

A disordered resonant circuit in the context of the theory of complex random variables

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The need to relate the probability distribution of a complex random variable to its *a priori* given random elements occurs very frequently in applied physics. However, the theory of probability is usually not emphasized in the education of physicists, even when the calculation of the probability distribution only involves the use of transformation of random variables. In this paper a survey is given of the facts concerning to nonlinear complex random transformations. We illustrate both the essential properties which characterize the method and its utility as a problem-solving procedure with an application from undergraduate physics: *disordered electric circuits*. The calculation of constrained probability distributions is also revisited. The results are then used as a basis for a discussion of the concept of constrained complex random variables in the context of electric circuits.

Keywords: Complex random variables; disordered electric circuits; constrained distributions

La necesidad de relacionar la distribución de probabilidades de una variable aleatoria compleja, con los valores de sus elementos aleatorios dados *a priori*, ocurre muy frecuentemente en física aplicada. Sin embargo, con frecuencia la teoría de probabilidad no está debidamente enfatizada en la formación de grado de los físicos, a pesar que el cálculo de la distribución de probabilidades sólo involucra el uso de transformaciones de variables aleatorias. En este artículo presentamos las características fundamentales de las transformaciones aleatorias complejas no-lineales. Ilustramos tanto las propiedades esenciales que caracterizan al método como su utilidad para la resolución de problemas, con una aplicación de física básica: *circuitos eléctricos desordenados*. El cálculo de distribuciones de probabilidad constreñidas es también abordado y sus resultados son analizados en el contexto de circuitos eléctricos.

Descriptor: Variables aleatorias complejas; circuitos eléctricos desordenados; distribuciones con vínculos

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1. Introduction

Random variables are essential in physics, chemistry, engineering, etc. They are fundamental objects in the theory of statistical mechanics [1, 2], transport in disordered media [3], kinetic theory [4], and they also play an important role in the analysis of uncertainties in experimental data [5]. Nevertheless, the theory of probability is usually not emphasized in the education of physicists, at least not in the same degree as with other fundamental subjects of mathematics. As a matter of fact, complex random variables are less studied and they are almost forgotten in most books for physicists, even when the probability distribution of these complex variables only involves the use of the theorem of transformation of random variables.

The occurrence of complex random variables is very common in the context of electric circuits. A parallel RCL circuit appears as a representation of a *real resistor*. This is so because an electric current through a wire will produce a magnetic field, so any real resistor will have some induc-

tance L . On the other hand, when a resistor has a potential difference across it, the density of electrons will change, thus the resistor will show also some capacitance C [6]. Thus at low frequencies a *real resistor* may be thought as a simple element R (frequently the *parasitic* elements L and C are taken as negligible parameters). As the frequency is increased, the reactive impedance ωL and capacitive impedance $1/\omega C$ start to be comparable to R , thus it begins to look like a resonant circuit [6], where the parameters L and C have some uncertainties. It is thus suitable to model those parameters as random variables, in this case the total impedance $Z \equiv Z(\omega)$ will be a complex random variable. Hence the probability distribution for a complex random variable must be worked out in order to solve the RCL system.

The study of ionic conductors has received increased attention in recent years due to the unusual physical properties observed in these systems [7, 8]. Interestingly, the analysis of the complex impedance in these materials, can be done in term of effective electric circuits, among them the simplest one is the parallel RC circuit [9], so it is necessary to characterize

the mean value and the dispersion of the different elements that take place in the effective electric circuits. To understand these facts, it is essential to introduce a formal theory of complex random variables.

2. A simple model for a complex random variable

In order to simplify the calculation we will model the circuit with the RC parallel one (*i.e.*, we take $L = 0$). Then, let us consider C as a random variable characterized by some probability distribution $P_C(c)$, therefore we get the following equation for the total impedance:

$$\frac{1}{Z} = \frac{1}{R} + i\omega C; \tag{1}$$

here R is taken as a *deterministic* quantity, thus the real and imaginary part of Z will acquire a random character due to the occurrence of the random variable C in its definition Eq. (1).

The complex random variable $Z = Z_1 + iZ_2$ will be characterized by a joint probability distribution for the *two* random variables (Z_1, Z_2) , which in terms of the theorem of transformation of random variables is given by [2]

$$P_Z(z_1, z_2) = \langle \delta [z_1 - Z_1(c)] \delta [z_2 - Z_2(c)] \rangle_{P_C(c)}, \tag{2}$$

where the functions $Z_j(c)$, ($j = 1, 2$) are given by

$$Z_1(c) = \frac{R}{1 + (R\omega c)^2}, \quad Z_2(c) = -\frac{R^2\omega c}{1 + (R\omega c)^2}. \tag{3}$$

Note that it is not possible to write a *cumulative distribution function* for a complex number $z_1 + iz_2$ [10]. In what follows we will simplify the notation dropping out all superfluous notations, *i.e.*, $P(c) \equiv P_C(c)$, $P(z_1, z_2) \equiv P_Z(z_1, z_2)$, etc.

2.1. Transforming the complex random variable

Taking into account Eq. (2) and the usual definition of the mean value over a distribution function we can write

$$P(z_1, z_2) = \int_{\mathcal{D}_c} \delta \left(z_1 - \frac{R}{1 + (R\omega c)^2} \right) \times \delta \left(z_2 + \frac{R^2\omega c}{1 + (R\omega c)^2} \right) P(c) dc. \tag{4}$$

In order to make an explicit calculation, we have to choose the distribution $P(c)$. If we take $P(c)$ as a uniform distribution over $\mathcal{D}_c = [0, 1]$ we arrive, after a little algebra, to the expression

$$P(z_1, z_2) = \frac{\delta \left(z_1^{1/2} \sqrt{R - z_1} + z_2 \right)}{2\omega z_1^{3/2} \sqrt{R - z_1}}, \tag{5}$$

which is the desired joint probability distribution. Note that the marginal distributions [11] $P(z_1)$ and $P(z_2)$,

$$P(z_1) = \left(2\omega z_1^{3/2} \sqrt{R - z_1} \right)^{-1},$$

$$z_1 \in \left[\frac{R}{1 + (R\omega)^2}, R \right] \equiv \mathcal{D}_1, \tag{6}$$

$$P(z_2) = 2 \left\{ \omega R^2 \sqrt{1 - \left(\frac{2z_2}{R} \right)^2} \left[1 + \sqrt{1 - \left(\frac{2z_2}{R} \right)^2} \right] \right\}^{-1},$$

$$z_2 \in \left[\frac{-R}{1 + (R\omega)^2}, 0 \right] \equiv \mathcal{D}_2, \tag{7}$$

allow us to find moments like $\langle z_1^m \rangle$ and $\langle z_2^m \rangle$, which in fact could also be obtained from Eq. (3) as $\langle z_1^m \rangle = \langle Z_1^m(c) \rangle_{P(c)}$, etc. For example, let us here calculate the first moments $\langle z_1 \rangle$ and $\langle z_2 \rangle$. From (6) and (7) we get respectively:

$$\langle z_1 \rangle = \int_{\mathcal{D}_1} z_1 P(z_1) dz_1 = \frac{1}{\omega} \arctan(R\omega) \tag{8}$$

$$\langle z_2 \rangle = \int_{\mathcal{D}_2} z_2 P(z_2) dz_2 = \frac{-1}{2\omega} \ln [1 + (R\omega)^2] \tag{9}$$

as we expected from (3) and the use of $P(c) = 1$ with $\mathcal{D}_c = [0, 1]$.

If we are interested in the average of a function like $g(z) \equiv g(z_1, z_2)$ we should use the joint probability distribution given in (5). As a matter of fact this probability distribution indicates that Z_1 and Z_2 are strongly correlated random variables, showing (for example) that $\langle z_2/z_1 \rangle \simeq \langle z_2 \rangle / \langle z_1 \rangle$ is a naive approximation which only works at very low frequency. In order to quantify this comment let us calculate the variance of the random function $g(z_1, z_2) = z_2/z_1$, from the distribution (5) we get

$$\sigma_{\text{exact}}^2 = \left\langle \left(\frac{z_2}{z_1} \right)^2 \right\rangle - \left\langle \left(\frac{z_2}{z_1} \right) \right\rangle^2 = \frac{R^2\omega^2}{12}, \tag{10}$$

but from the previous naive approximation we easily would obtain

$$\sigma_{\text{app.}}^2 = \langle z_2^2 \rangle \left\langle \left(\frac{1}{z_1} \right)^2 \right\rangle - \langle z_2 \rangle^2 \left\langle \left(\frac{1}{z_1} \right) \right\rangle^2$$

$$= \left[1 + \frac{2}{3}(R\omega)^2 + \frac{1}{5}(R\omega)^4 \right]$$

$$\times \left\{ \frac{\tan^{-1}(R\omega)}{2R\omega} - \frac{1}{2} [1 + (R\omega)^2]^{-1} \right\}$$

$$- \frac{\left(1 + \frac{(R\omega)^2}{3} \right)^2 \log [1 + (R\omega)^2]^2}{4(R\omega)^2}. \tag{11}$$

Note that this approximation is exact only in the particular case when Z_1 and Z_2 are independent random variables. But this is not the case characterized by the random transformations (3).

In Fig. 1 we have shown, for a fixed value of R , the comparison between both variances: σ_{exact}^2 and $\sigma_{\text{app.}}^2$, as a function of the frequency ω . Note that in the $\omega \rightarrow 0$ limit, $\sigma_{\text{app.}}^2$ can be approximated by:

$$\lim_{\omega \rightarrow 0} \sigma_{\text{app.}}^2 \approx \frac{(R\omega)^2}{12} - \frac{17(R\omega)^4}{180} + \mathcal{O}(\omega^6) \quad (12)$$

in agreement up to $\mathcal{O}(\omega^2)$ with the exact result (10), i.e., only at very low frequencies Z_1 and Z_2 can be thought as independent random variables.

3. A circuit with constraints

It may occur that the circuit has some constraints, for example the rms-current could externally be fixed to some value. Therefore it may be necessary to know the joint probability distribution, but constrained to such fixed conditions. Let Z_1, Z_2 be random variables characterized by the joint probability distribution $P(z_1, z_2)$. If these random variables are subjected to q constraints $h_j(z_1, z_2) = 0, (j = 1, 2, \dots, q)$, we may want to know the joint probability distribution of the constrained random variables $\mathcal{P}(z_1, z_2)$.

The concept of constrained random variables is very common in physics. This constrained probability can easily be understood in terms of marginal distributions [11] and conditional distributions [12]. We will show here how to calculate that constrained probability distribution for the case of a complex random variable Z .

First, we define $q + 2$ random variables as

$$\begin{aligned} Y_l &\equiv Z_l, & \text{if } 1 \leq l \leq 2 \\ Y_l &\equiv h_l(Z_1, Z_2), & \text{if } 3 \leq l \leq 2 + q \end{aligned} \quad (13)$$

then joint probability distribution for the $q + 2$ random variables is [2]

$$\begin{aligned} P(y_1, y_2, \dots, y_{q+2}) &= \left\langle \delta(y_1 - z_1) \delta(y_2 - z_2) \right. \\ &\quad \left. \times \prod_{l=1}^q \delta[y_{2+l} - h_l(z_1, z_2)] \right\rangle_{P(z_1, z_2)}, \end{aligned} \quad (14)$$

the average is easily done due to the occurrence of the first two deltas, then we get

$$\begin{aligned} P(y_1, y_2, \dots, y_{q+2}) &= P(y_1, y_2) \prod_{l=1}^q \delta(y_{2+l} - h_l(y_1, y_2)). \end{aligned} \quad (15)$$

Second, the joint probability distribution of Y_1, Y_2 given that $Y_3 = y_3, \dots, Y_{q+2} = y_{q+2}$, is written in terms of the conditional probability

$$\begin{aligned} P_{2|q+2}(y_1, y_2 | y_3, \dots, y_{q+2}) &= \frac{P(y_1, y_2, \dots, y_{q+2})}{\iint ds_1 ds_2 P(s_1, s_2, y_3, \dots, y_{q+2})}. \end{aligned} \quad (16)$$

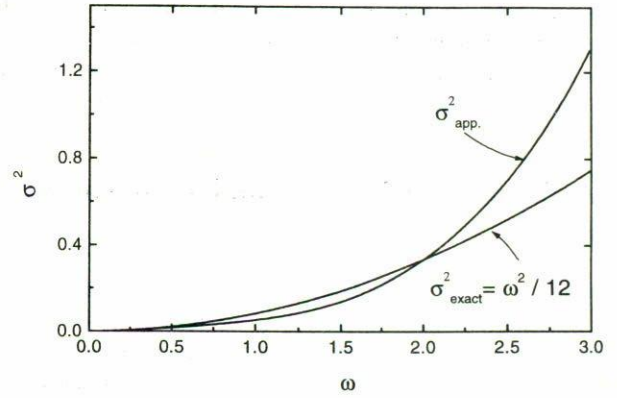


FIGURE 1. Variance of z_2/z_1 . The comparison between the exact result Eq. (10) and the naive approximation Eq. (11) is plotted, for a fixed value of $R = 1$, as a function of the frequency ω .

Then the constrained probability distribution $\mathcal{P}(z_1, z_2)$ is simply $P_{2|q+2}(z_1, z_2 | 0, \dots, 0)$, which in terms of Eqs. (15) and (16) can be written as

$$\mathcal{P}(z_1, z_2) = \mathcal{N}^{-1} P(z_1, z_2) \prod_{l=1}^q \delta(h_l(z_1, z_2)), \quad (17)$$

where the normalization constant is

$$\mathcal{N} = \iint dz_1 dz_2 P(z_1, z_2) \prod_{l=1}^q \delta(h_l(z_1, z_2)). \quad (18)$$

Equation (17) is the desired result.

3.1. The rms-current as a constraint in the probability distribution

Let the voltage, in a linear alternating-current circuit, be characterized by $V(t) = V_0 \cos(\omega t)$. Therefore, the root-mean-square current I_{rms} will be given in terms of the time-average over one cycle:

$$I_{\text{rms}} = \frac{V_0}{\sqrt{2}} \frac{1}{\sqrt{z_1^2 + z_2^2}}, \quad (19)$$

where $z_1^2 + z_2^2$ measures the length of the total random complex impedance Z on the circuit. If we are interested in electronic devices which may work with a fixed amount of rms-current, we shall be in presence of the physical constraint $I_{\text{rms}} = \text{cte}$. This means that we should know the probability distribution $\mathcal{P}(z_1, z_2)$ under the constraint $|z|^2 = \text{cte}$. Hence let us use the constrained probability distribution given in (17). First, we have to calculate the normalization (18) where $h(z_1, z_2) = |z|^2 - B$. Here the constant B is given in terms of the parameters that appear in (19). From (5) and (18) we get

$$\begin{aligned} \mathcal{N} &= \iint dz_1 dz_2 \frac{\delta(z_1^{1/2} \sqrt{R - z_1} + z_2)}{2\omega z_1^{3/2} \sqrt{R - z_1}} \delta(z_1^2 + z_2^2 - B) \\ &= \left(2\omega B^{3/2} \sqrt{1 - B/R^2}\right)^{-1}, \end{aligned} \quad (20)$$

then the constrained probability distribution will look like

$$P(z_1, z_2) = B^{3/2} \sqrt{1 - B/R^2} \times \frac{\delta(z_1^{1/2} \sqrt{R - z_1} + z_2)}{z_1^{3/2} \sqrt{R - z_1}} \delta(z_1^2 + z_2^2 - B), \quad (21)$$

which is the desired result.

4. Summary and Discussions

4.1. Characterizing a sample

The analysis of the steady state AC-response of solid and liquid electrolytes can be made by using impedance spectroscopy techniques. Since a detailed microscopic model of the response is usually lacking, the total impedance $Z = Z_1 + iZ_2$ is frequently fit to an equivalent electrical circuit. Therefore it is possible to obtain the parameters of the circuit using a complex nonlinear least square data fitting [8]. Besides that method we propose that the fit can be improved by introducing a dispersion in the mean values of those parameters. Thus the theory of complex random variables is the correct framework to tackle that problem.

In the previous section we have given a survey of that theory, in particular we have worked out the parallel RC circuit to show the importance of the knowledge of the joint probability distribution of Z_1 and Z_2 (when C is the only random variable in the circuit). Of course in an electrolytes sample there are other elements which also need to be characterized (for example L and R), in that case the theory of complex random variable is the appropriated framework to study disordered electric circuits.

If $P_C(c)$ were an exponential distribution of the form $P_C(c) = (1/c_0) \exp[-c/c_0]$, similar calculations, as we did in Sect. 2, could be done in terms of Laplace transform techniques. More complicated distributions like the Gamma probability distribution $P_C(c) = [a^\nu/\Gamma(\nu)]c^{\nu-1} \exp(-ac)$ should be worked out with some numerical help.

Note that, in our RC circuit, by measuring the probability distribution of the phase $\Phi = \tan^{-1}(Z_2/Z_1)$, it could experimentally be possible to infer the probability distribution of the random variable C . The connection between $P_\Phi(\varphi)$ and $P_C(c)$ is just given by

$$P_\Phi(\varphi) = \int \delta[\varphi - \Phi(c)] P_C(c) dc = \frac{1 + \tan^2(\varphi)}{R\omega} P_C \left[\frac{\tan(-\varphi)}{R\omega} \right]. \quad (22)$$

Thus let us say, for a moment, that the mean value of the capacity is $\langle C \rangle = c_0$, therefore by making a histogram of the probability distribution of the phase Φ it is possible to infer the dispersion of the mean value c_0 . This fact shows that a possible characterization of the RC circuit can be done in

term of (22) if the phase Φ is the suitable variable to be measured. Other random functions of Z_2, Z_1 can also be worked out in a similar way by using the joint probability distribution $P(z_1, z_2)$, as we have shown in the previous chapter.

In particular our approach could be adapted to the complex least square fitting algorithm in order to characterize the spectroscopy of solid and liquid electrolytes [8].

4.2. Summary

We have presented, from a pedagogical point of view, the general approach that must be used when the probability distribution of complex random variables is required. Emphasis has been made on alternating electric circuits with random components.

In the study of ionic materials it is necessary to characterize the mean value and the dispersion of the constituent elements that take place in the effective electric circuits. To study these quantities, it is essential to introduce the joint probability distribution of a complex random variable. We have shown that a possible characterization of C can be done in term of the random phase Φ . Other random functions of Z_2, Z_1 can also be worked out in a similar way by using the joint probability distribution $P(z_1, z_2)$. Of course, in a sample there could be other elements which also needed to be characterized, in any case the theory of complex random variable is the suitable framework to study disordered electric circuits.

In particular, we have worked out the RC parallel circuit, and found the joint probability distribution of the real and imaginary part of total impedance Z when C was a random variable with uniform distribution. In general we have shown that Z_1 and Z_2 are strongly correlated random variables. Hence the mean value of any function like $g(z_1, z_2)$ must be calculated in terms of the joint probability distribution $P(z_1, z_2)$. In particular, when $P(c) = 1, c \in [0, 1]$ we have done the calculation for the dispersion of z_2/z_1 [in many electronic devices we usually are interested in the phase $\varphi = \tan^{-1}(z_2/z_1)$]. We have compared the exact variance against a naive approximation, showing the usefulness of the joint probability distribution. Thus, for the parallel RC circuit, we have shown that only in the $\omega \rightarrow 0$ limit the random variables Z_1 and Z_2 look statistically independent. We have also presented the concept of constrained probability distributions in the context of a complex random variable. To exemplify some real situation, we have fixed the value for the rms-current in the RC circuit. The extension to the RCL circuit (when L is a deterministic parameter) is easily done by introducing a redefinition in the resistive element and a shift in the capacity on Eq. (1). The case when L and C are both random variables is just a mere generalization of the work presented in this article.

We have remarked that the present approach is the correct one to obtain the required statistical averaged of a function of the complex random impedance $Z(\omega)$.

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10. The *cumulative distribution function* (the total probability that the random variable X has any value $\leq x$) is defined as: $\mathbf{P}(x) = \int_{-\infty}^x P(x') dx'$. Mathematicians prefer $\mathbf{P}(x)$ to the probability density $P(x)$ because it does not involve Dirac-delta functions. From this definition it is simple to see that for a complex number $z = z_1 + iz_2$ the cumulative distribution function cannot be defined because a complex number cannot be ordered.
11. Let X be a random variable having r components X_1, \dots, X_r . The probability that the set of random variables X_1, \dots, X_s ($s < r$) have certain values (x_1, \dots, x_s) regardless of the values of the remaining X_{s+1}, \dots, X_r variables, is given by the *marginal probability distribution*: $P(x_1, \dots, x_s) = \int \dots \int P(x_1, \dots, x_s, x_{s+1}, \dots, x_r) dx_{s+1} \dots dx_r$.
12. The conditional probability distribution $P_{s|r-s}(x_1, \dots, x_s | x_{s+1}, \dots, x_r)$ *i.e.*: the probability distribution that the random variables X_1, \dots, X_r has the values (x_1, \dots, x_s) having the prescribed values (x_{s+1}, \dots, x_r) , is given in term of the marginal probability by Bayes' rule as: $P_{s|r-s}(x_1, \dots, x_s | x_{s+1}, \dots, x_r) = P(x_1, \dots, x_s, x_{s+1}, \dots, x_r) / P(x_{s+1}, \dots, x_r)$