Solution of the equation for elastic waves in an isotropic medium

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Recibido el 25 de noviembre de 1998; aceptado el 17 de junio de 1999

The solution of the equation for elastic waves in an isotropic medium is expressed in terms of three potentials that satisfy the scalar wave equation. Two such expressions are obtained, adapted to the cylindrical and the spherical coordinates, by explicitly integrating the wave equation in circular cylindrical coordinates and in spherical coordinates.

Keywords: Elastic waves; spin weight; Debye potentials

La solución de la ecuación para ondas elásticas en un medio isótropo se expresa en términos de tres potenciales que obedecen la ecuación escalar de onda. Se obtienen dos de tales expresiones, adaptadas a las coordenadas cilíndricas y esféricas, integrando explícitamente la ecuación de ondas en coordenadas cilíndricas circulares y en coordenadas esféricas.

Descriptores: Ondas elásticas; peso de espín; potenciales de Debye

PACS: 03.40.Dz; 02.30.Jr

1. Introduction

Most partial differential equations encountered in mathematical physics involve a single unknown function (*e.g.*, the Schrödinger equation, the Hamilton-Jacobi equation and the Laplace equation) and the procedure usually employed to solve them is the method of separation of variables. However, in the case of systems of partial differential equations, such as those governing vector or spinor fields, the usual method of separation of variables cannot be applied in a straightforward manner (see below). Nevertheless, various nonscalar linear equations in spherical or cylindrical coordinates can be solved by separation of variables if these equations are written in terms of spin-weighted components. Some examples are the Maxwell equations [1, 2], the Dirac equation [3–7], the spin-1 and spin-2 Helmholtz equation [2, 4, 5, 8], the curl eigenvalue equation [5, 9] and the equations of equilibrium for an isotropic elastic medium [10]. Among other things, this procedure leads to expressions for the solutions of these equations in terms of scalar potentials. For example, in the case of the equations of equilibrium for an isotropic elastic medium, one obtains an expression for the solution in terms of three scalar potentials that have to obey the Laplace equation; the expression so obtained is analogous to the Papkovich-Neuber solution [11–13] which involves four harmonic potentials (see also Ref. 14, Chap. I).

The equations for the elastic waves in an isotropic medium (see, *e.g.*, Ref. 14, Chap. III),

$$(1-2\sigma)\nabla^2\mathbf{u}+\nabla(\nabla\cdot\mathbf{u})-\frac{2(1+\sigma)(1-2\sigma)\rho}{E}\frac{\partial^2\mathbf{u}}{\partial t^2}=0, \quad (1)$$

where $\mathbf{u}$ is the displacement vector, σ is the Poisson ratio, E is the Young modulus and ρ is the mass density, constitute a system of three partial differential equations which, even in cartesian coordinates, couples the three components of the vector field $\mathbf{u}$. In this paper we solve Eqs. (1) in circular cylindrical and spherical coordinates, making use of spin-weighted functions. It is shown that there is an expression adapted to each of these coordinate systems for the elastic waves in terms of three scalar Debye potentials that obey scalar wave equations. As in the examples mentioned above, Eqs. (1) are solved by decomposing the vector field $\mathbf{u}$ and Eqs. (1) into spin-weighted components and then solving the resulting equations by means of separation of variables, making use of the corresponding spin-weighted harmonics.

In Sects. 2 and 3, Eqs. (1) are solved in circular cylindrical and spherical coordinates, respectively, and the basic notions about spin weight and the spin-weighted harmonics are also given there.

2. Solution by separation of variables in circular cylindrical coordinates

Let $\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\}$ be the orthonormal basis induced by the circular cylindrical coordinates ρ, ϕ, z. A quantity η has spin weight s if under the rotation about $\hat{e}_z$ given by

$$\hat{e}_\rho{}' + i\hat{e}_\phi{}' = e^{i\beta}(\hat{e}_\rho + i\hat{e}_\phi) \quad (2)$$

transforms according to

$$\eta' = e^{is\beta}\eta. \quad (3)$$

Any vector field **F** can be written as

$$\mathbf{F} = \tfrac{1}{2}F_{-1}\,(\hat{e}_\rho + i\hat{e}_\phi) + \tfrac{1}{2}F_1\,(\hat{e}_\rho - i\hat{e}_\phi) + F_0\hat{e}_z, \qquad (4)$$

where $F_{\pm 1} \equiv \mathbf{F}\cdot(\hat{e}_\rho \pm i\hat{e}_\phi)$ and $F_0 \equiv \mathbf{F}\cdot\hat{e}_z$. Then Eqs. (2) and (3) imply that $F_{\pm 1}$ and F_0 have spin weight ± 1 and 0, respectively.

The operators ð and $\bar{ð}$ acting on a quantity η with spin weight s are defined by [4]

$$ð\eta \equiv -\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi} - \frac{s}{\rho}\right)\eta = -\rho^s\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi}\right)(\rho^{-s}\eta),$$
$$\bar{ð}\eta \equiv -\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi} + \frac{s}{\rho}\right)\eta = -\rho^{-s}\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi}\right)(\rho^s\eta). \quad (5)$$

The quantities $ð\eta$ and $\bar{ð}\eta$ have spin weight $s+1$ and $s-1$, respectively, therefore

$$ð\bar{ð}\eta = \bar{ð}ð\eta = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{2is}{\rho^2}\frac{\partial}{\partial\phi} - \frac{s^2}{\rho^2}\right)\eta. \quad (6)$$

The gradient of a function f with spin weight 0 is given by

$$\nabla f = -\frac{1}{2}\bar{ð}f(\hat{e}_\rho + i\hat{e}_\phi) - \frac{1}{2}ðf(\hat{e}_\rho - i\hat{e}_\phi) + \frac{\partial f}{\partial z}\hat{e}_z \quad (7)$$

and the divergence and curl of a vector field **F** can be written as

$$\nabla\cdot\mathbf{F} = -\frac{1}{2}ðF_{-1} - \frac{1}{2}\bar{ð}F_1 + \frac{\partial F_0}{\partial z},$$
$$\nabla\times\mathbf{F} = \frac{1}{2i}\left(\bar{ð}F_0 + \frac{\partial F_{-1}}{\partial z}\right)(\hat{e}_\rho + i\hat{e}_\phi) - \frac{1}{2i}\left(ðF_0 + \frac{\partial F_1}{\partial z}\right)(\hat{e}_\rho - i\hat{e}_\phi) + \frac{1}{2i}(ðF_{-1} - \bar{ð}F_1)\hat{e}_z. \quad (8)$$

Then, from Eqs. (7) and (8) one finds that

$$\nabla^2 f = ð\bar{ð}f + \frac{\partial^2 f}{\partial z^2} \quad (9)$$

and

$$\nabla^2\mathbf{F} = \frac{1}{2}\left(\bar{ð}ðF_{-1} + \frac{\partial^2 F_{-1}}{\partial z^2}\right)(\hat{e}_\rho + i\hat{e}_\phi) + \frac{1}{2}\left(ð\bar{ð}F_1 + \frac{\partial^2 F_1}{\partial z^2}\right)(\hat{e}_\rho - i\hat{e}_\phi) + \left(\bar{ð}ðF_0 + \frac{\partial^2 F_0}{\partial z^2}\right)\hat{e}_z. \quad (10)$$

The cylindrical (or plane) harmonics [4] with spin weight s, ${}_sF_{\alpha m}$, are defined by

$$ð\bar{ð}({}_sF_{\alpha m}) = -\alpha^2\,{}_sF_{\alpha m},$$
$$-i\frac{\partial}{\partial\phi}({}_sF_{\alpha m}) = m\,{}_sF_{\alpha m}, \quad (11)$$

where α is a (real or complex) constant and m is an integer or half-integer according to whether s is an integer or a half-integer. Then, Eqs. (6) and (11) imply that, if α is different from zero,

$${}_sF_{\alpha m} = A\,{}_sJ_{\alpha m} + B\,{}_sN_{\alpha m}, \quad (12)$$

where A and B are arbitrary constants,

$${}_sJ_{\alpha m}(\rho,\phi) \equiv J_{m+s}(\alpha\rho)e^{im\phi},$$
$${}_sN_{\alpha m}(\rho,\phi) \equiv N_{m+s}(\alpha\rho)e^{im\phi} \quad (13)$$

and J_ν, N_ν are Bessel functions. From Eqs. (5) and the recurrence relations for the Bessel functions one finds that

$$ð({}_sZ_{\alpha m}) = \alpha\,{}_{s+1}Z_{\alpha m},$$
$$\bar{ð}({}_sZ_{\alpha m}) = -\alpha\,{}_{s-1}Z_{\alpha m}, \quad (14)$$

where ${}_sZ_{\alpha m}$ denotes ${}_sJ_{\alpha m}$ or ${}_sN_{\alpha m}$.

When $\alpha = 0$, the cylindrical harmonics are given by

$${}_sF_{0m} = A\,\rho^{m+s}e^{im\phi} + B\,\rho^{-m-s}e^{im\phi}, \qquad (\text{for } m+s \neq 0), \quad (15)$$

and

$${}_sF_{0,-s} = A\,e^{-is\phi} + B\,(\ln\rho)e^{-is\phi}, \qquad (\text{for } m = -s), \quad (16)$$

where A, B are constants. For a fixed value of s, the functions ${}_sF_{\alpha m}$ form a complete set, in the sense that any function with spin weight s can be expanded in terms of the ${}_sF_{\alpha m}$.

Making use of Eqs. (7), (8) and (10) one finds that the spin-weighted components of Eq. (1) are

$$(1-2\sigma)\left(ð\bar{ð}u_1 + \frac{\partial^2 u_1}{\partial z^2}\right) + ð\left(\frac{1}{2}ðu_{-1} + \frac{1}{2}\bar{ð}u_1 - \frac{\partial u_0}{\partial z}\right) - \kappa\frac{\partial^2 u_1}{\partial t^2} = 0,$$
$$(1-2\sigma)\left(\bar{ð}ðu_{-1} + \frac{\partial^2 u_{-1}}{\partial z^2}\right) + \bar{ð}\left(\frac{1}{2}ðu_{-1} + \frac{1}{2}\bar{ð}u_1 - \frac{\partial u_0}{\partial z}\right) - \kappa\frac{\partial^2 u_{-1}}{\partial t^2} = 0, \quad (17)$$
$$(1-2\sigma)\left(\bar{ð}ðu_0 + \frac{\partial^2 u_0}{\partial z^2}\right) - \frac{\partial}{\partial z}\left(\frac{1}{2}ðu_{-1} + \frac{1}{2}\bar{ð}u_1 - \frac{\partial u_0}{\partial z}\right) - \kappa\frac{\partial^2 u_0}{\partial t^2} = 0,$$

where we have introduced the abbreviation

$$\kappa \equiv \frac{2(1+\sigma)(1-2\sigma)\rho}{E}. \tag{18}$$

This system of equations admits solutions of the form

$$u_k = g_k(z)_k J_{\alpha m}(\rho,\phi) e^{-i\omega t} + G_k(z)_k N_{\alpha m}(\rho,\phi) e^{-i\omega t}, \tag{19}$$

with $k = 1, -1, 0$, and where the g_k and G_k are functions to be determined, α is a constant different from zero, m is an integer and ω is also a constant. Substituting Eq. (19) into Eqs. (17), making use of Eqs. (11) and (14) and the linear independence of ${}_sJ_{\alpha m}$ and ${}_sN_{\alpha m}$, one finds that the functions g_k must obey the system of ordinary differential equations

$$\begin{aligned}
&(1-2\sigma)\Big(\frac{d^2 g_1}{dz^2} - \alpha^2 g_1\Big) + \frac{1}{2}\alpha^2 g_{-1} - \frac{1}{2}\alpha^2 g_1 - \alpha\frac{dg_0}{dz} + \kappa\omega^2 g_1 = 0,\\
&(1-2\sigma)\Big(\frac{d^2 g_{-1}}{dz^2} - \alpha^2 g_{-1}\Big) - \frac{1}{2}\alpha^2 g_{-1} + \frac{1}{2}\alpha^2 g_1 + \alpha\frac{dg_0}{dz} + \kappa\omega^2 g_{-1} = 0,\\
&(1-2\sigma)\Big(\frac{d^2 g_0}{dz^2} - \alpha^2 g_0\Big) - \frac{1}{2}\alpha\frac{dg_{-1}}{dz} + \frac{1}{2}\alpha\frac{dg_1}{dz} + \frac{d^2 g_0}{dz^2} + \kappa\omega^2 g_0 = 0
\end{aligned} \tag{20}$$

and the functions G_k obey a system of the same form, with G_k in place of g_k.

Equations (20) can be rewritten as

$$(1-2\sigma)\Big(\frac{d^2 n}{dz^2} - \alpha^2 n\Big) + \kappa\omega^2 n = 0, \tag{21}$$

$$(1-2\sigma)\frac{d^2 H}{dz^2} - 2\alpha^2(1-\sigma)H - \alpha\frac{dg_0}{dz} + \kappa\omega^2 H = 0, \tag{22}$$

$$2(1-\sigma)\frac{d^2 g_0}{dz^2} - \alpha^2(1-2\sigma)g_0 + \alpha\frac{dH}{dz} + \kappa\omega^2 g_0 = 0, \tag{23}$$

with

$$\begin{aligned}
n &\equiv \tfrac{1}{2}(g_1 + g_{-1}),\\
H &\equiv \tfrac{1}{2}(g_1 - g_{-1}).
\end{aligned} \tag{24}$$

The solution of Eq. (21) is given by

$$n(z) = a_1 \exp\Big(i\sqrt{k_t^2 - \alpha^2}\,z\Big) + a_2 \exp\Big(-i\sqrt{k_t^2 - \alpha^2}\,z\Big), \tag{25}$$

where a_1, a_2 are arbitrary constants and

$$k_t^2 \equiv \frac{\kappa\omega^2}{1-2\sigma} = \frac{2(1+\sigma)\omega^2\rho}{E}, \tag{26}$$

provided that $k_t^2 \neq \alpha^2$, while if $k_t^2 = \alpha^2$,

$$n(z) = a_1 z + a_2. \tag{27}$$

By combining Eqs. (22) and (23) one can obtain a decoupled fourth-order equation (with constant coefficients) for H that can be easily solved and then, using Eqs. (22) and (23) again, one finds g_0. However, it is convenient to follow a different procedure, introducing the two auxiliary one-variable functions

$$\begin{aligned}
v &\equiv \alpha g_0 + \frac{dH}{dz},\\
w &\equiv \alpha H + \frac{dg_0}{dz}.
\end{aligned} \tag{28}$$

These combinations arise by considering the scalar functions $\hat{e}_z\cdot\nabla\times\nabla\times\mathbf{u}$ and $\nabla\cdot\mathbf{u}$, respectively; for instance, making use of Eqs. (8) and (14) one finds that for a vector field with components (19), $\nabla\cdot\mathbf{u} = (\frac{1}{2}\alpha(g_1 - g_{-1}) + dg_0/dz)_0 J_{\alpha m} e^{-i\omega t} + (\frac{1}{2}\alpha(G_1 - G_{-1}) + dG_0/dz)_0 N_{\alpha m} e^{-i\omega t}$, and from Eq. (1) it follows that $\hat{e}_z\cdot\nabla\times\nabla\times\mathbf{u}$, $\nabla\cdot\mathbf{u}$ and $\hat{e}_z\cdot\nabla\times\mathbf{u}$ obey the scalar wave equation [the function $n(z)$, defined by Eq. (24), is related to $\hat{e}_z\cdot\nabla\times\mathbf{u}$]. From Eqs. (22), (23) and (28) it follows that

$$\begin{aligned}
&\frac{d^2 v}{dz^2} + (k_t^2 - \alpha^2)v = 0,\\
&\frac{d^2 w}{dz^2} + (k_l^2 - \alpha^2)w = 0,
\end{aligned} \tag{29}$$

with

$$k_l^2 \equiv \frac{\kappa\omega^2}{2(1-\sigma)} = \frac{(1+\sigma)(1-2\sigma)\omega^2\rho}{(1-\sigma)E}. \tag{30}$$

The solutions of Eqs. (29) are of the same form as those of Eq. (21). (Actually, Eqs. (21) and (29) follow directly from the fact that $\hat{e}_z\cdot\nabla\times\mathbf{u}$, $\hat{e}_z\cdot\nabla\times\nabla\times\mathbf{u}$ and $\nabla\cdot\mathbf{u}$ obey scalar wave equations.)

On the other hand, from Eqs. (28) and (29) we obtain

$$\begin{aligned}
\Big(\frac{d^2}{dz^2} - \alpha^2\Big)g_0 &= \frac{dw}{dz} - \alpha v\\
&= \Big(\frac{d^2}{dz^2} - \alpha^2\Big)\Big(-\frac{1}{k_l^2}\frac{dw}{dz} + \frac{\alpha}{k_t^2}v\Big)
\end{aligned}$$

hence,

$$g_0 = \frac{\alpha}{k_t^2}v - \frac{1}{k_l^2}\frac{dw}{dz} + Ae^{\alpha z} + Be^{-\alpha z}, \tag{31}$$

where A and B are some constants. Then, from Eqs. (28) and (29) one has

$$H = \frac{\alpha}{k_l^2}w - \frac{1}{k_t^2}\frac{dv}{dz} - Ae^{-\alpha z} + Be^{-\alpha z}. \tag{32}$$

Substituting Eqs. (31) and (32) into Eqs. (22) and (23) one finds that if $\omega \neq 0$, then $A = B = 0$. Thus,

$$g_0 = \frac{\alpha}{k_t^2} v - \frac{1}{k_l^2} \frac{dw}{dz},$$
$$H = \frac{\alpha}{k_l^2} w - \frac{1}{k_t^2} \frac{dv}{dz}, \qquad (33)$$

with similar expressions for G_0 and $\frac{1}{2}(G_1 - G_{-1})$. Then, from Eqs. (19), (24), (33) and (14) we find that the spin-weighted components of the displacement vector $\mathbf{u}$ can be expressed as

$$u_1 = \eth\psi_1 - i\eth\psi_2 + \frac{\partial}{\partial z}\eth\psi_3,$$
$$u_{-1} = \bar{\eth}\psi_1 + i\bar{\eth}\psi_2 + \frac{\partial}{\partial z}\bar{\eth}\psi_3, \qquad (34)$$
$$u_0 = -\frac{\partial}{\partial z}\psi_1 + \eth\bar{\eth}\psi_3,$$

with

$$\psi_1 \equiv \frac{1}{k_l^2}[w(z)\,{}_0J_{\alpha m} + W(z)\,{}_0N_{\alpha m}]e^{-i\omega t},$$
$$\psi_2 \equiv \frac{i}{\alpha}[n(z)\,{}_0J_{\alpha m} + N(z)\,{}_0N_{\alpha m}]e^{-i\omega t}, \qquad (35)$$
$$\psi_3 \equiv -\frac{1}{\alpha k_t^2}[v(z)\,{}_0J_{\alpha m} + V(z)\,{}_0N_{\alpha m}]e^{-i\omega t},$$

where the functions $W(z)$, $N(z)$ and $V(z)$ obey the same equations as $w(z)$, $n(z)$ and $v(z)$, respectively [Eqs. (21) and (29)]. With the aid of Eqs. (9), (11), (21) and (29) one finds that the three scalar potentials (35) satisfy the wave equations

$$\nabla^2\psi_1 - \frac{1}{v_l^2}\frac{\partial^2\psi_1}{\partial t^2} = 0,$$
$$\nabla^2\psi_{2,3} - \frac{1}{v_t^2}\frac{\partial^2\psi_{2,3}}{\partial t^2} = 0, \qquad (36)$$

where

$$v_l \equiv \frac{\omega}{k_l} = \sqrt{\frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)\rho}},$$
$$v_t \equiv \frac{\omega}{k_t} = \sqrt{\frac{E}{2(1+\sigma)\rho}} \qquad (37)$$

and from Eqs. (7) it follows that Eqs. (34) amount to

$$\mathbf{u} = -\nabla\psi_1 + \hat{e}_z \times \nabla\psi_2 + \nabla \times (\hat{e}_z \times \nabla\psi_3), \qquad (38)$$

or, equivalently,

$$\mathbf{u} = -\nabla\psi_1 - \nabla \times (\psi_2\hat{e}_z) - \nabla \times \nabla \times (\psi_3\hat{e}_z). \qquad (39)$$

In a similar manner, one can show that Eqs. (17) admit separable solutions analogous to Eq. (19) in terms of the spin-weighted cylindrical harmonics with $\alpha = 0$ [Eqs. (15) and (16)], which can also be written in the form (38) with the potentials ψ_i satisfying Eqs. (36) (*cf.* Ref. 10). By virtue of the completeness of the spin-weighted cylindrical harmonics and the linearity of Eqs. (38) and (36), it follows that the general solution of Eq. (1) is given by Eq. (38) or (39), where the scalar potentials ψ_i are solutions of the wave equations (36).

Equation (39) shows that the displacement vector field, in effect, can be written as the sum of a vector field $(-\nabla\psi_1)$ with vanishing curl and a vector field $(-\nabla \times (\psi_2\hat{e}_z) - \nabla \times \nabla \times (\psi_3\hat{e}_z))$ with vanishing divergence (as assumed in Ref. 14). It is easy to verify directly that Eq. (39) satisfies Eq. (1) provided that the scalar potentials ψ_i obey the corresponding wave equations [Eqs. (36)]. If the potentials ψ_i are real, then the displacement vector field is also real. It should be remarked that the expressions (38) and (39) are not linked to a particular coordinate system, despite the fact that the circular cylindrical coordinates were employed to obtain these formulas; however, owing to the presence of the (constant) vector field $\hat{e}_z$, Eqs. (38) and (39) are adapted to the cartesian or the cylindrical coordinates (circular, parabolic or elliptic).

The solutions of Eq. (1) generated by the potential ψ_1 propagate with the velocity v_l, while those generated by ψ_2 or ψ_3 propagate with the velocity v_t. If the potentials ψ_i are plane waves, then the elastic waves generated by ψ_1 are longitudinal waves, whereas those generated by ψ_2 or ψ_3 are transverse. (This is the reason why the subscripts l and t have been employed in the definitions (26), (30) and (37).) In fact, substituting $\psi_1 = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$, with $|\mathbf{k}| = k_l$, into Eq. (38) one obtains $\mathbf{u} = A\sin(\mathbf{k}\cdot\mathbf{r} - \omega t)\,\mathbf{k}$, which represents a longitudinal elastic wave (with $\mathbf{u}$ parallel to $\mathbf{k}$); on the other hand, $\psi_2 = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$, with $|\mathbf{k}| = k_t$, yields $\mathbf{u} = -A\sin(\mathbf{k}\cdot\mathbf{r} - \omega t)\,\hat{e}_z \times \mathbf{k}$, which satisfies $\mathbf{u}\cdot\mathbf{k} = 0$ and, hence, is a transverse wave. Similarly, if $\psi_3 = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$, with $|\mathbf{k}| = k_t$, then $\mathbf{u} = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)\,(\hat{e}_z \times \mathbf{k}) \times \mathbf{k}$, which also satisfies $\mathbf{u}\cdot\mathbf{k} = 0$.

It may be noticed that, according to Eq. (13), a separable solution of the form $u_k = g_k(z)\,{}_kJ_{\alpha m}(\rho,\phi)e^{-i\omega t}$ [Eq. (16)] corresponds to

$$u_\rho = \frac{1}{2}(u_1 + u_{-1})$$
$$= \frac{1}{2}[g_1(z)J_{m+1}(\alpha\rho)$$
$$+ g_{-1}(z)J_{m-1}(\alpha\rho)]e^{im\phi}e^{-i\omega t}, \qquad (40)$$

which is not separable since $g_1(z)$ and $g_{-1}(z)$ are not independent. On the other hand, the presence of Bessel functions of order $m+1$ and $m-1$ accompanying the factor $e^{im\phi}$ in Eq. (40) arises in a natural way by expressing each spin-weighted component of $\mathbf{u}$ in terms of the spin-weighted harmonics of the corresponding weight [Eq. (19)].

We close this section pointing out that if one assumes that the potentials ψ_i do not depend on the time, Eq. (38) does not reduce to the expression found in Ref. 10 for the solution of the equations of equilibrium for an isotropic elastic medium. On the other hand, starting from Eq. (38) one can obtain the Green's function corresponding to Eq. (1) (*cf.* Ref. 15).

3. Solution by separation of variables in spherical coordinates

Let $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$ be the orthonormal basis induced by the spherical coordinates r, θ, φ. A quantity η has spin weight s if under the rotation about $\hat{e}_r$ given by

$$\hat{e}_\theta{}' + i\hat{e}_\varphi{}' = e^{i\beta}(\hat{e}_\theta + i\hat{e}_\varphi) \tag{41}$$

transforms according to

$$\eta' = e^{is\beta}\eta. \tag{42}$$

Any vector field $\mathbf{F}$ can be written as

$$\mathbf{F} = \tfrac{1}{2}F_{-1}\,(\hat{e}_\theta + i\hat{e}_\varphi) + \tfrac{1}{2}F_1\,(\hat{e}_\theta - i\hat{e}_\varphi) + F_0\hat{e}_r, \tag{43}$$

where $F_{\pm 1} \equiv \mathbf{F}\cdot(\hat{e}_\theta \pm i\hat{e}_\varphi)$ and $F_0 \equiv \mathbf{F}\cdot\hat{e}_r$. Then, from Eqs. (41) and (42) it follows that $F_{\pm 1}$ and F_0 have spin weight ± 1 and 0, respectively.

The operators $\eth$ and $\bar{\eth}$ acting on a quantity η with spin weight s are defined in this case by [16]

$$\begin{aligned}\eth\eta &\equiv -\Big(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} - s\cot\theta\Big)\eta \\ &= -\sin^s\theta\Big(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\Big)(\eta\sin^{-s}\theta), \\ \bar{\eth}\eta &\equiv -\Big(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + s\cot\theta\Big)\eta \\ &= -\sin^{-s}\theta\Big(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\Big)(\eta\sin^s\theta).\end{aligned} \tag{44}$$

The quantities $\eth\eta$ and $\bar{\eth}\eta$ have spin weight $s+1$ and $s-1$, respectively.

The gradient of a function f with spin weight 0 is given by

$$\nabla f = -\frac{1}{2r}\bar{\eth}f\,(\hat{e}_\theta + i\hat{e}_\varphi) - \frac{1}{2r}\eth f\,(\hat{e}_\theta - i\hat{e}_\varphi) + \frac{\partial f}{\partial r}\hat{e}_r \tag{45}$$

and the divergence and curl of a vector field $\mathbf{F}$ can be written as

$$\begin{aligned}\nabla\cdot\mathbf{F} &= -\frac{1}{2r}\eth F_{-1} - \frac{1}{2r}\bar{\eth}F_1 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0), \\ \nabla\times\mathbf{F} &= \frac{1}{2ir}\left[\frac{\partial}{\partial r}(rF_{-1}) + \bar{\eth}F_0\right](\hat{e}_\theta + i\hat{e}_\varphi) - \frac{1}{2ir}\left[\frac{\partial}{\partial r}(rF_1) + \eth F_0\right](\hat{e}_\theta - i\hat{e}_\varphi) + \frac{1}{2ir}(\eth F_{-1} - \bar{\eth}F_1)\hat{e}_r.\end{aligned} \tag{46}$$

Then, from Eqs. (45) and (46) one finds that

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\Big[r^2\frac{\partial f}{\partial r}\Big] + \frac{1}{r^2}\eth\bar{\eth}f \tag{47}$$

and

$$\begin{aligned}\nabla^2\mathbf{F} = &\Big[\frac{1}{2r}\frac{\partial^2}{\partial r^2}(rF_{-1}) + \frac{1}{2r^2}\bar{\eth}\eth F_{-1} - \frac{1}{r^2}\bar{\eth}F_0\Big](\hat{e}_\theta + i\hat{e}_\varphi) + \Big[\frac{1}{2r}\frac{\partial^2}{\partial r^2}(rF_1) + \frac{1}{2r^2}\eth\bar{\eth}F_1 - \frac{1}{r^2}\eth F_0\Big](\hat{e}_\theta - i\hat{e}_\varphi) \\ &+ \Big[\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0) + \frac{1}{r^2}\bar{\eth}\eth F_0 + \frac{1}{r^2}\eth F_{-1} + \frac{1}{r^2}\bar{\eth}F_1\Big]\hat{e}_r.\end{aligned} \tag{48}$$

The spin-weighted spherical harmonics with spin weight s, ${}_sY_{jm}$, are defined by

$$\begin{aligned}\bar{\eth}\eth({}_sY_{jm}) &= \big(s(s+1) - j(j+1)\big)\,{}_sY_{jm}, \\ -i\frac{\partial}{\partial\varphi}\,{}_sY_{jm} &= m\,{}_sY_{jm},\end{aligned} \tag{49}$$

where s, j and m are all integers or half-integers. The functions ${}_sY_{jm}$ can be normalized in such a way that

$$\begin{aligned}\eth({}_sY_{jm}) &= \sqrt{(j-s)(j+s+1)}\,{}_{s+1}Y_{jm}, \\ \bar{\eth}({}_sY_{jm}) &= -\sqrt{(j+s)(j-s+1)}\,{}_{s-1}Y_{jm},\end{aligned} \tag{50}$$

and the ${}_0Y_{jm}$ are the usual spherical harmonics Y_{jm}. The set of functions ${}_sY_{jm}$, with s fixed, is complete in the sense that any quantity with spin weight s can be expanded in a series of the ${}_sY_{jm}$ [16].

The (spherical) spin-weighted components of Eq. (1) are

$$\begin{aligned}&(1-2\sigma)\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru_1) + \frac{1}{r^2}\eth\bar{\eth}u_1 - \frac{2}{r^2}\eth u_0\right] - \frac{1}{r^3}\frac{\partial}{\partial r}r^2\eth u_0 + \frac{1}{2r^2}\eth\eth u_{-1} + \frac{1}{2r^2}\eth\bar{\eth}u_1 - \kappa\frac{\partial^2 u_1}{\partial t^2} = 0, \\ &(1-2\sigma)\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru_{-1}) + \frac{1}{r^2}\bar{\eth}\eth u_{-1} - \frac{2}{r^2}\bar{\eth}u_0\right] - \frac{1}{r^3}\frac{\partial}{\partial r}r^2\bar{\eth}u_0 + \frac{1}{2r^2}\bar{\eth}\eth u_{-1} + \frac{1}{2r^2}\bar{\eth}\bar{\eth}u_1 - \kappa\frac{\partial^2 u_{-1}}{\partial t^2} = 0, \\ &(1-2\sigma)\left[\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_0) + \frac{1}{r^2}\bar{\eth}\eth u_0 - \frac{1}{r^2}\eth u_{-1} + \frac{1}{r^2}\bar{\eth}u_1\right] + \frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_0) - \frac{1}{2}\frac{\partial}{\partial r}\frac{1}{r}\eth u_{-1} - \frac{1}{2}\frac{\partial}{\partial r}\frac{1}{r}\bar{\eth}u_1 - \kappa\frac{\partial^2 u_0}{\partial t^2} = 0\end{aligned} \tag{51}$$

[*cf.* Eqs. (17)]. This system of equations admits separable solutions of the form

$$u_k = g_k(r)\,{}_kY_{jm}(\theta,\varphi)\,e^{-i\omega t}, \qquad (k = 1, -1, 0), \tag{52}$$

where j is a non-negative integer and m is an integer such that $|m| \leq j$. The vector field (52) is an eigenfunction of the square of the total angular momentum with eigenvalue $j(j+1)$ [2]. By substituting Eq. (52) into Eqs. (51), by virtue of Eqs. (50), one finds that, if $j \neq 0$, the radial functions g_k are determined by

$$\begin{aligned}
&(1-2\sigma)\Big[\frac{1}{r}\frac{d^2}{dr^2}(rg_1) - \frac{\mu^2}{r^2}g_1 - \frac{2\mu}{r^2}g_0\Big] - \frac{\mu}{r^3}\frac{d}{dr}(r^2 g_0) - \frac{\mu^2}{2r^2}(g_1 - g_{-1}) + \kappa\omega^2 g_1 = 0,\\
&(1-2\sigma)\Big[\frac{1}{r}\frac{d^2}{dr^2}(rg_{-1}) - \frac{\mu^2}{r^2}g_{-1} + \frac{2\mu}{r^2}g_0\Big] + \frac{\mu}{r^3}\frac{d}{dr}(r^2 g_0) + \frac{\mu^2}{2r^2}(g_1 - g_{-1}) + \kappa\omega^2 g_{-1} = 0,\\
&(1-2\sigma)\Big[\frac{d}{dr}\frac{1}{r^2}\frac{d}{dr}(r^2 g_0) - \frac{\mu^2}{r^2}g_0 - \frac{\mu}{r^2}(g_1 - g_{-1})\Big] + \frac{d}{dr}\frac{1}{r^2}\frac{d}{dr}(r^2 g_0) + \frac{\mu}{2}\frac{d}{dr}\frac{1}{r}(g_1 - g_{-1}) + \kappa\omega^2 g_0 = 0,
\end{aligned} \tag{53}$$

where

$$\mu \equiv \sqrt{j(j+1)}. \tag{54}$$

Making use of the definitions

$$\begin{aligned}
M &\equiv \tfrac{1}{2}(g_1 + g_{-1}),\\
H &\equiv \tfrac{1}{2}(g_1 - g_{-1}),
\end{aligned} \tag{55}$$

the set of equations (53) can be rewritten as

$$\frac{d^2M}{dr^2} + \frac{2}{r}\frac{dM}{dr} + \Big(k_t^2 - \frac{\mu^2}{r^2}\Big)M = 0, \tag{56}$$

with k_t defined by Eq. (26), and

$$\begin{aligned}
&(1-2\sigma)\Big[\frac{1}{r}\frac{d^2}{dr^2}(rH) - \frac{\mu^2}{r^2}H - \frac{2\mu}{r^2}g_0\Big]\\
&\qquad - \frac{\mu}{r^3}\frac{d}{dr}(r^2 g_0) - \frac{\mu^2}{r^2}H + \kappa\omega^2 H = 0,
\end{aligned} \tag{57}$$

$$\begin{aligned}
&2(1-\sigma)\frac{d}{dr}\frac{1}{r^2}\frac{d}{dr}(r^2 g_0) + \kappa\omega^2 g_0 - \frac{(1-2\sigma)\mu^2}{r^2}g_0\\
&\qquad - \frac{2(1-2\sigma)\mu}{r^2}H + \mu\frac{d}{dr}\Big(\frac{H}{r}\Big) = 0.
\end{aligned} \tag{58}$$

Since $\mu^2 = j(j+1)$ [see Eq. (54)], the solution of Eq. (56) can be written as

$$M(r) = a_1\, j_j(k_t r) + a_2\, n_j(k_t r) \tag{59}$$

or

$$M(r) = A_1\, h_j^{(1)}(k_t r) + A_2\, h_j^{(2)}(k_t r), \tag{60}$$

where a_1, a_2, A_1 and A_2 are constants and j_j, n_j, $h_j^{(1)}$ and $h_j^{(2)}$ are spherical Bessel functions.

In order to solve Eqs. (57) and (58), we introduce the auxiliary functions

$$\begin{aligned}
v &\equiv \frac{\mu}{r}g_0 + \frac{1}{r}\frac{d}{dr}(rH),\\
w &\equiv \frac{\mu}{r}H + \frac{1}{r^2}\frac{d}{dr}(r^2 g_0).
\end{aligned} \tag{61}$$

(Note that, according to Eqs. (46), (52) and (55), $\nabla \cdot \mathbf{u} = w(r)\, Y_{jm}(\theta,\varphi)\, e^{-i\omega t}$; similarly, v and M are related to the radial parts of $\mathbf{r}\cdot\nabla\times\nabla\times\mathbf{u}$ and $\mathbf{r}\cdot\nabla\times\mathbf{u}$, respectively.) Then, by means of a straightforward computation, from Eqs. (57) and (58) one finds that

$$\begin{aligned}
&\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \Big(k_t^2 - \frac{\mu^2}{r^2}\Big)v = 0,\\
&\frac{d^2w}{dr^2} + \frac{2}{r}\frac{dw}{dr} + \Big(k_l^2 - \frac{\mu^2}{r^2}\Big)w = 0,
\end{aligned} \tag{62}$$

with k_l defined by Eq. (30). (Equations (56) and (62) also follow directly from the fact that $\mathbf{r}\cdot\nabla\times\mathbf{u}$, $\mathbf{r}\cdot\nabla\times\nabla\times\mathbf{u}$ and $\nabla\cdot\mathbf{u}$ obey wave equations as a consequence of Eq. (1).) Hence,

$$\begin{aligned}
v(r) &= b_1\, j_j(k_t r) + b_2\, n_j(k_t r),\\
w(r) &= c_1\, j_j(k_l r) + c_2\, n_j(k_l r),
\end{aligned} \tag{63}$$

where b_1, b_2, c_1 and c_2 are constants.

On the other hand, eliminating H from Eqs. (61), one finds that

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\mu^2}{r^2}\Big)(rg_0) = -\mu v + \frac{1}{r}\frac{d}{dr}(r^2 w). \tag{64}$$

Making use of Eqs. (62), the right-hand side of Eq. (64) can be expressed as

$$\frac{\mu}{k_t^2}\Big(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} - \frac{\mu^2}{r^2}v\Big) - \frac{1}{k_l^2}\frac{1}{r}\frac{d}{dr}r^2\Big(\frac{d^2w}{dr^2} + \frac{2}{r}\frac{dw}{dr} - \frac{\mu^2}{r^2}w\Big) = \Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\mu^2}{r^2}\Big)\Big(\frac{\mu}{k_t^2}v - \frac{1}{k_l^2}r\frac{dw}{dr}\Big)$$

thus,

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\mu^2}{r^2}\right)\left(rg_0 - \frac{\mu}{k_t^2}v + \frac{1}{k_l^2}r\frac{dw}{dr}\right) = 0,$$

therefore,

$$g_0 = \frac{\mu}{k_t^2}\frac{v}{r} - \frac{1}{k_l^2}\frac{dw}{dr} + D_1 r^{j-1} + D_2 r^{-j-2}, \qquad (65)$$

where D_1 and D_2 are constants. Substituting Eq. (65) into the second equation in (61), using Eq. (62), one finds that

$$H = \frac{\mu}{k_l^2}\frac{w}{r} - \frac{1}{k_t^2}\frac{1}{r}\frac{d}{dr}(rv) - (j+1)\frac{D_1}{\mu}r^{j-1} + j\frac{D_2}{\mu}r^{-j-2}. \qquad (66)$$

Substituting Eqs. (65) and (66) into Eqs. (57) and (58) one finds that if $\omega \neq 0$, then D_1 and D_2 must vanish, hence

$$g_0 = \frac{\mu}{k_t^2}\frac{v}{r} - \frac{1}{k_l^2}\frac{dw}{dr},$$
$$H = \frac{\mu}{k_l^2}\frac{w}{r} - \frac{1}{k_t^2}\frac{1}{r}\frac{d}{dr}(rv) \qquad (67)$$

and, from Eqs. (52), (55), (67) and (50) one finds that

$$u_1 = \frac{1}{r}\eth\psi_1 - i\eth\psi_2 + \frac{1}{r}\frac{\partial}{\partial r}r\eth\psi_3,$$
$$u_{-1} = \frac{1}{r}\bar{\eth}\psi_1 + i\bar{\eth}\psi_2 + \frac{1}{r}\frac{\partial}{\partial r}r\bar{\eth}\psi_3, \qquad (68)$$
$$u_0 = -\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\eth\bar{\eth}\psi_3,$$

where

$$\psi_1 \equiv \frac{1}{k_l^2}w(r)\,Y_{jm}\,e^{-i\omega t},$$
$$\psi_2 \equiv \frac{i}{\mu}M(r)\,Y_{jm}\,e^{-i\omega t}, \qquad (69)$$
$$\psi_3 \equiv -\frac{1}{\mu k_t^2}v(r)\,Y_{jm}\,e^{-i\omega t}.$$

According to Eqs. (45) and (46), Eqs. (68) amount to the simple expression

$$\mathbf{u} = -\nabla\psi_1 + \mathbf{r}\times\nabla\psi_2 + \nabla\times(\mathbf{r}\times\nabla\psi_3) \qquad (70)$$

[*cf.* Eq. (38)] and, by virtue of Eqs. (56) and (62), the scalar potentials (69) obey the wave equations (36).

In the case of the separable solutions (52) with $j = 0$ (*i.e.*, $\mu = 0$), the only nonvanishing component is u_0, which is a function of r and t only, and from Eqs. (51) one obtains

$$\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_0) + k_l^2 u_0 = 0 \qquad (71)$$

therefore, using the recurrence relations for the spherical Bessel functions, we have

$$u_0 = (a\,j_1(k_l r) + b\,n_1(k_l r))\,e^{-i\omega t}$$
$$= -\frac{\partial}{\partial r}\frac{1}{k_l}(a\,j_0(k_l r) + b\,n_0(k_l r))\,e^{-i\omega t}, \qquad (72)$$

which is of the form (68) with $\psi_1 = (a\,j_0(k_l r) + b\,n_0(k_l r))e^{-i\omega t}/k_l$ and $\psi_2 = \psi_3 = 0$, and these potentials also satisfy the wave equations (36). Thus, owing to the completeness of the spin-weighted spherical harmonics and the linearity of Eqs. (36) and (70), it follows that the most general solution of Eq. (1) can be expressed in the form (70), where the scalar potentials ψ_i are solutions of the wave equations (36).

Equation (70) can be also written as

$$\mathbf{u} = -\nabla\psi_1 - \nabla\times(\psi_2\mathbf{r}) - \nabla\times\nabla\times(\psi_3\mathbf{r}) \qquad (73)$$

[*cf.* Eq. (39)], which again shows that the displacement vector $\mathbf{u}$ is the sum of an elastic wave with vanishing curl propagafing with velocity v_l [Eq. (37)] and an elastic wave with vanishing divergence propagating with velocity v_t. If the potentials ψ_i are real, $\mathbf{u}$ is also real. Expression (70) is known in the literature; a recent application of it can be found in Ref. 17.

4. Concluding remarks

Apart from the simplifications coming from the use of spin-weighted functions, in the example considered here, the existence of certain differential expressions (such as $\nabla\cdot\mathbf{u}$) that obey decoupled equations helps to solve the systems of ordinary differential equations (20) and (53) by relating them to simpler decoupled equations of second order.

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