The daily variation of the incident solar radiation is assumed to be

$$j_r = j_o + j_o \sin\left[\left(\frac{2\pi t}{\tau_r}\right) + \varphi_o\right], \quad \text{for} \quad 0 \leq t \leq \tau_r, \quad (3)$$

$$j_r = 0, \qquad \text{for} \quad \tau_r \leq t \leq 2\tau_r. \quad (4)$$

The conditions of solar incident radiant flux during a complete day are closely satisfied by selecting $\varphi_o = -\pi/2$ and $\tau_r = 43.2 \times 10^3$ s (half of the daily period τ).

In the same way, a boundary condition at $x = L$, similar to Eq. (2) without incident radiant heat flux, j_r, can be established,

$$\begin{aligned} j_{x=L} &= -k\left(\frac{\partial T}{\partial x}\right)_{(L,t)} \\ &= h(T_{x=L} - T_a) + \varepsilon\sigma(T^4_{x=L} - T^4_\infty). \quad (5)\end{aligned}$$

The ambient air temperature T_a may be aproximated as [1]

$$T_a = T_m + T_n \sin\left[\left(\frac{2\pi t}{\tau}\right) + \varphi_1\right], \quad (6)$$

where T_m y T_n are the mean value and amplitude of the temperature oscillations, respectively, and φ_1 is its initial phase. Thermal conductivity is assumed to vary polynomially with temperature

$$k = k_0 + k_1 T + k_2 T^2 + k_3 T^3 + \ldots + k_n T^n, \quad (7)$$

where $k_i (i = 0, 1, 2, \ldots n)$ are constants.

The initial condition is given by

$$T_{(x,0)} = T_0. \quad (8)$$

The general procedure for obtaining the network model representative of a transport process consists of dividing the physical region of interest into volume elements. Thus, according to Fourier's law, for the compartment i, of thickness Δx, Eq. (1) can be written as

$$\rho c_e \left(\frac{dT_i}{dt}\right) = -\frac{\Delta j}{\Delta x} = \frac{j'_{i-\Delta} - j'_{i+\Delta}}{\Delta x}, \quad (9)$$

and, since $j_{i\pm\Delta}$ are the fluxes leaving and entering the compartment i,

$$j'_{i\pm\Delta} = \pm k_{i\pm\Delta}\frac{T_i - T_{i\pm\Delta}}{\Delta x/2}, \quad (10)$$

Eq. (9) is expressed as

$$\begin{aligned} \rho c_e \left(\frac{dT_i}{dt}\right) = \left(\frac{1}{\Delta x}\right)\Bigg[k_{i-\Delta}\frac{T_{i-\Delta} - T_i}{\Delta x/2} \\ - k_{i+\Delta}\frac{T_i - T_{i+\Delta}}{\Delta x/2}\Bigg]. \quad (11)\end{aligned}$$

$T_{i\pm\Delta}$ and T_i being the temperatures at the ends and at the central point of the compartment i, respectively (Fig. 2). Conductivity $k_{i\pm\Delta} = k(T_{i\pm\Delta})$ are the values of the k at these ends. Substituting Eq. (7) into Eq. (11) yields the following equation

$$\begin{aligned} \rho c_o \Delta x \left(\frac{dT_i}{dt}\right) = \frac{(T_{i-\Delta} - T_i)}{(\frac{\Delta x}{2k_0})} + 2(T_{i-\Delta} - T_i)\left[\left(\frac{k_1}{\Delta x}\right)T_{i-\Delta} + \left(\frac{k_2}{\Delta x}\right)(T_{i-\Delta})^2 + \ldots + \left(\frac{k_n}{\Delta x}\right)(T_{i-\Delta})^n\right] \\ - \frac{(T_i - T_{i+\Delta})}{(\frac{\Delta x}{2k_0})} - 2(T_i - T_{i+\Delta}\left[\left(\frac{k_1}{\Delta x}\right)T_{i+\Delta} + \left(\frac{k_2}{\Delta x}\right)(T_{i+\Delta})^2 + \ldots + \left(\frac{k_n}{\Delta x}\right)(T_{i+\Delta})^n\right]. \quad (12)\end{aligned}$$

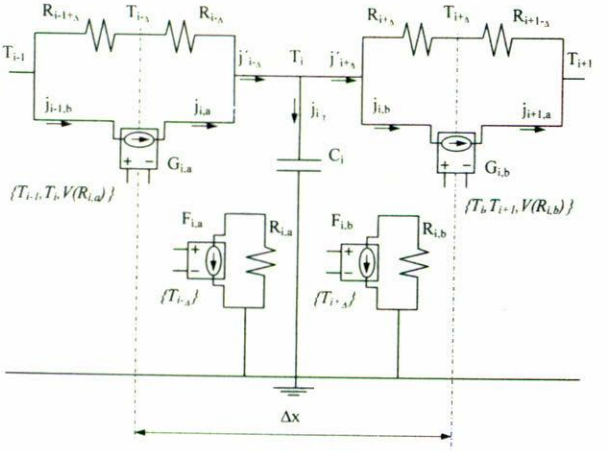

FIGURE 2. Network model for transient heat conduction in a volume element of material with temperature-dependent thermal conductivity.

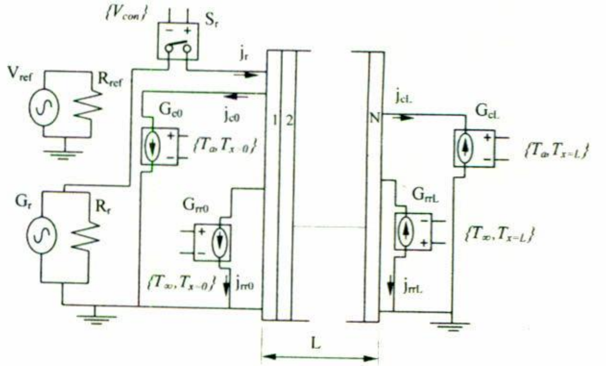

FIGURE 3. Network model for transient heat conduction in the entire plate. Boundary conditions are explicit in the model. j_{c0} and j_{cL} are the convective fluxes at $x = 0$ and $x = L$, respectively; j_{rr0} and j_{rrL} are the re-radiation fluxes also at $x = 0$ and $x = L$.

This equation can be written as

$$j_{i-\Delta} - j_{i+\Delta} - j_{i\gamma} + j_{i,a} - j_{i,b} = 0, \tag{13}$$

where

$$j_{i\pm\Delta} = \pm\frac{T_i - T_{i\pm\Delta}}{\Delta x/(2k_0)}, \tag{14}$$

$$j_{i\gamma} = \rho c_e \Delta x\left(\frac{dT_i}{dt}\right), \tag{15}$$

$$j_{i,a} = 2(T_{i-\Delta} - T_i)\left[\left(\frac{k_1}{\Delta x}\right)T_{i-\Delta} + \left(\frac{k_2}{\Delta x}\right)(T_{i-\Delta})^2 + \ldots + \left(\frac{k_n}{\Delta x}\right)(T_{i-\Delta})^n\right], \tag{16}$$

$$j_{i,b} = 2(T_i - T_{i+\Delta})\left[\left(\frac{k_1}{\Delta x}\right)T_{i+\Delta} + \left(\frac{k_2}{\Delta x}\right)(T_{i+\Delta})^2 + \ldots + \left(\frac{k_n}{\Delta x}\right)(T_{i+\Delta})^n\right]. \tag{17}$$

Equation (13) may be considered to be Kirchhoff's current law and the temperature T as a variable satisfying Kirchhoff's voltage law, because T is a continous and single-valued variable. Thus, it is posible to analyse a heat conduction process by an electric network, characteristic variables of which are j and T. In this sense, Eqs. (14)–(17) define resistors with resistance $R_{i\pm\Delta} = \Delta x/2k_0$, a capacitor with capacitance $\gamma = \rho c_e \Delta x$, and two multivariable voltage-dependent current sources $G_{i,a}$ and $G_{i,b}$, of output $j_{i,a}$ and $j_{i,b}$ respectively, which are connected in accordance with the Kirchhoffs current law, Eq. (13). According to the topology, $T_{i-\Delta} \equiv T_{i-1+\Delta}$, $T_{i+\Delta} \equiv T_{i+1-\Delta}$, and $|j_{i-\Delta}| = |j_{i-1+\Delta}|$, $|j_{i+\Delta}| = |j_{i+1-\Delta}|$; for that, $(T_{i-\Delta}-T_i) = (T_{i-1}-T_{i-1+\Delta})$, $(T_i-T_{i+\Delta}) = (T_{i+1-\Delta}-T_{i+1})$, and from Eqs. (16) and (17), $|j_{i,a}| = |j_{i-1,b}|$, and $|j_{i,b}| = |j_{i+1,a}|$. In this way, the sources $G_{i,a}$ and $G_{i,b}$ can be connected between points $i-1$ and i, and i and $i+1$, respectively, as it is shown in Fig. 2.

The voltage-dependent current source $G_{i,a}$ is controlled by $V^*_{i,a} = (T_{i-\Delta} - T_i)$ and $V^{**}_{i,a} = 2(k_1/\Delta x)T_{i-\Delta} + (k_2/\Delta x)(T_{i-\Delta})^2 + \ldots + (k_n/\Delta x)(T_{i-\Delta})^n$. The first, $V^*_{i,a}$, is obtained directly on its associated nodes, while $V^{**}_{i,a}$ is evaluated by another auxiliary circuit with a voltage-dependent current source $F_{i,a}$ controlled by $T_{i-\Delta}$ through a polynomial expression, and the resistor $R_{i,a} = 1$. In the same way, $G_{i,b}$ needs to be implemented by means of an auxiliary circuit with the voltage-dependent current source, $F_{i,b}$, of output $2[(k_1/\Delta x)T_{i+\Delta} + (k_2/\Delta x)(T_{i+\Delta})^2 + \ldots + (k_n/\Delta x)(T_{i+\Delta})^n]$ and the resistor $R_{i,b} = 1$. The other control voltage, $(T_i - T_{i+\Delta})$, is obtained directly from the circuit. Control variables for the control sources and switches are shown in Figs. 2 and 3, in italics and brackets.

For network modelling purposes, any number N of elemental circuits, such as that in Fig. 2, must be connected in series to form the network model for the entire physical region undergoing a transient heat conduction process. Fig. 3 shows the entire ladder network model for modelling Eq. (12).

The next step is to include initial and boundary conditions in the network model. Initial temperatures throughout the system have been incorporated by means of the initial potential of the capacitors in the network model. The incident radiant heat flux, Eq. (3), is modelled by a sinusoidal variable current source G_r, of output j_r. It is important to note that any periodic solar radiant flux which satisfies specific conditions of a particular geographical area can be considered in the problem and incorporated into the network model. The heat fluxes of the natural convection at $x = 0$ and $x = L$ are modelled by the voltage-dependent current sources, G_{c0} and G_{cL}, of outputs $j_{c0} = h(T_{x=0} - T_a)$ and $j_{cL} = h(T_{x=L} - T_a)$ respectively, which are controlled by the voltages $T_{x=0}$, $T_{x=L}$ and T_a. These sources satisfy any variations in temperature, which are generally of the form $h = h(\Delta T)^p$, with ΔT being the temperature difference between slab surface and the fluid, and $p < 1$. The temporal variation of the ambient air temperature T_a, Eq. (6), has been modelled by an auxiliary

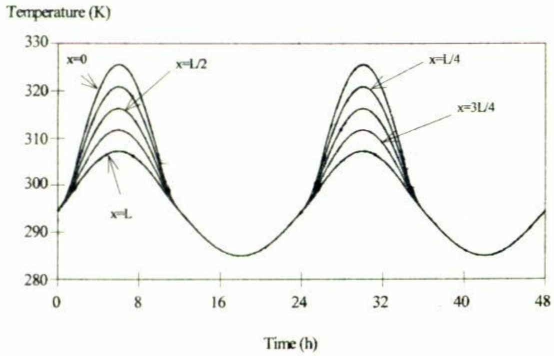

FIGURE 4. Simulated time-dependent temperature profiles at $x = 0, L/4, L/2, 3L/4$ and L.

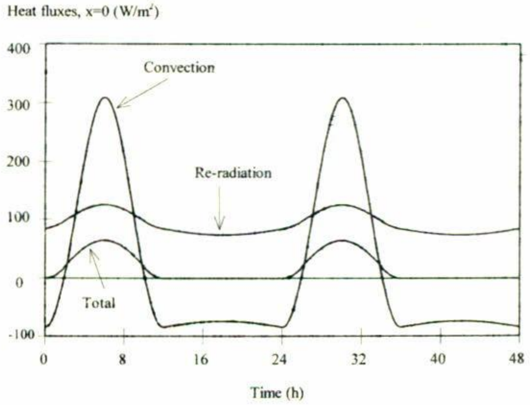

FIGURE 5. Temporal variations of the convective, re-radiative and total instantaneous heat fluxes through the front surface.

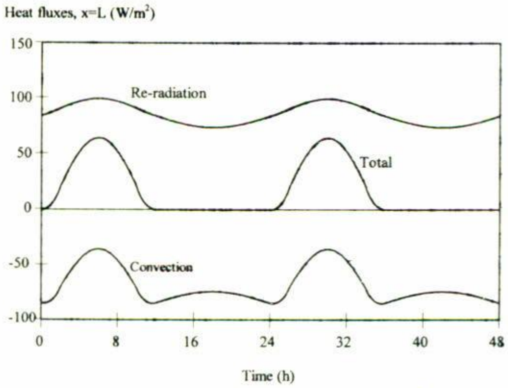

FIGURE 6. Temporal dependence of the convective, re-radiative and total instantaneous heat fluxes through the rear surface.

circuit, with a variable voltage source $V_{\rm ref}$, of output T_a, and its associated resistor $R_{\rm ref}$ equal to unity (Fig. 3). The heat fluxes caused by the exchange with the background radiation sink, at $x = 0$ and $x = L$, which are strongly non-linear, are also easily implemented in the network model by the voltage-dependent current sources G_{rr0} and G_{rrL}, whose outputs are $j_{rr0} = \varepsilon\sigma(T^4_{x=0} - T^4_\infty)$ and $j_{rrL} = \varepsilon\sigma(T^4_{x=L} - T^4_\infty)$, respectively. Once the general network model is obtained, the temporal evolution of heat flows through any section and temperatures anywhere can be simulated simultaneously.

3. Numerical results

Implementation of the network model of Fig. 3 in terms of an electrical network simulation routine such as PSPICE gives the theoretical response of the system. PSPICE is a member of the SPICE family of non-linear circuit simulators that can calculate the behavior of analogue circuits with speed and accuracy even in personal computers. The program uses a modification of the Newton-Raphson algorithm to solve non-linear algebraic equations and contains a stiffly stable predictor-corrector integration technique that is a combination of trapezoidal and Gear's algorithm with a truncation-error time-step control [10].

Simulations of the above considered system were carried out for the following values corresponding to an iron plate and to typical ambient conditions: $c_e = 4.49 \times 10^{-4}$ Jkg^{-1}K^{-1}, $\rho = 7.78 \times 10^3$ kgm^{-3}, $k_0 = 1.47 \times 10^{-2}$ Wm^{-1}K^{-1}, $k_1 = -2.42 \times 10^{-5}$ Wm^{-1}K^{-2}, $k_j = 0$, $j > 1$, $T_0 = 300$ K, $h = h_o(\Delta T)^p$ Wm^{-2}K^{-1}, with $h_o = 10$ and $p = 0.25$, $\varepsilon\sigma = 1.12 \times 10^{-8}$ Wm^{-2}K^{-4}, $T_m = 300$ K, $T_n = 10$ K, $\phi_1 = 0$, $T_\infty = 0$ K, $j_o = 250$ Wm^{-2}, $L = 2 \times 10^{-3}$ m, $N = 40$. Simulations were carried out on a SUN Sparcstation 2, 4/50 GX.

Figure 4 shows temperature profiles in the slab at $x = 0$, $L/4$, $L/2$, $3L/4$ and L, during the two first cycles. There is no significant transitory contribution because of the very low thermal inertia of the slab. Figs. 5 and 6 show the time dependence of the total, convection and re-radiation fluxes at $x = 0$ and $x = L$, respectively.

According to the temperature profiles obtained, the maximum and minimum conductivities are 78.0×10^{-4} and 68.2×10^{-4} Wm^{-1}K^{-1}, respectively, corresponding to 285.0 and 325.6 K. To analyze the influence of the non-linear term we have chosen $k_1 = -4.48 \times 10^{-5}$ Wm^{-1}K^{-2}. Figure 7 shows the temperature profiles for that value, the maximum and minimum temperature of the slab being 327.7 and 285.0 K and the associated conductivity for those temperatures 0.167×10^{-4} and 19.3×10^{-4} Wm^{-1}K^{-1}. Figure 8 shows the temperature profiles for a constant conductivity of 0.167×10^{-4} Wm^{-1}K^{-1}.

The strains and stresses induced by the thermal fields depend on the mechanical constrictions at the boundaries and the slab dimensions. By means of ANSYS 5.3, sofware of Ansys Inc., we evaluated the stresses for the maximum thermal gradient (at noon) and obtained the following values: $T(0, y, z) = 325.6$ K and $T(x = L, y, z) = 307$ K and $E = 2.06 \times 10^{11}$ N/m^2, $\alpha = 1.2 \times 10^{-5}$ K^{-1}, $\nu = 0.3$, $L_y = 0.5$ m and $L_z = 0.05$ m, where E, α and ν are Young's modulus, thermal expansion coefficient and Poisson's ratio, respectively.

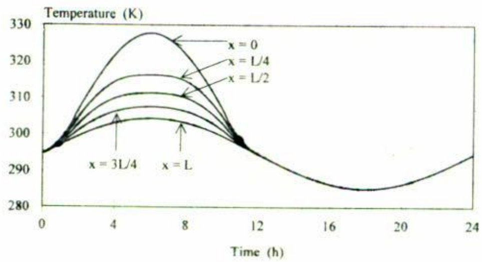

FIGURE 7. Temperature profiles for $k_o = 1.47 \times 10^{-2}$ $\text{Wm}^{-1}\text{K}^{-1}$, $k_1 = -4.48 \times 10^{-5}$ $\text{Wm}^{-1}\text{K}^{-2}$.

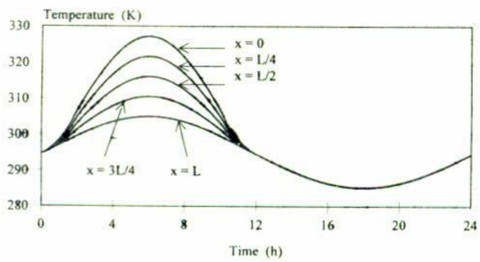

FIGURE 8. Temperature profiles for $k = 0.167 \times 10^{-4}\text{Wm}^{-1}\text{K}^{-1}$.

By preventing the displacement of the boundaries $y = \pm 0.25$ m and leaving free the other edges, maximum stresses are $\sigma_y(0,0,0) = 3.7 \times 10^7$ N/m^2, $\sigma_z(0,0,0) = 1.9 \times 10^6$ N/m^2 and $\sigma_x(0,0,0) = 1.9 \times 10^5$ N/m^2. Without displacements at $y = \pm 0.25$ m and $z = \pm 0.025$ m, the maximum stresses are $\sigma_y(0,0,0) = 8.4 \times 10^7$ N/m^2, $\sigma_z(0,0,0) = 5.7 \times 10^7$ N/m^2 and $\sigma_x(0,0,0) = 4.9 \times 10^5$ N/m^2. Without restrictions (free slab) the stresses are negligible and the displacement and lateral deflection are $\Delta L_y = 1.5 \times 10^{-4}$ m and $\delta = 3.5 \times 10^{-3}$ m, respectively.

4. Conclusions

In the light of the results, the temperatures at $x = 0$ and $x = L$ are almost independent of the conductivity whatever its value or its degree of dependence on T, the non-linearities imposed by the boundary conditions (convection, radiation, re-radiation) determining these profiles. However, the thermal profile inside of the slab is sensitive to changes in conductivity with temperature, as was to be expected.

The maximum stresses induced by these thermal fields, in the least favourable conditions (zero displacement at the four edges) remain far from the elastic limit of steel.

To simulate the network model proposed, including capture and complete analysis, a highly developed integrated environment must be used. PSPICE has two options which are easy to learn and use for capturing the network: graphically or by programing; otherwise PSPICE can be implemented in PC or work stations.

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