Non hamiltonian autonomous systems

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The dynamical equations for an autonomous system are expressed in terms of constants of motion of the system. The general change of variable transformation of the dynamical system is studied including action-angle like transformations. Examples of this non-Hamiltonian approach are given, and the approach is applied to the study of radiation damping suffered by a charged particle inside the beam circulating around an accelerator ring.

Keywords: Non-Hamiltonian systems; dissipation; constant of motion

Las ecuaciones dinámicas para un sistema autónomo son expresadas en términos de las constantes de movimiento del sistema. Se estudia la transformación general de un cambio de variable del sistema, incluyendo las llamadas transformaciones de acción-ángulo. Se dan ejemplos de este procedimiento no hamiltoniano y se aplica al estudio del amortiguamiento por radiación de las oscilaciones que experimenta una partícula cargada cuando ésta circula dentro de un haz alrededor del anillo de un acelerador.

Descriptores: Sistemas no hamiltonianos; disipación; constantes de movimiento

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1. Introduction

The motion of a single particle in a n-dimensional space is described by Newton's equations

$$\frac{d^2x_i}{dt^2} = F_i \qquad i = 1, \ldots, n, \tag{1}$$

where the mass of the particle has been included in the definition of the external force, $\mathbf{F} = (F_1, \ldots, F_n)$. The variables x_i, $i = 1, \ldots, n$ present the coordinates of the particle, and the variable t represents the time. One says that the system (1) is autonomous if the force $\mathbf{F}$ does not depend explicitly on time. Then, the system (1) can be written as the following autonomous dynamical system

$$\frac{dx_i}{dt} = v_i, \qquad i = 1, \ldots, n \tag{2a}$$

and

$$\frac{dv_i}{dt} = F_i(\mathbf{x}, \mathbf{v}), \qquad i = 1, \ldots, n; \tag{2b}$$

where v_i is the ith-component of the velocity of the particle, $\mathbf{v} = (v_1, \ldots, v_n)$. For one dimensional autonomous systems, the first integral of motion represents the constant of motion which is associated with the energy of the particle. This constant of motion is closely related with the Hamiltonian and the Lagrangian of the system [1]. In particular, many dissipative systems are written in the form (2) and represent a challange for a consistent Hamiltonian and Lagrangian formulation. These systems have a great deal of interest in classical mechanics, electrical network theory, statistical mechanics and quantum mechanics. In order to understand the theoretical problems presented by these type of system, a new approach will be formulated on this paper based on the constants of motion of the system (2). In Ref. 1, the constant of motion was used mainly to stablish its relationship with the Lagrangian and Hamiltonian. In this paper, one wants to study the whole dynamics (without using Lagrangian or Hamiltonians) of the system using the constants o f motion of the system. Firstly, the Eqs. (2) are given in terms of the constants of motion of the system. Secondly, the transformations associated with the system are studied, including the action-angle transformations. Finally, several examples are shown, in particular, the study of one-dimensional classical radiation damping (suffered by a charge particle which is circulating within a beam in a synchrotron accelerator).

2. Constant of motion

A constant of motion K is a position and velocity dependent function, $K = K(\mathbf{x}, \mathbf{v})$, which satisfies the equation

$$\frac{dK}{dt} = 0. \tag{3}$$

According to Eq. (2), this means that the following partial differential equation is satisfied:

$$\sum_{i=1}^{n} \left[v_i \frac{\partial K}{\partial x_i} + F_i(\mathbf{x}, \mathbf{v}) \frac{\partial K}{\partial v_i} \right] = 0. \tag{4}$$

This equation can be solve by the characteristics method [2], where the equations for the characteristic surfaces are given by

$$\frac{dx_1}{v_1} = \cdots = \frac{d_n}{v_n} = \frac{dv_1}{F_1(\mathbf{x}, \mathbf{v})} = \cdots = \frac{dv_n}{F_n(\mathbf{x}, \mathbf{v})} = \frac{dK}{0}. \tag{5}$$

The last term in (5) means that the function K will be given by an arbitrary function of the $2n-1$ independent characteristic surfaces obtained from the other terms of (5). These characteristic surfaces, ξ_i, $i = 1, \ldots, 2n-1$, are also constant of motion. Therefore, they satisfy Eq. (4) too, and by selecting m of these $2n-1$ characteristics, the following system is formed

$$\sum_{i=1}^{n} \left(v_i \frac{\partial \xi_k}{\partial x_i} + F_i \frac{\partial \xi_k}{\partial v_i} \right) = 0, \qquad k = 1, \ldots, m; \tag{6}$$

which can be written in the form

$$A\mathbf{v}^t + B\mathbf{F}^t = 0, \tag{7}$$

where $\mathbf{v}^t$ and $\mathbf{F}^t$ are the velocity and force vectors

$$\mathbf{v}^t = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{8a}$$

and

$$\mathbf{F}^t = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}, \tag{8b}$$

A is the $m \times n$ matrix defined as

$$A = \begin{pmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \cdots & \dfrac{\partial \xi_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial \xi_m}{\partial x_1} & \cdots & \dfrac{\partial \xi_m}{\partial x_n} \end{pmatrix}. \tag{9}$$

The $m \times n$ matrix B is defined as

$$B = \begin{pmatrix} \dfrac{\partial \xi_1}{\partial v_1} & \cdots & \dfrac{\partial \xi_1}{\partial v_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial \xi_m}{\partial v_1} & \cdots & \dfrac{\partial \xi_m}{\partial v_n} \end{pmatrix}. \tag{10}$$

3. Dynamical equations using n-constants of motion

Choosing $m = n$ in (9) and (10) and using the fact that these constants of motion are functionally independent, the Jacobian $\partial(\xi_1, \ldots, \partial\xi_n)/\partial(v_1, \ldots, v_n) = \det(B)$, is different from zero, it follows from (7)

$$F_i(\mathbf{x}, \mathbf{v}) = -\sum_{l=1}^{n} (B^{-1}A)_{il}\, v_l, \tag{11}$$

where $(B^{-1}A)$ is a $n \times n$ matrix. The dynamical system (2) can be written in terms of theses n-constants of motion as

$$\frac{dx_i}{dt} = v_i \tag{12a}$$

and

$$\frac{dv_i}{dt} = -\sum_{l=1}^{n} (B^{-1}A)_{il}\, v_l, \qquad i = 1, \ldots, n. \tag{12b}$$

In addition, if ρ is any arbitrary function of $\mathbf{x}$ and $\mathbf{v}$, its time evolution,

$$\frac{d\rho}{dt} = \sum_{i=1}^{n} \left(v_i \frac{\partial \rho}{\partial x_i} + F_i \frac{\partial \rho}{\partial v_i} \right), \tag{13}$$

can be written as

$$\frac{d\rho}{dt} = \sum_{i=1}^{n} \left[v_i \frac{\partial \rho}{\partial x_i} - (B^{-1}A)_{ii} \frac{\partial \rho}{\partial v_i} \right] - \sum_{i=1}^{n} \sum_{l \neq i}^{n} (B^{-1}A)_{il} v_l \frac{\partial \rho}{\partial v_i}. \tag{14}$$

4. Transformations

Having m constants of motion ξ_k, $k = 1, \ldots, m$ $(1 \le m \le 2n-1)$, consider a vector function $\vec{\eta} = (\eta_1, \ldots, \eta_{2n-m})$ which depends on $\mathbf{x}$ and $\mathbf{v}$, and it is independent of $\vec{\xi} = (\xi_1, \ldots, \xi_m)$. In addition, it is assumed that the Jacobian $\partial(\vec{\xi}, \vec{\eta})/\partial(\mathbf{x}, \mathbf{v})$ is different from zero. In this case, the set $\{\vec{\xi}, \vec{\eta}\}$ represents new set of variables, and the following transformation is possible:

$$x_i = x_i(\vec{\xi}, \vec{\eta}), \qquad i = 1, \ldots, n \tag{15a}$$

and

$$v_i = v_i(\vec{\xi}, \vec{\eta}), \qquad i = 1, \ldots, n. \tag{15b}$$

Using these variables, the dynamical system (2) is written as

$$\frac{d\xi_k}{dt} = 0 \qquad k = 1, \ldots, m \tag{16}$$

and

$$\frac{d\eta_l}{dt} = G_l(\vec{\eta}) \qquad l = 1, \ldots, 2n-m, \tag{17}$$

where the function $G_l(\vec{\eta})$ is given by

$$G_l(\vec{\eta}) = \sum_{i=1}^{n} \left\{ v_i(\vec{\xi}, \vec{\eta}) \frac{\partial \eta_l}{\partial x_i} + F_i[\mathbf{x}(\vec{\xi}, \vec{\eta}), \mathbf{v}(\vec{\xi}, \vec{\eta})] \frac{\partial \eta_l}{\partial v_i} \right\}. \tag{18}$$

There are, of course, a non-numerable ways of choosing these functions η_l, $\quad l = 1, \ldots, 2n-m$ since the only conditions they have to satisfy is

$$\frac{\partial(\vec{\xi}, \vec{\eta})}{\partial(\mathbf{x}, \mathbf{v})} \neq 0, \tag{19a}$$

which can be written explicity as

$$\sum_{\sigma\in S(2n)} \in(\sigma)\xi_{1,\sigma(1)}\xi_{2,\sigma(2)}\cdots\xi_{m,\sigma(m)}\eta_{1,\sigma(m+1)}\cdots\eta_{2n-m,\sigma(2n)} \neq 0, \tag{19b}$$

where $S(2n)$ is the group of permutations of $2n$ elements, σ is a permutation, $\xi(\sigma)$ is the sign associated to this permutation, $\xi_{i\sigma(i)}$ and $\eta_{i\sigma(m=i)}$ represent the partial differentiation with respect to the variables [if $1 \leq \sigma(i) \leq n$ or $1 \leq \sigma(m+i) \leq n$ the variables are $x's$ wise the differentiation is with respect to $v's$].

In particular, for an action-angle like transformation of the system (2), it is understood the transformation (15) such that $m = n$, and the Eqs. (16) and (17) are of the following form

$$\frac{d\xi_k}{dt} = 0, \qquad k = 1,\ldots,n \tag{20a}$$

and

$$\frac{d\eta_l}{dt} = \lambda_l, \qquad l = 1,\ldots,n, \tag{20b}$$

where λ_l for $l = 1,\ldots,n$ are constants. The expression (20b) can be written as

$$\sum_{i=1}^{n}\left[v_i\frac{\partial\eta_l}{\partial x_i} + F_i(\mathbf{x},\mathbf{v})\frac{\partial\eta_l}{\partial v_i}\right] = \lambda_l, \qquad l = 1,\ldots,n. \tag{21}$$

Since the constants of motion ξ_k, $k = 1,\ldots,n$ are characteristics of Eq. (21), they can be used to have

$$x_i = x_i(\vec{\xi},\mathbf{v}) \quad \text{or} \quad v_i = v_i(\vec{\xi},\mathbf{x}), \quad i = 1,\ldots,n, \tag{22}$$

which help to solve the equations

$$\frac{dx_1}{v_1} = \ldots = \frac{dx_n}{v_n} = \frac{dv_1}{F_1} = \ldots = \frac{dv_n}{F_n} = \frac{d\eta_l}{\lambda_l}. \tag{23}$$

Then, in principle, it is possible to substitute (22) in (23) to obtain the solution

$$\eta_l = g_l^{(1)}(\vec{\xi},\mathbf{v}) \quad \text{or} \quad \eta_l = g_l^{(2)}(\vec{\xi},\mathbf{x}), \quad l = 1,\ldots,n. \tag{24}$$

So, using (20b), it follows

$$\eta_l = \lambda_l t + \phi_l \qquad l = 1,\ldots,n; \tag{25}$$

where $\phi_l, l = 1,\ldots,n$ are constants. Substituting this in (24) and doing the inverse transformation (this is possible because $\partial(\vec{\eta})/\partial(\mathbf{v}) \neq 0$ or $\partial(\vec{\eta})/\partial(\mathbf{x}) \neq 0$), one gets

$$v_l = \psi(\vec{\xi},t), \qquad l = 1,\ldots,n \tag{26}$$

and

$$x_l = \psi'(\vec{\xi},t), \qquad l = 1,\ldots,n. \tag{27}$$

To get the action-like variable for the ith-component of position and velocity, (x_i, v_i), of the particle, one may select $2n-2$ characteristics

$$(\xi_1^{(i)},\xi_2^{(i)},\ldots,\xi_{2n-2}^{(i)}) = \vec{\xi}^{(i)} \tag{28a}$$

of the set $\{\xi_1,\ldots,\xi_{2n-1}\}$ such that

$$\frac{\partial(\xi_1^{(i)},\ldots,\xi_{2n-2}^{(i)})}{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n,v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n)} \neq 0. \tag{28b}$$

This allows to have

$$x_k = x_k(\vec{\xi}^{(i)}), \quad k = 1,\ldots,n \quad \text{and} \quad k \neq i; \tag{29a}$$

and

$$v_k = v_k(\vec{\xi}^{(i)}), \quad k = 1,\ldots,n \quad \text{and} \quad k \neq i. \tag{29b}$$

One can substitute (29) in the remainning characteristic surface

$$\xi_i^{(i)} = \xi_i^{(i)}(x_1,\ldots,x_n,v_1,\ldots,v_n), \tag{30}$$

where $\xi_i^{(i)} \neq \xi_j^{(i)}$, $j = 1,\ldots,2n-2$, to obtain a relation among the variables x_i, v_i and the constants (28a) only,

$$\xi_i^{(i)} = g_i(\vec{\xi}^{(i)},x_i,v_i). \tag{31}$$

From this relation, one may have the expression

$$v_i = f_i(\vec{\xi}^{(i)},\xi_i^{(i)},x_i), \tag{32}$$

and can define the action-like variable as

$$I_i = \oint v_i\,dx_i = \int_{x_i^-}^{x_i^+} f_i(\vec{\xi}^{(i)},\xi_i^{(i)},x_i)\,dx_i, \tag{33}$$

where the points x_i^- and x_i^+ are determined from (31) with the implicit function

$$\xi^{(i)} = g_i(\vec{\xi}^{(i)},x_i^{\pm},0). \tag{34}$$

since $\vec{\xi}^{(i)}$ and $\xi_i^{(i)}$ are constant of motion, the action-like variables I_i, $i = 1,\ldots,n$ are also constant of motion. Once the integration is done, there is only dependency on the characteristics,

$$I_i = I_i(\xi_1,\ldots,\xi_{2n-1}), \qquad i = 1,\ldots,n. \tag{35}$$

In addition, because of the independency of $\{\xi_i\}_{i=1,\ldots,n}$, it follows

$$\frac{\partial(I_1,\ldots,I_n)}{\partial(x_1,\ldots,x_n)} \neq 0 \qquad \text{or} \qquad \frac{\partial(I_1,\ldots,I_n)}{\partial(v_1,\ldots,v_n)} \neq 0, \tag{36}$$

and they can be used as new variables such that in (20a), one may have instead

$$\frac{dI_i}{dt} = 0, \qquad i = 1, \ldots, n. \tag{37}$$

5. One-dimensional case

Let ξ be the constant of motion associated to (2) for $n = 1$,

$$v\frac{\partial \xi}{\partial x} + F(x,v)\frac{\partial \xi}{\partial v} = 0. \tag{38}$$

In this case, $A = \partial\xi/\partial x$, $B = \partial\xi/\partial v$, and the force is written in terms of the constant of motion as

$$F(x,v) = -\frac{\xi_x}{\xi_v}. \tag{39}$$

So, the dynamical system is written in terms of this constant of motion as

$$\frac{dx}{dt} = v \tag{40a}$$

and

$$\frac{dv}{dt} = -\frac{v}{\xi_v}\xi_x. \tag{40b}$$

Thus, one needs just one function $\eta = \eta(x,v)$ such that

$$\frac{\partial(\xi,\eta)}{\partial(x,v)} \neq 0 \tag{41}$$

to have ξ and η as the new variables with the dynamical system given by

$$\frac{d\xi}{dt} = 0 \tag{42a}$$

and

$$\frac{d\eta}{dt} = v(\xi,\eta)\frac{\partial \eta}{\partial x} + F\left[x(\xi,\eta), v(\xi,\eta)\right]\frac{\partial \eta}{\partial v}. \tag{42b}$$

The condition (41) allows to find the function η in a simple way. Let λ be a nonzero number, then, choosing

$$\frac{\partial(\xi,\eta)}{\partial(x,v)} = \lambda, \tag{43}$$

where ξ is the known of motion, the following partial differential equation arises for η:

$$\xi_x\frac{\partial \eta}{\partial v} - \xi_v\frac{\partial \eta}{\partial x} = \lambda. \tag{44}$$

The same constant of motion ξ brings about the expression

$$v = v(\xi, x), \tag{45}$$

which can be used in (44) to get the solution

$$\eta(x,v) = g(\xi) - \lambda\int\frac{dx}{\xi_v\left[x, v(\xi,x)\right]}, \tag{46}$$

where $g(\xi)$ is an arbitrary function.

Note, from (40a) and (40b), that the

$$\frac{v}{\xi_v} = 1 \tag{47}$$

brings about the Hamilton-like systems

$$\frac{dx}{dt} = \xi_v \tag{48a}$$

and

$$\frac{dv}{dt} = -\xi_x. \tag{48b}$$

The condition (47) implies that the constant of motion must be of the form

$$\xi = \frac{v^2}{2} + V(x), \tag{49}$$

where the function $V(x)$ is arbitrary. Note from (42b) also that one may select an action-angle like transformation by making this expression equal to a constant and solving it.

6. Examples

6.1. Example A

The dynamical system for the harmonic oscillator is given by

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -\omega^2 x. \tag{50}$$

It is well known that the constant of motion asociated to this system is

$$\xi = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2x^2, \tag{51}$$

where m represents the mass of the particle. From (51) follows the expression

$$v = \pm\sqrt{\frac{2\xi}{m} - \omega^2x^2}. \tag{52}$$

Using this expression and the relation (46) , one gets

$$\eta = g(\xi) \mp \frac{\lambda}{m}\int\frac{dx}{\sqrt{2\dfrac{\xi}{m} - \omega^2x^2}}. \tag{53a}$$

Solving this equation, it is obtained

$$\eta(x,v) = g(v^2+\omega^2x^2) \mp \frac{\lambda}{m\omega}\arcsin\left(\frac{\omega x}{\sqrt{v^2+\omega^2x^2}}\right). \tag{53b}$$

So, Eqs. (51) and (53b) form the new variables, and the inverse transformation is given by

$$x = \sqrt{\frac{2\xi}{m\omega^2}}\sin\left\{\frac{\sqrt{2\xi m}}{\lambda}\left[\eta \pm g(\xi)\right]\right\} \tag{54a}$$

and

$$v = \sqrt{\frac{2\xi}{m}}\cos\left\{\frac{\sqrt{2\xi m}}{\lambda}\left[\eta \pm g(\xi)\right]\right\}. \tag{54b}$$

The dynamical system formed with variables ξ and η is of the type action-angle,

$$\frac{d\xi}{dt} = 0 \tag{55a}$$

and

$$\frac{d\eta}{dt} = -\frac{\lambda}{m}. \tag{55b}$$

The solution of (55b) is

$$\eta = -\frac{\lambda}{m}\, t + \eta_o, \tag{56}$$

where η_o is a constant. Substituting this solution in (54), it brings about the time evolution of the system. The constant η_o and ξ are determined by initial conditions, and the constant λ can be chosen as $\lambda = 1$. Using (51), (52), (34), and (33), the action-like variable is given by

$$I = \frac{\pi\xi}{m\omega} = \frac{\pi}{2\omega}\left(v^2 + \omega^2 x^2\right). \tag{57}$$

6.2. Example B

Considere the following two-dimensional autonomous system:

$$\frac{dx}{dt} = v_1, \tag{58a}$$

$$\frac{dy}{dt} = v_2, \tag{58b}$$

$$\frac{dv_1}{dt} = -\frac{x}{(x^2+y^2)^{3/2}} \tag{58c}$$

$$\frac{dv_2}{dt} = -\frac{y}{(x^2+y^2)^{3/2}}. \tag{58d}$$

Writing this system in polar coordinates, $x = \rho\cos\theta$ and $y = \rho\sin\theta$, it is given by

$$\frac{d\rho}{dt} = v_\rho, \tag{59a}$$

$$\frac{d\theta}{dt} = v_\theta, \tag{59b}$$

$$\frac{dv_\rho}{dt} = -\frac{1}{\rho^2} + v_\theta^2\rho \tag{59c}$$

and

$$\frac{dv_\theta}{dt} = -\frac{2v_\rho v_\theta}{\rho}. \tag{59d}$$

The functions ξ_1, ξ_2, ξ_3, and ξ_4 given by

$$\xi_1 = \frac{(v_\rho^2 + v_\theta^2\rho^2)}{2} - \frac{1}{\rho}, \tag{60a}$$

$$\xi_2 = v_\theta\rho^2, \tag{60b}$$

$$\xi_3 = \cos\theta - v_\theta^2\rho^3\cos\theta - v_\theta v_\rho\rho^2\sin\theta \tag{60c}$$

and

$$\xi_4 = \sin\theta - v_\theta^2\rho^3\sin\theta + v_\theta v_\rho\rho^2\cos\theta \tag{60d}$$

are constant of motion of the system (59). However, they are not independent since the Jacobian of the transformation is zero,

$$\frac{\partial(\xi_1,\xi_2,\xi_3,\xi_4)}{\partial(\rho,\theta,v_{\rho,v_\theta},v_\theta)} = 0. \tag{61}$$

Nevertheless, one can take three of them as part of a new set of variables. The remainning variable, η, will be found such that the following relation is satisfied:

$$\frac{d\eta}{dt} = \lambda, \tag{62}$$

which can be written as

$$v_\rho\frac{\partial\eta}{\partial\rho} + v_\theta\frac{\partial\eta}{\partial\theta} + \left(-\frac{1}{\rho^2} + v_\theta^2\rho\right)\frac{\partial\eta}{\partial v_\rho} - \frac{2v_\rho v_\theta}{\rho}\frac{\partial\eta}{\partial v_\theta} = \lambda. \tag{63}$$

This equation can be solved by the characteristics method, where the equations for the characteristics are given by

$$\frac{d\rho}{v_\rho} = \frac{d\theta}{v_\theta} = \frac{dv_\rho}{-\frac{1}{\rho^2} + v_\theta^2\rho} = \frac{dv_\theta}{-\frac{2v_\theta v_\rho}{\rho}} = \frac{d\eta}{\lambda}. \tag{64}$$

Since ξ_1 and ξ_2 are characteristics of (63), they can be used to obtain the relations

$$v_\theta = \frac{\xi_2}{\rho^2} \tag{65a}$$

and

$$v_\rho = \sqrt{2\xi_1 + \frac{2}{\rho} - \frac{\xi_2^2}{\rho^2}}. \tag{65b}$$

Using these relations in the first two terms of (64), one can get the relationship between the variables ρ and θ,

$$\frac{1}{\rho} = \frac{1}{\xi_2^2} - \frac{\sqrt{2\xi_1\xi_2^2+1}}{\xi_2^2}\sin(\theta+\theta_o) \tag{66}$$

which can be used in (65a) and (64) to obtain

$$\eta = \eta_o + \lambda\int\frac{d\theta}{\left[1 - \sqrt{2\xi_1\xi^2+1}\,\sin^2(\theta+\theta_o)\right]^2}. \tag{67}$$

Solving this equation, it follows

$$\eta = \eta_o + \frac{\cos(\theta+\theta_o)}{2\xi_1\xi_2^2\left[1 - \sqrt{2\xi_1\xi_2^2+1}\,\sin(\theta+\theta_o)\right]} - \frac{1}{(2\xi_1\xi_2^2)^2}\log R(\xi_1,\xi_2,\theta), \tag{68a}$$

where η_o and θ_o are constant, and R is defined as

$$R(\xi_1,\xi_2,\theta) = \frac{\tan\left(\frac{\theta+\theta_o}{2}\right) + \sqrt{2\xi_1\xi_2^2+1} - \sqrt{2\xi_1\xi_2^2}}{\tan\left(\frac{\theta+\theta_o}{2}\right) + \sqrt{2\xi_1\xi_2^2+1} + \sqrt{2\xi_1\xi_2^2}}. \tag{68b}$$

Since the solution of Eq. (62) is

$$\eta = \lambda t + \eta_{o'}, \tag{69}$$

the time evolution of the system is found. On the other hand, the functions ξ_1, ξ_2, ξ_3 and η are independent since the Jacobian, $\partial(\xi_1,\xi_2,\xi_3,\eta)/\partial(\rho,\theta,v_\rho,v_\theta)$, is different from zero.

6.3. Example C

The equation that describes the evolution of energy particle (E) with respect the energy of the synchrounous particle (E_s) in a beam, $x = E - E_s$, of an accelerator ring is given by

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \Omega_o^2 x = 0, \tag{70}$$

where α is the damping coefficient and Ω_o is is the synchronous frequency. These parameters are determined by the characteristics of the accelerator ring and the rf-voltage used to accelerate the particles [3]. The motion of the individual charge particle is damped relative to that of the synchronous particle, bringing about a reduction of the beam phase space size (called emittance). This phenomenon is called radiation damping. Equation (60) can be written as the following dynamical system:

$$\frac{dx}{dt} = v \tag{71a}$$

and

$$\frac{dv}{dt} = -(\Omega_o^2 x + 2\alpha v). \tag{71b}$$

The constant of motion associated to this system is given by [1]

$$\xi = \frac{1}{2}\Big[v^2 + \Omega_o^2 x^2 + \alpha x v\Big]\exp\Big[-\alpha G\Big(\alpha, \Omega_o, \frac{v}{x}\Big)\Big], \tag{72a}$$

where the function G is defined as

$$G\Big(\alpha, \Omega_o, \frac{v}{x}\Big) = \begin{cases} \dfrac{1}{2\sqrt{\Delta}}\log\left[\dfrac{\alpha + \frac{v}{x} - \sqrt{\Delta}}{\alpha + \frac{v}{x} + \sqrt{\Delta}}\right], & \text{if} \quad \Omega_o^2 < \alpha^2, \\ \dfrac{1}{\alpha + \frac{v}{x}}, & \text{if} \quad \Omega_o^2 = \alpha^2, \\ \dfrac{1}{\sqrt{-\Delta}}\arctan\left[\dfrac{\alpha + \frac{v}{x}}{\sqrt{-\Delta}}\right], & \text{if} \quad \Omega_o^2 > \alpha^2; \end{cases} \tag{72b}$$

with Δ defined as $\Delta = \alpha^2 - \Omega_o^2$. F or very weak dissipation levels, $\alpha^2/\Omega_o^2 \ll 1$, the constant of motion can be given at first order of approximation in α as

$$\xi = \frac{1}{2}[v^2 + \Omega_o^2 x^2\,] + \alpha\left[xv - \frac{v^2 + \Omega_o^2 x^2}{2\Omega_o}\arctan\left(\frac{v}{\Omega_o x}\right)\right]. \tag{73}$$

The curve $\xi =$ constant has a gap per cycle in the x-v space for $v = 0$. This is originated by the non-single value "arctan" function appearing in this constant of motion. The gap size is a measure of the energy lost per cycle of ion of the charge. This size can be calculated taking the limit of (73) for the velocity approaching to $v = 0$ from both sides. Observe that, since the curve $\xi =$ constant must be continuous for $v = 0$, there is a change by π in the argument of the "arctan" function. The limits bring about the following expression

$$(\delta x) = x\left[\sqrt{1 - \frac{6\pi\alpha/\Omega_o}{1 + 3\pi\alpha/\Omega_o}} - 1\right], \qquad x > 0 \tag{74}$$

which can be approximated as

$$(\delta x) = -x\left[\frac{3\pi\alpha}{\Omega_o}\right]. \tag{75}$$

To make a numerical acalculation, take the former pp-SSC accelerator at 20 TeV. For this accelerator one has [5] $\alpha/\Omega_o \approx 10^{-8}$. This means that it would take about 10^8 cyckes of oscillations for a proton beam to shrink to about 63% of its initial emittance. this shrinking effect may not continue any longer beacause of the electrostatic repulssion of the charges and quantum fluctuations of radiation [3,4]. This example points out the practical use of motion for a dissipative system.

7. Conclusion

The constants of motion of an autonomous dynamical system are used to express the equations of motion of the system. The transformation of the equations due to change of variables was studied. Some examples were presented to point out the use of the constants of motion in the dynamical equations.

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