The MIC-Kepler problem with positive energy and the 4D harmonic oscillator

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It is shown that the extended Kustaanheimo-Stiefel transformation, which relates the four-dimensional isotropic harmonic oscillator to the Kepler problem with a magnetic monopole field and a centrifugal potential (MIC-Kepler problem), can be derived in an elementary way noting that the Hamilton-Jacobi, or the Schrödinger, equation for the oscillator in terms of polar coordinates is essentially that for the MIC-Kepler problem in parabolic coordinates. Making use of this fact, the solution of the MIC-Kepler problem with positive energy is obtained. It is also shown that the MIC-Kepler problem is obtained from the four-dimensional harmonic oscillator in classical mechanics by a dimensional reduction analogous to the one employed in the Kaluza-Klein theory.

Keywords: MIC-Kepler problem; Kustaanheimo-Stiefel transformation; magnetic monopole; dimensional reduction

Se muestra que la transformación de Kustaanheimo-Stiefel extendida, la cual relaciona el oscilador armónico isótropo en cuatro dimensiones y el problema de Kepler con el campo de un monopolo magnético y un potencial centrífugo (problema de MIC-Kepler), puede derivarse en una forma elemental notando que la ecuación de Hamilton-Jacobi, o de Schrödinger, para el oscilador en términos de coordenadas polares es esencialmente la del problema de MIC-Kepler en coordenadas parabólicas. Usando este hecho, se obtiene la solución del problema de MIC-Kepler con energía positiva. Se muestra también que el problema de MIC-Kepler se obtiene del oscilador armónico en cuatro dimensiones por una reducción dimensional análoga a la empleada en la teoría de Kaluza-Klein.

Descriptores: Problema de MIC-Kepler; transformación de Kustaanheimo-Stiefel; monopolo magnético; reducción dimensional

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1. Introduction

The Kepler problem in $n+1$ dimensions can be related to an isotropic harmonic oscillator in $2n$ dimensions, for $n = 1, 2, 4$ and 8 (see, *e.g.*, Ref. 1 and the references cited therein). In the most relevant case, corresponding to the Kepler problem in three dimensions, the transformation employed to relate this problem with a four-dimensional (4D) isotropic harmonic oscillator (the Kustaanheimo-Stiefel transformation [2]) can be extended in such a way that, in the same manner, the 4D harmonic oscillator is related to the so-called MIC-Kepler problem where, in addition to the $1/r$ potential there is a magnetic monopole field and a centrifugal potential (see Ref. 3 and the references cited therein). As shown in Ref. 4, if the magnetic charge and the strength of the centrifugal potential are suitably related, the orbits are plane, but the center of force is not contained in the plane of the orbit.

Since the orbits of the harmonic oscillator are bounded, the Kustaanheimo-Stiefel (KS) transformation only reproduces the bounded motion of the MIC-Kepler problem, *i.e.*, the motion with negative energy; however, if the 4D harmonic oscillator has an imaginary frequency, it can be related to the MIC-Kepler problem with positive energy [5]. (The MIC-Kepler problem with zero energy is related in this way with the 4D harmonic oscillator of zero frequency, *i.e.*, a free particle in four dimensions [3].) Moreover, since the energy of a "harmonic oscillator" with imaginary frequency can be any real number, this system can be related to the MIC-Kepler where the $1/r$ potential is attractive, repulsive or absent. (By contrast, if the frequency is real, the energy of the oscillator cannot be negative and only an attractive $1/r$ potential can be reproduced in this way.)

In this paper we show that the extended KS transformation mentioned above can be derived in an elementary way using polar coordinates in the Hamilton-Jacobi (or the Schrödinger) equation for the 4D harmonic oscillator, which are identified with parabolic coordinates in the three-dimensional space. Making use of this transformation we obtain the solution of the MIC-Kepler problem in the case where the energy is positive, which was not considered in Ref. 3. We also show that the MIC-Kepler problem can be obtained from the 4D harmonic oscillator in classical mechanics by means of a dimensional reduction analogous to that employed in the Kaluza-Klein theory.

2. Derivation of the KS transformation in classical mechanics

We start by considering the Hamilton-Jacobi (HJ) equation (for the Hamilton characteristic function) corresponding to the 4D "harmonic oscillator" with imaginary frequency in cartesian coordinates

$$\frac{1}{2M}\left[\left(\frac{\partial W}{\partial u_1}\right)^2+\left(\frac{\partial W}{\partial u_2}\right)^2+\left(\frac{\partial W}{\partial u_3}\right)^2+\left(\frac{\partial W}{\partial u_4}\right)^2\right] - \frac{M\omega^2}{2}(u_1^2+u_2^2+u_3^2+u_4^2) = E, \quad (1)$$

with ω and E real. (By abuse of language, we call this system "oscillator", even though there is no oscillatory motion.) Replacing the pairs of cartesian coordinates (u_1, u_2) and (u_3, u_4) by the corresponding polar coordinates (ρ_1, θ_1) and (ρ_2, θ_2), respectively (*e.g.*, $u_1 = \rho_1 \cos\theta_1$, $u_2 = \rho_1 \sin\theta_1$), from Eq. (1) we have

$$\frac{1}{2M}\left[\left(\frac{\partial W}{\partial \rho_1}\right)^2+\left(\frac{\partial W}{\partial \rho_2}\right)^2+\frac{1}{\rho_1^2}\left(\frac{\partial W}{\partial \theta_1}\right)^2+\frac{1}{\rho_2^2}\left(\frac{\partial W}{\partial \theta_2}\right)^2\right] - \frac{M\omega^2}{2}(\rho_1^2+\rho_2^2) = E. \quad (2)$$

As will be shown below, it is convenient to make use of the new variables

$$w = \theta_1, \quad \phi = \theta_2 - \theta_1. \quad (3)$$

Then, dividing Eq. (2) by $4(\rho_1^2+\rho_2^2)$ one finds that

$$\frac{1}{2M}\left\{\frac{1}{4(\rho_1^2+\rho_2^2)}\left[\left(\frac{\partial W}{\partial \rho_1}\right)^2+\left(\frac{\partial W}{\partial \rho_2}\right)^2\right] + \frac{1}{4\rho_1^2\rho_2^2}\left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{1}{4\rho_1^2(\rho_1^2+\rho_2^2)}\left[\left(\frac{\partial W}{\partial w}\right)^2 - 2\frac{\partial W}{\partial \phi}\frac{\partial W}{\partial w}\right]\right\} - \frac{E}{4(\rho_1^2+\rho_2^2)} = \frac{M\omega^2}{8}. \quad (4)$$

Since w is an ignorable coordinate, we look for solutions of Eq. (4) of the form

$$W(\rho_1,\rho_2,\phi,w) = \widetilde{W}(\rho_1,\rho_2,\phi) + Kw, \quad (5)$$

where K is some constant. Substituting Eq. (5) into Eq. (4) one obtains

$$\frac{1}{2M}\left\{\frac{1}{4(\rho_1^2+\rho_2^2)}\left[\left(\frac{\partial \widetilde{W}}{\partial \rho_1}\right)^2+\left(\frac{\partial \widetilde{W}}{\partial \rho_2}\right)^2\right] + \frac{1}{4\rho_1^2\rho_2^2}\left(\frac{\partial \widetilde{W}}{\partial \phi} - \frac{K\rho_2^2}{\rho_1^2+\rho_2^2}\right)^2 + \frac{K^2}{4(\rho_1^2+\rho_2^2)^2}\right\} - \frac{E}{4(\rho_1^2+\rho_2^2)} = \frac{M\omega^2}{8}. \quad (6)$$

When the separation constant K is equal to zero, Eq. (6) reduces to

$$\frac{1}{2M}(\nabla\widetilde{W})^2 - \frac{E}{4r} = \frac{M\omega^2}{8}, \quad (7)$$

provided that (ρ_1, ρ_2, ϕ) are *parabolic coordinates* in three-dimensional space, related to the cartesian coordinates (x, y, z) by means of

$$x = 2\rho_1\rho_2\cos\phi, \quad y = 2\rho_1\rho_2\sin\phi, \quad z = \rho_1^2 - \rho_2^2. \quad (8)$$

In effect, from Eqs. (8) it follows that

$$(dx)^2 + (dy)^2 + (dz)^2 = 4(\rho_1^2+\rho_2^2)[(d\rho_1)^2 + (d\rho_2)^2] + 4\rho_1^2\rho_2^2(d\phi)^2 \quad (9)$$

and

$$r^2 \equiv x^2 + y^2 + z^2 = (\rho_1^2+\rho_2^2)^2. \quad (10)$$

Equation (7) is the HJ equation for a particle with positive energy

$$\epsilon = \frac{M\omega^2}{8} \quad (11)$$

with the potential $-E/(4r)$, which may be attractive (if $E > 0$) or repulsive (if $E < 0$). If $E = 0$, the particle is free.

Similarly, when K is different from zero, Eq. (6) can be written as

$$\frac{1}{2M}\left(\nabla\widetilde{W} - \frac{q}{c}\mathbf{A}\right)^2 + \frac{K^2}{8Mr^2} - \frac{E}{4r} = \frac{M\omega^2}{8}, \quad (12)$$

which is the HJ equation for a particle with electric charge q in the magnetic field corresponding to the vector potential $\mathbf{A}$ given by

$$\frac{q}{c}\mathbf{A}\cdot d\mathbf{r} = \frac{K\rho_2^2}{\rho_1^2+\rho_2^2}d\phi = \frac{K(x dy - y dx)}{2r(r+z)}, \quad (13)$$

and the central potential

$$V(r) = -\frac{E}{4r} + \frac{K^2}{8Mr^2}. \quad (14)$$

The magnetic field generated by the vector potential (13) is that of a magnetic monopole of charge

$$g = \frac{Kc}{2q}, \quad (15)$$

placed at the origin. Equation (12) is the HJ equation for the MIC-Kepler problem with positive energy.

From Eqs. (3) and (8) one obtains

$$x = 2\rho_1\rho_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = 2(u_1u_3 + u_2u_4), \quad (16)$$

$$y = 2\rho_1\rho_2(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2) = 2(u_1u_4 - u_2u_3), \quad (17)$$

$$z = \rho_1^2 - \rho_2^2 = u_1^2 + u_2^2 - u_3^2 - u_4^2, \quad (18)$$

$$w = \arctan(u_2/u_1), \quad (19)$$

which are the transformation formulas employed in Ref. 3; Eqs. (16)-(18) correspond to the original KS transformation [2].

As in Ref. 3, the orbits of the MIC-Kepler problem can be obtained from those of the 4D harmonic oscillator. By suitably choosing the coordinate axes, we can assume that the motion of the 4D harmonic oscillator is given by $u_1 = a_1 \cosh \omega t$, $u_2 = a_2 \sinh \omega t$, $u_3 = \mu a_2 \sinh \omega t$, $u_4 = \mu a_1 \cosh \omega t$, where a_1, a_2 and μ are arbitrary constants. Then,

$$E = \frac{M\omega^2}{2}(1+\mu^2)(a_2^2 - a_1^2) \tag{20}$$

and from Eq. (5) it follows that $K = \partial W/\partial w = p_w$ (the momentum conjugate to w); hence, using Eqs. (3),

$$\begin{aligned} K = p_w &= p_1 \frac{\partial u_1}{\partial w} + p_2 \frac{\partial u_2}{\partial w} + p_3 \frac{\partial u_3}{\partial w} + p_4 \frac{\partial u_4}{\partial w} \\ &= u_1 p_2 - u_2 p_1 + u_3 p_4 - u_4 p_3 \\ &= M\omega(1-\mu^2) a_1 a_2. \end{aligned} \tag{21}$$

Therefore, $|\mu| = 1$ corresponds to $K = 0$ (in which case the monopole field and the centrifugal potential are absent). Making use now of Eqs. (16)-(18) one obtains

$$\begin{aligned} x &= 2\mu a_1 a_2 \sinh(2\omega t), \\ y &= \mu[(a_1^2 - a_2^2)\cosh(2\omega t) + a_1^2 + a_2^2], \\ z &= \frac{1}{2}(1-\mu^2)[(a_1^2 + a_2^2)\cosh(2\omega t) + a_1^2 - a_2^2], \end{aligned} \tag{22}$$

which, if $|\mu| \neq 1$, are parametric equations of the hiperbola given by the intersection of the cone

$$x^2 + y^2 = \left(\frac{2\mu}{1-\mu^2}\right)^2 z^2 \tag{23}$$

and the plane

$$(1-\mu^2)(a_1^2 + a_2^2)y - 2\mu(a_1^2 - a_2^2)z - 4\mu(1-\mu^2)a_1^2 a_2^2 = 0. \tag{24}$$

(Note that while t represents the time for the harmonic oscillator, it does not correspond to the time of the MIC-Kepler problem which, however, can be related to t [3].)

Equation (6) and, hence, Eq. (2), can be solved by separation of variables. Assuming that $\widetilde{W}(\rho_1, \rho_2, \phi) = F(\rho_1) + G(\rho_2) + L_z \phi$, where L_z is some constant, one finds that the functions $F(\rho_1)$ and $G(\rho_2)$ must satisfy the equations

$$\left(\frac{dF}{d\rho_1}\right)^2 + \frac{(K-L_z)^2}{\rho_1^2} - ME - M^2\omega^2\rho_1^2 = \lambda, \tag{25}$$

$$\left(\frac{dG}{d\rho_2}\right)^2 + \frac{L_z^2}{\rho_2^2} - ME - M^2\omega^2\rho_2^2 = -\lambda, \tag{26}$$

where λ is another constant. Eliminating the terms proportional to ω^2 (which is related to the energy of the MIC-Kepler problem [Eq. (11)]) from these equations one finds that

$$\begin{aligned} \lambda = \frac{1}{\rho_1^2 + \rho_2^2}\Big[\rho_2^2\left(\frac{dF}{d\rho_1}\right)^2 - \rho_1^2\left(\frac{dG}{d\rho_2}\right)^2 \\ + \frac{\rho_2^2}{\rho_1^2}(K-L_z)^2 - \frac{\rho_1^2}{\rho_2^2}L_z^2 + ME(\rho_1^2 - \rho_2^2)\Big]. \end{aligned} \tag{27}$$

On the other hand, using Eqs. (8), it follows that

$$\begin{aligned} \frac{dF}{d\rho_1} = \frac{\partial W}{\partial \rho_1} &= \frac{\partial W}{\partial x}\frac{\partial x}{\partial \rho_1} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial \rho_1} + \frac{\partial W}{\partial z}\frac{\partial z}{\partial \rho_1} \\ &= p_x \frac{x}{\rho_1} + p_y \frac{y}{\rho_1} + p_z 2\rho_1 \end{aligned}$$

and, similarly,

$$\frac{dG}{d\rho_2} = \frac{\partial W}{\partial \rho_2} = p_x \frac{x}{\rho_2} + p_y \frac{y}{\rho_2} - p_z 2\rho_2,$$

thus the expression (27) for the constant of motion λ amounts to

$$\begin{aligned} \lambda = -4z(p_x^2 + p_y^2) + 4(xp_x + yp_y)p_z \\ + ME\frac{z}{r} + (K^2 - 2KL_z)\frac{r-z}{r(r+z)}, \end{aligned}$$

which is, apart from a factor (-4), the z-component of the generalization of the Runge-Lenz vector given in Ref. 3. Hence, the conservation of this vector is related to the separability of the MIC-Kepler problem in parabolic coordinates.

3. The MIC-Kepler problem via dimensional reduction à la Kaluza-Klein

According to Jacobi's principle, the orbits in configuration space of a particle with Hamiltonian

$$H = \frac{1}{2M} g^{\alpha\beta} p_\alpha p_\beta + V(q^\alpha) \tag{28}$$

are the geodesics of the metric

$$(E - V) g_{\alpha\beta}\, dq^\alpha dq^\beta, \tag{29}$$

where $(g_{\alpha\beta})$ is the inverse of $(g^{\alpha\beta})$, E is the value of H corresponding to the initial conditions and there is sum over repeated indices. Thus, the orbits in configuration space of the 4D harmonic oscillator with imaginary frequency are the geodesics of the metric

$$\left[E + \frac{M\omega^2}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2)\right](du_1^2 + du_2^2 + du_3^2 + du_4^2)$$

which, expressed in the coordinates (x, y, z, w) [see Eqs. (16)-(19)], takes the form

$$\left(\frac{E}{4r} + \frac{M\omega^2}{8}\right)\left\{dx^2 + dy^2 + dz^2 + 4r^2\left[dw + \frac{x\,dy - y\,dx}{2r(r+z)}\right]^2\right\}, \tag{30}$$

where $r^2 \equiv x^2 + y^2 + z^2$.

The metric (30) is of the form

$$f\left[\gamma_{ij}dx^i dx^j + \beta(dx^0 + \alpha_i dx^i)^2\right] \equiv g_{\alpha\beta}dx^\alpha dx^\beta, \quad (31)$$

where the functions f, β, α_i and γ_{ij} depend on x^1, x^2, x^3 only, $i, j, \ldots = 1, 2, 3$; $\alpha, \beta, \ldots = 0, 1, 2, 3$ and there is sum over repeated indices. Hence, Eq. (31) implies that

$$g_{00} = f\beta, \quad g_{0i} = f\beta\alpha_i, \quad g_{ij} = f(\gamma_{ij} + \beta\alpha_i\alpha_j) \quad (32)$$

and the inverse of $(g_{\alpha\beta})$ is given by

$$g^{00} = \frac{1}{f\beta} + \frac{\gamma^{ij}\alpha_i\alpha_j}{f}, \quad g^{0i} = -\frac{\gamma^{ij}\alpha_j}{f}, \quad g^{ij} = \frac{\gamma^{ij}}{f}, \quad (33)$$

where (γ^{ij}) denotes the inverse of (γ_{ij}). Using again Jacobi's principle, the geodesics of the metric (31) are the orbits of the Hamiltonian

$$H' = \frac{E'}{2M} g^{\alpha\beta} p_\alpha p_\beta$$

[*cf.* Eqs. (28) and (29)], where E' is the value of the Hamiltonian H' given by the initial conditions. The orbits determined by the Hamiltonian H' can be obtained by means of the HJ equation

$$\frac{E'}{2M} g^{\alpha\beta} \frac{\partial W}{\partial x^\alpha}\frac{\partial W}{\partial x^\beta} = E';$$

therefore, making use of Eqs. (33), this equation becomes

$$\frac{1}{2M}\gamma^{ij}\left(\frac{\partial W}{\partial x^i} - \alpha_i\frac{\partial W}{\partial x^0}\right)\left(\frac{\partial W}{\partial x^j} - \alpha_j\frac{\partial W}{\partial x^0}\right) - f + \frac{1}{2M\beta}\left(\frac{\partial W}{\partial x^0}\right)^2 = 0. \quad (34)$$

Looking for a solution of the form $W(x^0, x^i) = Kx^0 + \widetilde{W}(x^i)$, where K is some constant [*cf.* Eq. (5)], from Eq. (34) one finds that $\widetilde{W}$ must satisfy

$$\frac{1}{2M}\gamma^{ij}\left(\frac{\partial \widetilde{W}}{\partial x^i} - K\alpha_i\right)\left(\frac{\partial \widetilde{W}}{\partial x^j} - K\alpha_j\right) - f + \frac{K^2}{2M\beta} = 0, \quad (35)$$

which is the HJ equation corresponding to the Hamiltonian

$$\frac{1}{2M}\gamma^{ij}(p_i - K\alpha_i)(p_j - K\alpha_j) - f + \frac{K^2}{2M\beta} = 0. \quad (36)$$

This Hamiltonian can be regarded as that of a particle with electric charge q in the magnetic field generated by the vector potential $\mathbf{A}$ defined by

$$\frac{q}{c}\mathbf{A}\cdot d\mathbf{r} = K\alpha_i dx^i \quad (37)$$

and the potential

$$V = -f + \frac{K^2}{2M\beta}. \quad (38)$$

Thus, as in the Kaluza-Klein theory, where the geodesics of a five-dimensional space reproduce the motion of a particle subject to an electromagnetic field in the four-dimensional space-time, the geodesic equations of the metric (31) are equivalent to the equations of motion of a charged particle in a magnetic field and a velocity-independent potential (see also Ref. 6 and the references cited therein).

A comparison of Eqs. (30) and (31) shows that in the case of the 4D harmonic oscillator with imaginary frequency, $f = E/(4r) + M\omega^2/8$, $\beta = 4r^2$, $\alpha_i dx^i = (xdy - ydx)/[2r(r+z)]$ and $\gamma_{ij} = \delta_{ij}$; by substituting these expressions into Eq. (35) one obtains Eq. (12).

4. Derivation of the KS transformation in quantum mechanics

The time-independent Schrödinger equation for the 4D "harmonic oscillator" with imaginary frequency in cartesian coordinates is

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}\right)\Psi - \frac{M\omega^2}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2)\Psi = E\Psi. \quad (39)$$

Introducing again polar coordinates (ρ_1, θ_1) and (ρ_2, θ_2) in the u_1u_2 and u_3u_4 planes, we obtain

$$-\frac{\hbar^2}{2M}\left(\frac{1}{\rho_1}\frac{\partial}{\partial\rho_1}\rho_1\frac{\partial}{\partial\rho_1} + \frac{1}{\rho_2}\frac{\partial}{\partial\rho_2}\rho_2\frac{\partial}{\partial\rho_2} + \frac{1}{\rho_1^2}\frac{\partial^2}{\partial\theta_1^2} + \frac{1}{\rho_2^2}\frac{\partial^2}{\partial\theta_2^2}\right)\Psi - \frac{M\omega^2}{2}(\rho_1^2 + \rho_2^2)\Psi = E\Psi \quad (40)$$

or, using the variables w and ϕ [Eq. (3)] in place of θ_1 and θ_2,

$$-\frac{\hbar^2}{2M}\left[\frac{1}{4\rho_1(\rho_1^2+\rho_2^2)}\frac{\partial}{\partial\rho_1}\rho_1\frac{\partial}{\rho_1} + \frac{1}{4\rho_2(\rho_1^2+\rho_2^2)}\frac{\partial}{\partial\rho_2}\rho_2\frac{\partial}{\partial\rho_2} + \frac{1}{4\rho_1^2\rho_2^2}\frac{\partial^2}{\partial\phi^2} - \frac{1}{2\rho_1^2(\rho_1^2+\rho_2^2)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial w} + \frac{1}{4\rho_1^2(\rho_1^2+\rho_2^2)}\frac{\partial^2}{\partial w^2}\right]\Psi - \frac{E}{4(\rho_1^2+\rho_2^2)}\Psi = \frac{M\omega^2}{8}\Psi. \quad (41)$$

Equation (41) admits separable solutions of the form

$$\Psi(\rho_1, \rho_2, \phi, w) = \psi(\rho_1, \rho_2, \phi)e^{iKw/\hbar}, \quad (42)$$

where K is a constant [*cf.* Eq. (5)] and ψ is a solution of

$$-\frac{\hbar^2}{2M}\left[\frac{1}{4\rho_1(\rho_1^2+\rho_2^2)}\frac{\partial}{\partial\rho_1}\rho_1\frac{\partial}{\partial\rho_1} + \frac{1}{4\rho_2(\rho_1^2+\rho_2^2)}\frac{\partial}{\partial\rho_2}\rho_2\frac{\partial}{\rho_2} + \frac{1}{4\rho_1^2\rho_2^2}\left(\frac{\partial}{\partial\phi} - \frac{iK}{\hbar}\frac{\rho_2^2}{\rho_1^2+\rho_2^2}\right)^2 - \frac{K^2}{4\hbar^2(\rho_1^2+\rho_2^2)^2}\right]\psi - \frac{E}{4(\rho_1^2+\rho_2^2)}\psi = \frac{M\omega^2}{8}\psi, \quad (43)$$

which, by virtue of Eq. (9), is the time-independent Schrödinger equation for the MIC-Kepler problem with positive energy in parabolic coordinates.

Equation (40) admits separable solutions of the form $\Psi(\rho_1, \theta_1, \rho_2, \theta_2) = F(\rho_1) e^{im_1\theta_1} G(\rho_2) e^{im_2\theta_2}$, where m_1 and m_2 are integers,

$$\frac{1}{\rho_1}\frac{d}{d\rho_1}\rho_1\frac{dF}{d\rho_1} + \left(\frac{M^2\omega^2}{\hbar^2}\rho_1^2 - \frac{m_1^2}{\rho_1^2} + \frac{ME}{\hbar^2} + \lambda\right) F = 0,$$

$$\frac{1}{\rho_2}\frac{d}{d\rho_2}\rho_2\frac{dG}{d\rho_2} + \left(\frac{M^2\omega^2}{\hbar^2}\rho_2^2 - \frac{m_2^2}{\rho_2^2} + \frac{ME}{\hbar^2} - \lambda\right) G = 0 \quad (44)$$

and λ is another separation constant (which is, apart from a constant factor, the eigenvalue of the z-component of the quantum-mechanical version of the generalized Runge-Lenz vector). It can be verified that the solutions of Eqs. (44) can be expressed as

$$F(\rho_1) = \rho_1^{|m_1|}\left[A_1 e^{iM\omega\rho_1^2/(2\hbar)} L^{|m_1|}_{(ib_1-2|m_1|-2)/4}\left(-\frac{iM\omega\rho_1^2}{\hbar}\right) + B_1 e^{-iM\omega\rho_1^2/(2\hbar)} L^{|m_1|}_{(-ib_1-2|m_1|-2)/4}\left(\frac{iM\omega\rho_1^2}{\hbar}\right)\right], \quad (45)$$

where $b_1 = (ME + \lambda\hbar^2)/(M\hbar\omega)$, L_n^k are associated Laguerre functions and A_1, B_1 are arbitrary constants. Similarly,

$$G(\rho_2) = \rho_2^{|m_2|}\left[A_2 e^{iM\omega\rho_2^2/(2\hbar)} L^{|m_2|}_{(ib_2-2|m_2|-2)/4}\left(-\frac{iM\omega\rho_2^2}{\hbar}\right) + B_2 e^{-iM\omega\rho_2^2/(2\hbar)} L^{|m_2|}_{(-ib_2-2|m_1|-2)/4}\left(\frac{iM\omega\rho_2^2}{\hbar}\right)\right], \quad (46)$$

where $b_2 = (ME - \lambda\hbar^2)/(M\hbar\omega)$, and A_2, B_2 are arbitrary constants.

On the other hand, from Eqs. (3) we have

$$e^{i(m_1\theta_1 + m_2\theta_2)} = e^{i(m_1+m_2)w} e^{im_2\phi};$$

hence, from Eq. (42) it follows that

$$K = m_1 + m_2 \quad (47)$$

and that

$$\psi(\rho_1, \rho_2, \phi) = F(\rho_1) G(\rho_2) e^{im_2\phi} \quad (48)$$

is a solution of Eq. (43).

5. Concluding remarks

Following the procedure that leads from Eq. (1) to Eq. (4) (or from Eq. (39) to Eq. (41) in the quantum-mechanical case), by means of an arbitrary coordinate transformation one can transform any given problem into another (locally) equivalent problem, in a possibly curved space; however, only some special transformations relate two problems interesting in their own. (Note that the extended KS transformation is two-to-one [see Eqs. (16)-(19)].)

In classical mechanics, the Jacobi principle allows us to consider the motion of a conservative system with a velocity-independent potential as a free motion by modifying the metric of the configuration space. The existence of a continuous symmetry of this metric then enables us to make a dimensional reduction that amounts to the presence of a magnetic interaction. However, in quantum mechanics, an obvious analog of Jacobi's principle only holds in a two-dimensional space.

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