The ecological efficiency of a thermal finite time engine

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Within the framework of the finite time thermodynamics, the efficiency of the Curzon and Ahlborn cycle with finite time adiabats is found using the ecological function instead of the power as maximization criterion. It is shown that the ecological efficiency is given by a power series expansion in the parameter $\lambda \sim 1/(\ln V_{\max} - \ln V_{\min})$, where $V_{\max}$ and $V_{\min}$ are the maximum and the minimum volumes spanned by the cycle, respectively. It is shown that when using instantaneous adiabats the obtained efficiency is the zero order term of the power series in λ and that it represents an upper bound on the possible value of the ecological efficiency.

Keywords:

En el esquema de la termodinámica de tiempos finitos, se encuentra la eficiencia del ciclo de Curzon y Ahlborn con adiabatas no instantáneas usando la función ecológica en vez de la potencia como criterio de maximización. Se muestra que la eficiencia ecológica se expresa como una serie de potencias en el parámetro $\lambda \sim 1/(\ln V_{\max} - \ln V_{\min})$, en donde $V_{\max}$ y $V_{\min}$ representan el volumen máximo y el volumen mínimo subtendido por el ciclo, respectivamente. Se muestra que cuando se calcula la eficiencia ecológica usando adiabatas instantáneas, ésta constituye el término de orden cero de la serie en potencias de λ y que constituye una cota superior para el valor de la eficiencia ecológica.

Descriptores:

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1. Introduction

Endoreversible finite-time thermodynamics (EFTT) is an extension of classical equilibrium thermodynamics for thermal engines that includes time dependence of the interaction processes with the external sources while excluding irreversible effects within the working substance [1,2]. The exclusion of intrinsec irreversible effects in the substance, or endoreversible hypothesis, is considered for cases in which the internal relaxation times of the working substance are negligibly short compared to the total time of the cycle. The Curzon and Alhborn (CA) engine [3] (see Fig. 1) is a Carnot-type cycle, in which there is no thermal equilibrium between the working fluid and the reservoirs at the isothermal branches of the cycle and in which the adiabatic branches are taken as instantaneuos. The CA cycle working at the maximum power regime has been shown to have an efficiency given by

$$\eta_{\mathrm{CA}} = 1 - \sqrt{\frac{T_2}{T_1}}, \tag{1}$$

where T_2 and T_1 are the absolute temperatures of the cold and hot reservoirs respectively. In contrast, the efficiency of the zero power output engine, the Carnot cycle, is given by

$$\eta_{\mathrm{C}} = 1 - \frac{T_2^*}{T_1^*}, \tag{2}$$

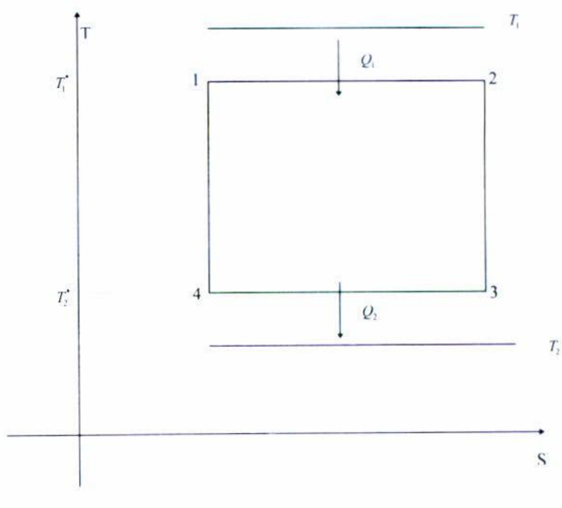

FIGURE 1. Curzon and Ahlborn cycle in the entropy S and absolute temperature T plane. Q_1 represents absorbed heat at absolute temperature T_1 and Q_2 rejected heat at absolute temperature T_2.

where the T_2^* and T_1^* are the absolute temperatures of the working substance at the isothermal branches ($T_2^* < T_1^*$). Notice that since by definition $\eta \equiv 1 - Q_2/Q_1$, the endoreversibility condition namely $Q_1/T_1^* = Q_2/T_2^*$ makes Eq. (2)

aplicable to any endoreversible cycle. Upon calling

$$z \equiv \frac{T_2^*}{T_1^*} \tag{3}$$

we have that

$$\eta = 1 - z\,; \tag{4}$$

this expression will be shown itself useful in what follows. The Carnot efficiency is recovered from Eq. (4) because for a Carnot cycle, $T_2/T_1 = T_2^*/T_1^*$. The Curzon and Ahlborn efficiency is recovered because in this case, $z = \sqrt{T_2/T_1}$. For endoreversible cycles with two isothermal branches, one has $z = z(T_2/T_1)$.

Since the publication of the Curzon and Ahlborn paper, an extensive work has been done in the field of the EFTT. To obtain Eq. (1) some authors use the formalism of differential and variational calculus [4–6]. Others use the formalism of the optimal control theory [7–11].

Angulo-Brown [12, 13] has advanced an optimization criterion for the CA cycle that combines the power output of the cycle P and the total entropy production σ; it has been called ecological function and is defined as follows:

$$E \equiv P - T_2\sigma. \tag{5}$$

Upon operation of a CA cycle, entropy is produced which depends on the temperature differences between the working substance and the reservoirs at the isothermal branches [5]. The use of the power ouput as an optimization criterion ignores the entropy production. On the other hand, it is only for the Carnot cycle that the entropy production is zero, but at the price of having zero power output. As an optimization criterion the function E represents a compromise to obtain the greatest possible power with the least entropy production. While a number of different optimization criteria might be proposed, the ecological function represents a first step towards obtaining a criterion to evaluate the performance of thermal plants wich will include ecological considerations, *i.e.* considerations concerning the thermodynamic degradation of the environment.

It has been shown [12] that for instantaneous adiabats, $E = E(T_2^*/T_1^*)$ is a convex function with a single maximum that leads to the following efficiency in terms of $\beta \equiv T_2/T_1$:

$$\eta_{\rm E} = \eta_{\rm C}\frac{1 + 2\beta + (3/2)\sqrt{2}(\beta + \beta^2)^{1/2}}{1 + 3\beta + 2\sqrt{2}(\beta + \beta^2)^{1/2}}. \tag{6}$$

The looked for compromise has been achieved inasmuch as the behavior of the $\eta_{\rm E}$ *versus* β is almost coincident with the semisum [12],

$$\eta_s = \tfrac{1}{2}(\eta_{\rm C} + \eta_{\rm CA}). \tag{7}$$

Gutkowics-Krusin, Procaccia, and Ross [14] have shown, by explicitely taking into consideration the time for all and each one of the branches of the CA cycle in terms of thermodynamic properties, that $\eta_{\rm CA}$ has an upper bound in the limit when $V_3/V_1 \to \infty$. It is the purpose of the present paper to exhibit an analogous result for the ecological efficiency.

2. The Gutkowics-Krusin, Procaccia, and Ross model

To make this paper self-contained we include in this section a review of some results from Ref. 14 that we need for our present purposes. In this model, the authors consider the ideal and a van der Waals gas as a working substance in a cylinder with a piston engine for which they assume the piston has mass; this fact has no influence on the endoreversible character of the CA cycle.

The power, defined by the quotient of the total work output W and the total time $t_{\rm tot}$ is found to be

$$P \equiv \frac{W}{t_{\rm tot}} = \frac{\alpha}{\ln\dfrac{V_3}{V_1}}\,\frac{(T_1^* - T_2^*)\left(\ln\dfrac{V_3}{V_1} + \upsilon\ln\dfrac{T_2^*}{T_1^*}\right)}{\dfrac{T_1^*}{T_1 - T_1^*} + \dfrac{T_2^*}{T_2^* - T_2}}. \tag{8}$$

Here, α denotes the thermal conductivity, $\upsilon \equiv 1/(\gamma - 1)$ where γ denotes the quotient of specific heats of the working substance, $\gamma \equiv c_p/c_v$; $t_{\rm tot}$ is the cycle period and the adiabatic branches are not instantaneous. In fact,

$$t_{\rm tot} = t_1 + t_2 + t_3 + t_4 \tag{9}$$

where the times for the isothermal branches have been found to be,

$$t_1 = f_1\ln\frac{V_2}{V_1} \quad \text{and} \quad t_3 = f_2\ln\frac{V_3}{V_4}, \tag{10}$$

and the times for the adiabatic branches have been assumed to be

$$t_2 = f_1\ln\frac{V_3}{V_2} \quad \text{and} \quad t_4 = -f_2\ln\frac{V_1}{V_4}, \tag{11}$$

with

$$f_1 = \frac{R}{\alpha}\frac{T_1^*}{T_1 - T_1^*} \quad \text{and} \quad f_2 = \frac{R}{\alpha}\frac{T_2^*}{T_2^* - T_2}, \tag{12}$$

R is the general constant of gases. The heat flows, Q_i have been assumed to be given by Newton's law of heat transfer, namely,

$$\frac{dQ_i}{dt} = \alpha(T_i - T_i^*), \quad i = 1, 2. \tag{13}$$

The power output, Eq. (8) is rewritten in terms of the variables $x \equiv T_1^*/T_1$ and $z \equiv T_2^*/T_1^*$; the ensuing function $P = P(x, z)$ namely,

$$P = \frac{\alpha T_1}{\ln\frac{V_3}{V_1}}\,\frac{(1 - z)\left(\ln\dfrac{V_3}{V_1} + \upsilon\ln z\right)}{\dfrac{1}{1 - x} + \dfrac{z}{zx - \beta}} \tag{14}$$

leads through the conditions for the maximum, namely $\partial P/\partial x = 0$ and $\partial P/\partial z = 0$, to

$$x = \frac{z+\beta}{2z}$$

and to

$$[\lambda(1-z) - z(1+\lambda\ln z)](z-\beta)(zx-\beta) = z(1-x)(z-1)(1+\lambda\ln z)\beta,$$

where λ represents the external parameter,

$$\lambda \equiv \frac{v}{\ln\frac{V_3}{V_1}} \tag{15}$$

meaning that

$$P_{\max} = P_{\max}\left(x(z), z\right).$$

That is, $P_{\max}$ is a projection on the (z, P) plane. It is also found that at the maximum power condition z is given by a power series in λ:

$$z = \sqrt{\beta} + \frac{1}{2}\left(1-\sqrt{\beta}\right)^2\lambda + \frac{1}{4}\left(1-\sqrt{\beta}\right)^2\left[\frac{\left(1-\sqrt{\beta}\right)^2}{2\sqrt{\beta}} - \ln\beta\right]\lambda^2 + O(\lambda^3). \tag{16}$$

Upon substituing Eq. (16) in Eq. (4) and because the terms in the series (16) are positive, an upper bound for the efficiency η_m is obtained when $\lambda = 0$, *i.e.*, when the machine size goes to infinity, it is the following one:

$$\eta_m = 1 - z(\lambda = 0) = \eta_{\rm CA}.$$

In the next section we construct the equation analogous to (16) for the ecological function following the Gutkowics-Krusin, Procaccia, and Ross model outlined here.

3. The ecological function

In the ecological function (5), we take P from Eq. (8) and the entropy production term σ as $\sigma = \Delta S/t_{\rm tot}$ [5], where ΔS represents the entropy change caused at the isothermal branches because of the heat transfers Eq. (13):

$$\sigma = \frac{1}{t_{tot}}\left(\frac{Q_2}{T_2} - \frac{Q_1}{T_1}\right). \tag{17}$$

Here, $t_{\rm tot}$ is given by Eqs. (9)–(12). In terms of the variables (x, z, β) σ becomes

$$\sigma = \alpha\frac{T_1}{T_2}\frac{(1+\lambda\ln z)(z-\beta)}{\dfrac{1}{1-x} + \dfrac{z}{zx-\beta}} \tag{18}$$

where thanks to the condition of endoreversibility, we have used the thermostatic results $V_2/V_1 = V_3/V_4$ and

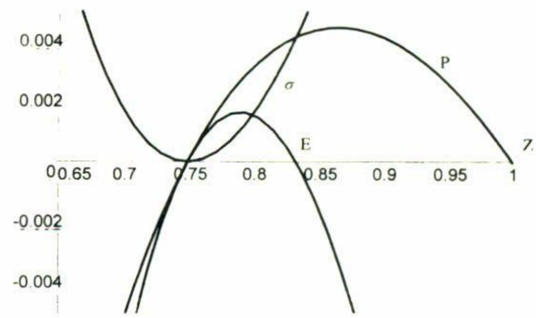

FIGURE 2. Power function $\frac{P}{\alpha T_1}(z)$, entropy production $\frac{\sigma}{\alpha T_1}(z)$ and the ecological function $\frac{E}{\alpha T_1}(z)$, see Eqs. (14), (18) and (19) respectively. Here $T_1 = 400$ K, $\beta = 0.75$ and $\lambda = 0$.

$V_2 = V_3\left(T_2^*/T_1^*\right)^v$ and where λ is given by Eq. (15). With Eqs. (18) and (8) the expression for the ecological function becomes

$$E(x,z) = \alpha T_1\frac{(1+\lambda\ln z)(1+\beta-2z)}{\dfrac{1}{1-x} + \dfrac{z}{zx-\beta}}. \tag{19}$$

Figure 2 shows the behavior of $P/\alpha T_1$, $\sigma/\alpha T_1$ and $E/\alpha T_1$ in the x constant plane, at $\lambda = 0$ and β a given constant value. It is apparent that the maximum power output is achieved with high production of entropy, it is also apparent that zero entropy production is achieved with zero power output, while the function E represents the maximum possible power output with the minimum possible entropy production.

Upon maximizing the two variables function $E = E(x, z)$ (β defined positive and λ defined semipositive, being external parameteres), we obtain for $\partial E/\partial x = 0$ and $\partial E/\partial z = 0$,

$$x = \frac{z+\beta}{2z} \tag{20}$$

and the following relation beween the variables z and x:

$$[2(1+\lambda\ln z)z - \lambda(1+\beta-2z)](z-\beta)(zx-\beta) = (1+\beta-2z)(1+\lambda\ln z)(1-x)\beta z, \tag{21}$$

respectively.

The substitution of x in Eq. (21) gives the equation that z obeys at the maximum of the ecological function, henceforth we shall denote it by the capital letter Z:

$$[2(1+\lambda\ln Z)Z - \lambda(1+\beta-2Z)](Z-\beta) = (1+\beta-2Z)(1+\lambda\ln Z)\beta. \tag{22}$$

If we suppose that $Z \equiv Z(\lambda)$ is given by a power expansion

$$Z \equiv b_0 + b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + \ldots \tag{23}$$

we find, upon taking the implicit succesive derivatives of Z with respect to λ in Eq. (22) and equating them with the

TABLE I. Comparison of various efficiencies for some plants. η_C Carnot's, Eq. (2); η_{CA} Curzon and Ahlborn's, Eq. (1); the semisum η_s, Eq. (7); the ecological efficiency, η_{gi}, Eq. (27) and the observed one η_{obs}.

Power Plant	$T_2(K)$	$T_1(K)$	η_C	η_{CA}	η_s	η_{gi}	η_{obs}
West Thurrock 1962 conventional coal fired steam plant	298	838	0.6444	0.403367	0.5240	0.5090	0.3600
Lardarello (Italy) geothermal steam plant	353	523	0.3250	0.1784	0.2517	0.2482	0.1600
Central steam power stations UK, 1936–1940	298	698	0.5731	0.3466	0.4598	0.4481	0.2800
Steam power plant USA 1956	298	923	0.6771	0.4318	0.5545	0.5379	0.4000
Combined-cycle (steam and mercury) USA plant 1949	298	753	0.6194	0.3831	0.5012	0.4874	0.3400
Doel 4 Belgium	283	56	0.5000	0.2929	0.3964	0.3876	0.3500

coefficients b_i in Eq. (23), that,

$$Z(\lambda) = \sqrt{\frac{1}{2}(\beta+\beta^2)}\left\{1 + \left[\frac{1}{4}(1+3\beta)\sqrt{\frac{2}{\beta+\beta^2}} - 1\right]\lambda + \left[\frac{\frac{1}{16}(1+3\beta)}{\beta+\beta^2} - \frac{1}{2}\sqrt{\frac{2}{\beta+\beta^2}}\ln\sqrt{\frac{1}{2}(\beta+\beta^2)}\right]\left[1+3\beta-4\sqrt{\frac{1}{2}(\beta+\beta^2)}\right]\lambda^2 + O(\lambda^3)\right\}. \quad (24)$$

Furthermore, using (3), we can write the efficiency as a power series in λ,

$$\eta_g \equiv 1 - Z(\lambda). \quad (25)$$

In the particular case when $\lambda = 0$ we find the value

$$Z(\lambda = 0) = \sqrt{\frac{1}{2}\beta(1+\beta)} \quad (26)$$

and the corresponding value for the efficiency

$$\eta_{gi} = 1 - Z(\lambda = 0) = 1 - \sqrt{\beta}\sqrt{\frac{1}{2}(1+\beta)}, \quad (27)$$

which is the maximum possible one, since all the terms in Eq. (24) are positive. From the expressions given here for the different efficiencies, Eq. (3), with $z = \sqrt{T_2/T_1}$, for η_{CA}, Eq. (7) for η_s, and Eq. (27) for the maximum ecological efficency η_{gi}, it is easy to prove that $\eta_s > \eta_{gi}$ and that $\eta_{gi} > \eta_{CA}$. Table I shows a comparison between η_C, η_{CA}, η_s and η_{gi} for different heat engines taken from Ref. 12. It is seen that $\eta_{CA} < \eta_{gi} < \eta_s$ as expected.

We must point out that the efficiencies η_E given in Eq. (6) and η_{gi} given in Eq. (27), both calculated at the maximum of the ecological function E, are the same function of β as can be seen upon substituting $\eta_C = 1 - \beta$ in Eq. (6) and taking out the common factor $\sqrt{1/2(\beta+\beta^2)}$ in the numerator of the fraction. Thus we find that in the same way as it happens with the efficiency at maximum power [14], the efficiency at maximum ecological function, calculated with the assumption of instantaneous adiabats corresponds to the efficiency at order $\lambda = 0$, *i.e.*, corresponds to an infinite machine size.

Notice that the previous calculations for an ideal gas, can be made for a different equation of state; in particular, for a van der Waals gas, the equations replacing (8), (18) and (19) have the same form with V replaced by $V - b$ and with the parameter λ Eq. (15), replaced by $\lambda_{vw} \equiv v/\ln[(V_3-b)/(V_1-b)]$, here b is the van der Waals constant that corrects the volume and depends on the substance. The relevant equations for the van der Waals gas are the following:

$$P \equiv \frac{W}{t_{tot}} = \frac{\alpha}{\ln\dfrac{V_3-b}{V_1-b}} \frac{(T_1^* - T_2^*)\left(\ln\dfrac{V_3-b}{V_1-b} + v\ln\dfrac{T_2^*}{T_1^*}\right)}{\dfrac{T_1^*}{T_1-T_1^*} + \dfrac{T_2^*}{T_2^*-T_2}},$$

$$\sigma = \alpha\frac{T_1}{T_2}\frac{(1+\lambda_{vW}\ln z)(z-\beta)}{\dfrac{1}{1-x} + \dfrac{z}{zx-\beta}},$$

$$E(x,z) = \alpha T_1 \frac{(1+\lambda_{vW}\ln z)(1+\beta-2z)}{\dfrac{1}{1-x} + \dfrac{z}{zx-\beta}}.$$

The time associated to the adiabatic branches is a characteristic that depends on the particulars of the engine, thus if we generalize the time of the cycle by taking $t_{tot} = t_1 + kt_2 + t_3 + kt_4$, with k a real number, we allow the time of the adiabatic branches to be varied as we vary the value of k; with this modification we find the same results (21) and (22) with $\lambda_k \equiv (1-k)\lambda$ instead of λ and Eq. (24) modified in the following manner:

$$Z_k(\lambda) = \sqrt{\tfrac{1}{2}(\beta+\beta^2)}\left[1 + k(b_1\lambda + b_2\lambda^2 + ...)\right]. \quad (28)$$

4. Conclusions

We have shown that the ecological efficiency $\eta_g(\lambda)$ given in Eq. (25) with Eq. (24) or Eq. (28), which is obtained by considering a general finite time dependency of the adiabats for ideal or van der Waals gases in a Curzon and Ahlborn machine can be written as a power series in the parameter $\lambda \equiv v/(\ln V_3 - \ln V_1)$, Eq. (15). This parameter represents the maximum distance in volume that the cycle spans. We have found that and upper bound for the ecological efficiency occurs when $\lambda = 0$, *i.e.*, when $V_3/V_1 \rightarrow \infty$, or in words when the machine size is infinite. The maximum value attainable by the ecological efficiency is the one found when one assumes instantaneous adiabats. The ecological efficiency then satisfies the same mathematical properties that the Curzon and Ahlborn efficiency η_{CA}, satisfies [14].

Although the η_{CA} compares better with observed data, see Table I, the ecological efficiency could be considered as a goal towards which new and future plants should operate, since it takes into account a compromise between power output and degradation of the environment.

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