

The Feynman-Gell-Mann reduction via the method of adjoint operators

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It is shown that the Feynman-Gell-Mann reduction, which transforms the Dirac equation with an external electromagnetic field into a second-order equation for a two-component spinor, can be obtained, along with other similar reductions, by means of Wald's method of adjoint operators. By means of these reductions some differential operators that map the set of solutions of the Dirac equation into itself are found when there is no electric field.

Keywords: Dirac equation; Feynman-Gell-Mann reduction; symmetry operators

Se muestra que la reducción de Feynman-Gell-Mann, la cual transforma la ecuación de Dirac con un campo electromagnético externo en una ecuación de segundo orden para un espinor de dos componentes, puede obtenerse, junto con otras reducciones similares, por medio del método de operadores adjuntos de Wald. Por medio de estas reducciones se hallan algunos operadores diferenciales que mapean el conjunto de soluciones de la ecuación de Dirac en sí mismo cuando no hay campo eléctrico.

Descriptores: Ecuación de Dirac; reducción de Feynman-Gell-Mann; operadores de simetría

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1. Introduction

The Dirac equation with an external electromagnetic field is one of the systems of linear partial differential equations encountered in mathematical physics whose solution may be difficult because of the coupling of the unknowns. Feynman and Gell-Mann [1] found that, by assuming that the solution of the Dirac equation with an external electromagnetic field can be expressed in terms of a two-component spinor, the latter obeys a second-order differential equation that can be solved in a straightforward manner for some simple electromagnetic fields (see, *e.g.*, Ref. 2).

On the other hand, there exists a method of wide applicability (the method of adjoint operators [3, 4]) that allows one to express the solution of a system of linear partial differential equations in terms of one or several potentials which obey differential equations that are simpler than the original system. The essential step in the application of this method consists in obtaining from the original system another with a lesser number of equations and unknowns by means of linear operations.

In this paper we show that the method of adjoint operators applied to the Dirac equation with an external electromagnetic field leads to the Feynman-Gell-Mann reduction and to other similar reductions. We also show that these reductions allow us to find first-order operators that map a solution of the Dirac equation into another solution. The case considered here illustrates the application of the method of adjoint

operators to obtain symmetry operators of systems of partial differential equations.

In Sect. 2 the basic ideas concerning the Feynman-Gell-Mann reduction and the method of adjoint operators are summarized. In Sect. 3, making use of the method of adjoint operators, we find expressions for the solution of the Dirac equation in terms of two-component spinors that obey second-order equations; one of these expressions coincides with the one considered by Feynman and Gell-Mann. In Sect. 4 we show that these expressions for the solution of the Dirac equation can be combined to find operators that map a solution of the Dirac equation with a magnetic field into another solution of the same equation.

2. The Feynman-Gell-Mann reduction and the method of adjoint operators

The Dirac equation with an external electromagnetic field can be written in the covariant form

$$\left[\gamma^\mu \left(i\hbar \partial_\mu - \frac{q}{c} A_\mu \right) - mc \right] \Psi = 0, \quad (1)$$

where q and m are the charge and rest-mass of the particle, A_μ is the four-potential of the electromagnetic field, the γ^μ are constant 4×4 matrices such that $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$, with $(\eta^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$, and $\partial_\mu = \partial/\partial x^\mu$ ($\mu, \nu = 0, 1, 2, 3$) (see, *e.g.*, Ref. 5). In what follows we shall

employ the standard representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (2)$$

where the σ^i are the Pauli matrices and I is the 2×2 unit matrix.

Assuming that the solution of Eq. (1) can be expressed as

$$\Psi = \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}, \quad (3)$$

where φ is a two-component spinor and using the fact that

$$\begin{aligned} & \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) - mc \right] \left[\gamma^\nu \left(i\hbar\partial_\nu - \frac{q}{c}A_\nu \right) + mc \right] \\ &= \eta^{\mu\nu} \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) \left(i\hbar\partial_\nu - \frac{q}{c}A_\nu \right) \\ & \quad - m^2c^2 + \frac{\hbar q}{c} \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{B} & -i\boldsymbol{\sigma} \cdot \mathbf{E} \\ -i\boldsymbol{\sigma} \cdot \mathbf{E} & \boldsymbol{\sigma} \cdot \mathbf{B} \end{pmatrix}, \end{aligned} \quad (4)$$

where \mathbf{E} and \mathbf{B} are the electric and magnetic fields corresponding to the four-potential A_μ , it follows that φ obeys the second-order equation [1, 2]

$$\begin{aligned} & \left[\eta^{\mu\nu} \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) \left(i\hbar\partial_\nu - \frac{q}{c}A_\nu \right) - m^2c^2 \right. \\ & \quad \left. + \frac{\hbar q}{c} \boldsymbol{\sigma} \cdot (\mathbf{B} + i\mathbf{E}) \right] \varphi = 0. \end{aligned} \quad (5)$$

Thus, any solution of Eq. (5) acts as a potential for the solution of the Dirac equation given by Eq. (3) and, in this manner, the problem of solving Eq. (1) is transformed into the problem of solving Eq. (5) for a two-component spinor.

Equation (3) is similar to the expressions obtained by means of the method of adjoint operators, which yields the solution of a system of partial differential equations in terms of potentials [3, 4]. If

$$\mathcal{E}(f) = 0 \quad (6)$$

is a system of linear partial differential equations, where \mathcal{E} is a linear partial differential operator that maps tensor or spinor fields into fields of the same type, and by combining linearly Eqs. (6) and their derivatives one obtains a new system with a lesser number of equations and unknowns,

$$\mathcal{O}(\chi) = 0, \quad (7)$$

then there exist linear operators \mathcal{T} and \mathcal{S} such that $\chi = \mathcal{T}(f)$ and the identity

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T} \quad (8)$$

holds.

Assuming that the adjoint, \mathcal{A}^\dagger , of any linear operator \mathcal{A} is defined in such a way that \mathcal{A}^\dagger is also a linear operator and

$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger$, whenever the composition $\mathcal{A}\mathcal{B}$ makes sense, then from Eq. (8) it follows that

$$\mathcal{E}^\dagger\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger. \quad (9)$$

Therefore, if ψ is a solution of

$$\mathcal{O}^\dagger(\psi) = 0, \quad (10)$$

Eq. (9) implies that $\mathcal{E}^\dagger(\mathcal{S}^\dagger(\psi)) = 0$, so that if \mathcal{E} is self-adjoint, $\mathcal{E}^\dagger = \mathcal{E}$, or if \mathcal{E}^\dagger is proportional to \mathcal{E} , one concludes that $\mathcal{S}^\dagger(\psi)$ is a solution of the original system (6).

In the next section we shall derive operator identities of the form (8), starting from the Dirac equation and it will be shown that Eq. (5) is a special case of Eq. (10).

3. Potentials for the solutions of the Dirac equation

As pointed out in Ref. 4, the operator $i\hbar\gamma^\mu\partial_\mu - mc$ is self-adjoint provided that the adjoint of a linear partial differential operator, \mathcal{A} , mapping four-component spinors into four-component spinors, is the linear partial differential operator, \mathcal{A}^\dagger , such that

$$\Phi^{*t}\gamma_0\mathcal{A}\Psi = [\mathcal{A}^\dagger\Phi]^{*t}\gamma_0\Psi + \partial_\mu s^\mu, \quad (11)$$

for any pair of four-component spinors Φ and Ψ , where $*$ denotes complex conjugation, the superscript t denotes transposition and s^μ is some four-vector. It may be noticed that if one defines $\bar{\Phi} \equiv \Phi^{*t}\gamma_0$, as in Ref. 5, then Eq. (11) amounts to

$$\bar{\Phi}\mathcal{A}\Psi = \overline{(\mathcal{A}^\dagger\bar{\Phi})}\Psi + \partial_\mu s^\mu. \quad (12)$$

Alternatively, if one writes the Dirac equation with an external electromagnetic field in the form

$$\mathcal{E}(\Psi) = 0, \quad (13)$$

where

$$\mathcal{E} \equiv \frac{i\hbar}{c}\partial_t - \frac{q}{c}\phi + \left(i\hbar\nabla + \frac{q}{c}\mathbf{A} \right) \cdot \boldsymbol{\alpha} - \beta mc, \quad (14)$$

and the Dirac matrices are taken in the usual way

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad (15)$$

(consistently with Eq. (2)) then \mathcal{E} is self-adjoint according to the usual definitions employed in quantum mechanics (which make, e.g., $-i\hbar\nabla$, $\boldsymbol{\alpha}$, β , $\boldsymbol{\sigma}$, self-adjoint). In other words, if, instead of Eq. (11), one defines \mathcal{A}^\dagger by

$$\Phi^{*t}\mathcal{A}\Psi = [\mathcal{A}^\dagger\Phi]^{*t}\Psi + \partial_\mu s^\mu, \quad (16)$$

then $\mathcal{E}^\dagger = \mathcal{E}$. The definition (16) is more convenient than the one given by Eqs. (11) and (12) because it allows us to make use of several well-known results. In what follows it will be assumed that the adjoints of the linear operators are given by the standard rules.

The operator \mathcal{E} [Eq. (14)] can also be written as

$$\mathcal{E} = \gamma_0 \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) - mc \right]. \quad (17)$$

Two systems of equations of the form (7) can be obtained by means of the operator

$$\mathcal{R} \equiv \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0. \quad (18)$$

Indeed,

$$\mathcal{R}\mathcal{E} = \begin{pmatrix} \mathcal{A} & i\mathcal{B} \\ i\mathcal{B} & \mathcal{A} \end{pmatrix} \quad (19)$$

[see Eq. (4)], where \mathcal{A} and \mathcal{B} are the linear operators

$$\begin{aligned} \mathcal{A} &\equiv \eta^{\mu\nu} \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) \left(i\hbar\partial_\nu - \frac{q}{c}A_\nu \right) \\ &\quad - m^2c^2 + \frac{\hbar q}{c} \boldsymbol{\sigma} \cdot \mathbf{B}, \\ \mathcal{B} &\equiv -\frac{\hbar q}{c} \boldsymbol{\sigma} \cdot \mathbf{E}, \end{aligned} \quad (20)$$

which map two-component spinors into two-component spinors and are self-adjoint

$$\mathcal{A}^\dagger = \mathcal{A}, \quad \mathcal{B}^\dagger = \mathcal{B}. \quad (21)$$

By expressing the solution of the Dirac equation $\mathcal{E}(\Psi) = 0$, in the form

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (22)$$

where u and v are two-component spinors, the equation $\mathcal{R}\mathcal{E}(\Psi) = 0$ amounts to

$$\mathcal{A}u + i\mathcal{B}v = 0, \quad i\mathcal{B}u + \mathcal{A}v = 0. \quad (23)$$

Adding Eqs. (23) one finds that

$$\mathcal{O}_1(\chi) = 0, \quad (24)$$

where

$$\mathcal{O}_1 \equiv \mathcal{A} + i\mathcal{B} \quad (25)$$

and

$$\chi = u + v = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \Psi \equiv \mathcal{T}_1(\Psi). \quad (26)$$

Thus, we have the operator identity $\mathcal{T}_1\mathcal{R}\mathcal{E} = \mathcal{O}_1\mathcal{T}_1$, which can be written as $S_1\mathcal{E} = \mathcal{O}_1\mathcal{T}_1$ [cf. Eq. (8)] with

$$S_1 \equiv \mathcal{T}_1\mathcal{R} = \mathcal{T}_1\gamma_0\gamma_0 \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0. \quad (27)$$

Since γ_0 , \mathcal{A} , \mathcal{B} and $\gamma_0[\gamma^\mu(i\hbar\partial_\mu - (q/c)A_\mu) + mc]$ are self-adjoint [see Eq. (17)] and $\mathcal{T}_1^\dagger = \mathcal{T}_1^t$, the adjoints of \mathcal{O}_1 and S_1 are given by

$$\begin{aligned} \mathcal{O}_1^\dagger &= \mathcal{A} - i\mathcal{B}, \\ S_1^\dagger &= \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0\mathcal{T}_1^t, \end{aligned} \quad (28)$$

hence, if ψ is a two-component spinor such that

$$\mathcal{O}_1^\dagger(\psi) = 0, \quad (29)$$

then $\Psi = S_1^\dagger(\psi)$ is a solution of the Dirac equation $\mathcal{E}(\Psi) = 0$. In a more explicit form, since $\gamma_0\mathcal{T}_1^t\psi = \begin{pmatrix} \psi \\ -\psi \end{pmatrix}$,

$$\Psi = \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \begin{pmatrix} \psi \\ -\psi \end{pmatrix} \quad (30)$$

is a solution of the Dirac equation (1) provided that ψ obeys Eq. (29), i.e., $(\mathcal{A} - i\mathcal{B})(\psi) = 0$ or, equivalently,

$$\left[\eta^{\mu\nu} \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) \left(i\hbar\partial_\nu - \frac{q}{c}A_\nu \right) - m^2c^2 + \frac{\hbar q}{c} \boldsymbol{\sigma} \cdot (\mathbf{B} + i\mathbf{E}) \right] \psi = 0. \quad (31)$$

Equations (30) and (31) coincide with Eqs. (3) and (5), respectively.

Subtraction of Eqs. (23) gives

$$\mathcal{O}_2(\xi) = 0, \quad (32)$$

where

$$\mathcal{O}_2 \equiv \mathcal{A} - i\mathcal{B} \quad (33)$$

and

$$\xi = u - v = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \Psi \equiv \mathcal{T}_2(\Psi). \quad (34)$$

Hence, we obtain the identity $S_2\mathcal{E} = \mathcal{O}_2\mathcal{T}_2$, where

$$S_2 \equiv \mathcal{T}_2\mathcal{R} \quad (35)$$

and, therefore,

$$\begin{aligned} \mathcal{O}_2^\dagger &= \mathcal{A} + i\mathcal{B}, \\ S_2^\dagger &= \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0\mathcal{T}_2^t. \end{aligned} \quad (36)$$

Thus, if ψ is a two-component spinor such that

$$\mathcal{O}_2^\dagger(\psi) = 0, \quad (37)$$

i.e.,

$$\left[\eta^{\mu\nu} \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) \left(i\hbar\partial_\nu - \frac{q}{c}A_\nu \right) - m^2c^2 + \frac{\hbar q}{c} \boldsymbol{\sigma} \cdot (\mathbf{B} - i\mathbf{E}) \right] \psi = 0, \quad (38)$$

the four-component spinor

$$\Psi = S_2^\dagger(\psi) = \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \begin{pmatrix} \psi \\ \psi \end{pmatrix} \quad (39)$$

is a solution of the Dirac equation (1) [cf. Eqs. (31) and (30)].

It should be clear that Eqs. (24) and (32) are not the only systems of two equations that can be derived from Eq. (1) and that with any equation of this type one obtains an expression for the solutions of the Dirac equation (see also Sect. 4). As we show in the next section, the expressions for the solutions of a system of partial differential equations in terms of potentials is not only a convenient way of solving the system of equations, but also may be useful to find internal symmetries of the system (see also Refs. 3, 6 and 7).

4. Symmetry operators

When the electric field is absent, the operator \mathcal{B} is equal to zero [see Eq. (20)] and $\mathcal{O}_1^\dagger = \mathcal{O}_1$ [see Eqs. (25) and (28)]; hence, if Ψ is a solution of the Dirac equation, according to Eq. (24), $\chi = \mathcal{T}_1(\Psi)$ satisfies $\mathcal{O}_1(\chi) = 0$ and, at the same time, $\mathcal{O}_1^\dagger(\chi) = 0$, which means that we can take $\psi = \chi$ as the solution of Eq. (29) and therefore, $\tilde{\Psi} = \mathcal{S}_1^\dagger(\chi) = \mathcal{S}_1^\dagger\mathcal{T}_1(\Psi)$ is also a solution of the Dirac equation. It can be seen that $\mathcal{T}_1^\dagger\mathcal{T}_1 = 1 + \gamma_5$, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, thus we conclude that the first-order differential operator

$$\mathcal{S}_1^\dagger\mathcal{T}_1 = \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0(1 + \gamma_5) \quad (40)$$

maps solutions of the Dirac equation with a given magnetic field into solutions of the same equation.

In an entirely similar manner, if $\mathbf{E} = 0$, then $\mathcal{O}_2^\dagger = \mathcal{O}_2$ and it follows that the operator

$$\mathcal{S}_2^\dagger\mathcal{T}_2 = \left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0(1 - \gamma_5) \quad (41)$$

also maps solutions of the Dirac equation with a given magnetic field into solutions of the same equation. It may be noticed that for any electromagnetic field, $\mathcal{O}_1^\dagger = \mathcal{O}_2$ and therefore $\mathcal{S}_1^\dagger\mathcal{T}_2$ and $\mathcal{S}_2^\dagger\mathcal{T}_1$ map solutions of the Dirac equation (1) into solutions of the same equation; however, since $\gamma_0\mathcal{T}_1^\dagger\mathcal{T}_2 = \mathcal{T}_2^\dagger\mathcal{T}_2 = 1 - \gamma_5$ and $\gamma_0\mathcal{T}_2^\dagger\mathcal{T}_1 = \mathcal{T}_1^\dagger\mathcal{T}_1 = 1 + \gamma_5$, taking into account the fact that γ_5 anticommutes with γ^μ , one finds that the effect of $\mathcal{S}_1^\dagger\mathcal{T}_2$ and $\mathcal{S}_2^\dagger\mathcal{T}_1$ on a solution of Eq. (1) is equivalent to a multiplication by $2mc$.

In the case where $\mathbf{E} = 0$, one can easily derive an infinity of equations of the form (7). Since $\mathcal{B} = 0$ [see Eq. (20)], from Eqs. (23) we find that, for any pair of complex constants, λ, μ ,

$$\mathcal{A}(\lambda u + \mu v) = 0, \quad (42)$$

which is of the form (7) with $\mathcal{O} = \mathcal{A}$ and $\chi = \lambda u + \mu v = \mathcal{T}(\Psi)$, with

$$\mathcal{T} = \begin{pmatrix} \lambda & 0 & \mu & 0 \\ 0 & \lambda & 0 & \mu \end{pmatrix} \quad (43)$$

[cf. Eqs. (26) and (34)] and therefore $\mathcal{T}\mathcal{R}\mathcal{E} = \mathcal{O}\mathcal{T}$. Thus, it follows that if ψ is a two-component spinor such that $\mathcal{O}^\dagger(\psi) = 0$, i.e., $\mathcal{A}(\psi) = 0$, which is just Eq. (31) or (38) with $\mathbf{E} = 0$, then $\mathcal{R}^\dagger\mathcal{T}^\dagger(\psi)$ is a solution of the Dirac equation. Furthermore, if Ψ is a solution of the Dirac equation

with an arbitrary (static) magnetic field, $\mathcal{R}^\dagger\mathcal{T}^\dagger\mathcal{T}(\Psi)$ is also a solution of the same equation. Thus

$$\left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0 \begin{pmatrix} |\lambda|^2 I & \lambda^* \mu I \\ \mu^* \lambda I & |\mu|^2 I \end{pmatrix} \quad (44)$$

maps solutions of the Dirac equation with a given magnetic field into solutions of the same equation. [Note that Eqs. (40) and (41) are special cases of Eq. (44).] Moreover, the same conclusion holds if in the composition $\mathcal{R}^\dagger\mathcal{T}^\dagger\mathcal{T}$, the two operators \mathcal{T} are of the form (43), with different pairs of constants. In this manner, one concludes that

$$\left[\gamma^\mu \left(i\hbar\partial_\mu - \frac{q}{c}A_\mu \right) + mc \right] \gamma_0 \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix} \quad (45)$$

also maps solutions of the Dirac equation with a given magnetic field into solutions of the same equation, for any set of complex constants a, b, c, d .

For any system of linear partial differential equations such that $\mathcal{O}^\dagger = \mathcal{O}$, the composition $\mathcal{S}^\dagger\mathcal{T}$ maps the set of solutions of the system of equations into itself. An additional example is provided by the source-free Maxwell equations for which two decoupled equations can be obtained with $\mathcal{O} = \nabla^2 = \mathcal{O}^\dagger$ [7].

5. Concluding remarks

As we have shown, the method of adjoint operators allows us to find systematically expressions for the solutions of the Dirac equation in terms of spinor potentials. This method can be also applied with other fields or when the background spacetime is not flat. Among the by-products of this approach is the finding of operators that map the set of solutions of the original problem into itself.

The symmetry operators found in Sect. 4 do not depend on the symmetries of the magnetic field; apart from the condition that the electric field be equal to zero in some inertial frame, there are no further restrictions on the magnetic field. If $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{B}^2 - \mathbf{E}^2 > 0$, then there exists an inertial frame where the electric field vanishes and the results of the preceding section apply.

The examples mentioned here show that the method of adjoint operators is not only useful to simplify the solution of systems of linear partial differential equations, but in some cases it can also be used to find internal symmetries.

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