Calculation of the QCD chiral coefficients from VMD models

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The Chiral conserving E^4-Chiral coefficients L_1, L_2 and L_3 are calculated for three different models of Vector Meson Dominance (VMD). It is found that the result is the same for all three, in agreement with general considerations shown the equivalence between different VMD models.

Keywords: QCD; Chiral Lagrangians; VMD

Se calculan los Coeficientes Quirales L_1, L_2 y L_3 para tres modelos diferentes de Dominancia Vectorial (VMD). Se encuentra el mismo resultado para los tres, en concordancia con cosideraciones generales que muestran la equivalencia entre diferentes modelos de Dominancia Vectorial.

Descriptores: QCD; lagrangianos Quirales; dominancia vectorial

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1. Introduction

Chiral Lagrangians [1–3] are a rigorous way to obtain QCD predictions in the low energy limit($E \ll 1$ GeV) limit up undeterminated constants, the Chiral Coefficients. At the moment it is unfortunately not possible to obtain those constants directly from QCD even in Lattice Calculations [4] (as most of the hadronic phenomenology) so they have to be determinated experimentally [2]. At higher energies ($E \sim 1$ GeV) there is not a rigorous way to obtain theoretical QCD predictions and we have to use phenomenological models like Vector Meson Dominance (VMD) [6–9], Constituent Quark Models (QM) [10], Extended Nambu Jona-Lasinio Models (ENJL) [11], Non-local Lagrangians (NLL) [12], Nonrelativistic Quark Models [13], Sum Rules [14] etc. Naturally these models can be extrapolate to the low energy limit to predict values for the Chiral Coefficients [15] so their predictions can be compared with the experimental values. It is also possible to extrapolate Chiral Lagrangians predictions, with a given unitarization scheme to the high energy region to obtain the resonances [$\rho(770)$, $f_0(980)$ and so on] [16]. Thus the Chiral Coefficients and the resonance parameters are deeply related: they are two different ways to describe the same low energy physics. Of particular interest are VMD models, or more generally resonance models because they permit to parametrize all the experimental results as long as we take enough parameters. It is found, too that the Chiral coefficients are saturated by few resonances, being the most important the Vector [$\rho(770)$], Scalar [$f_0(980)$] and Tensor [$f_2(1270)$] ones [15] (see Table I for a list of the known SU(3) multiplets: Pseudoscalar octet and resonance ones). The contributions of the last two tend to cancel between themselves (see Table II) so the Vector Meson Resonances are the most important. Several VMD [7, 8, 2] models have been proposed and it has been shown [9] that there is a more general description and that the different versions are only particular cases and sometimes they are equivalent. The purpose of this note is to compute the E^4 Chiral Symmetry conserving coefficients for the three different formulations of VMD and show that in spite the very different appearance the results are the same for the there models.

The Chiral Lagrangian for SU(3) (Chiral Symmetry conserving terms without external currents) is [2, 3]

$$\mathcal{L} = \frac{F_\pi^2}{4}\left[\mathrm{Tr}\left(X^\mu X_\mu^\dagger\right) + \mathrm{Tr}\left(\chi^\dagger U + U^\dagger \chi\right)\right] + L_1[\mathrm{Tr} X_\mu X^\mu]^2 + L_2[\mathrm{Tr}(X_\mu X_\nu)]^2 + L_3 \mathrm{Tr}\left[(X_\mu X^\mu)^2\right], \quad (1)$$

with (see Appendix for definitions and transformations) $\chi \equiv 2Bm_q \simeq \mathrm{diag.}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$, $U = \exp(i\pi/F_\pi)$, $F_\pi = 94$ MeV, $X_\mu = \partial_\mu U U^\dagger$ and

$$\pi = \pi^i \lambda^i = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\overline{K}^0 & -\frac{2}{\sqrt{3}}\eta_8 \end{pmatrix}. \quad (2)$$

2. Vector meson dominance (hidden symmetry version)

One of the most popular VMD version is the Hidden-Symmetry (VMD-HS) one [7]. Its Lagrangian is given by

$$\mathcal{L} = \frac{F_\pi^2}{4}\left[\mathrm{Tr}\,|D_\mu U|^2 + a\mathrm{Tr}\,|2(\Gamma_\mu - iV_\mu)|^2\right] - \frac{1}{2g^2}\mathrm{Tr}\left[F_{\mu\nu}(V)F^{\mu\nu}(V)\right], \quad (3)$$

TABLE I. Known Resonance Octets and the Psudoscalar one (Chiral Symmetry Goldstone Bosons). There are several resonances without a clear SU(3) partners: $f_0(400\text{–}1200)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$, $f_0(2020)$, $f_0(2060)$, $f_0(2200)$ for the scalar sector; and $f_2(1270)$, $f_2(1430)$, $f_2'(1525)$, $f_2(1565)$, $f_2(1640)$, $f_2(1710)$, $f_2(1810)$, $f_2(1950)$, $f_2(2010)$, $f_2(2150)$, $f_J(2020)$, $f_2(2300)$ and $f_2(2340)$ for the tensors.

Octet	$I=1$	$I=1/2$	$I=0$	NRQM qua. numb.
Scalar	$a_0(980)$	$K_0^*(1430)$	$f_0(980)$, $f_0(1300)$?	1^3P_0
Scalar*	$a_0(1450)$	$K_0^*(1950)$	$f_0(1500)$?, $f_0(2020)$?	2^3P_0
Pseudoscalar	π	K	η, η'	1^1S_0
Pseudoscalar*	$\pi(1300)$	$K(1460)$	$\eta(1295)$, $\eta(1760)$	2^3S_0
Pseudoscalar**	$\pi(1770)$	$K(1830)$	$\eta(1400-1470)$, $\eta(2225)$	3^3S_0
vector	$\rho(770)$	$K^*(892)$	ω, ϕ	1^3S_1
vector*	$\rho(1450)$	$K^*(1410)$	$\omega(1420)$, $\phi(1680)$	2^3S_1
axial vector	$a_1(1260)$	$K_{1_A}(K_1(1270), K_1(1400))$	$f_1(1285)$, $f_1(1510)$?	1^3P_1
axial vector*	?	$K_1(1650)$	?, ?	2^3P_1
Tensor	$a_2(1320)$	$K_2^*(1430)$	$f_2(1270)$, $f_2'(1430)$?	1^3P_2
Tensor*	?	$K_2(1980)$	$f_2(1525)$,	2^3P_2

where $V_\mu^i = g\rho_\mu^i$ and $m_\rho^2 = ag^2F_\pi^2$. With $a=2$ and $g=g_\rho$ (as it will be shown to be the case) we have the KSRF relation $m_\rho^2 = 2g_\rho^2F_\pi^2$ [7, 17]. An additional term $(F_V \mathrm{Tr}\,[F(V)_{\mu\nu}f_-^{\mu\nu}]$) can be added but up to irrelevant no-mixing terms can be absorbed in Eq. (3), using the equation of motion for V_μ. It can be shown that the integral over the vector fields can be done at three level by shifting the fields as: $V_\mu \to V_\mu - i\Gamma_\mu + \frac{1}{2}\xi^\dagger L_\mu\xi + \frac{1}{2}\xi R_\mu \xi^\dagger$, so

$$\mathcal{L}^{\text{eff.}} \simeq \frac{F_\pi^2}{4}\left[\mathrm{Tr}\,|D_\mu U|^2 + a\mathrm{Tr}\,|-2iV_\mu|^2\right] - \frac{1}{2g_\rho^2}\mathrm{Tr}\,[F_{\mu\nu}(-i\Gamma_\mu)]^2 + \mathcal{O}(E^6) - \text{terms} \quad (4)$$

Using the relation $2\Gamma_{\mu\nu} = 2(\partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu) = -[A_\mu, A_\nu] - [B_\mu, B_\nu]$ we can show that

$$F_{\mu\nu}(-i\Gamma_\mu) = -i(\partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu) - i[\Gamma_\mu, \Gamma_\nu] = \frac{i}{4}\xi^\dagger\left[(D_\mu U)U^\dagger, (D_\nu U)U^\dagger\right]\xi \quad (5)$$

and the final result is [15]

$$\mathcal{L}_4^{\text{eff}} = \frac{g_\rho^2F_\pi^4}{8m_\rho^4}\left[(\mathrm{Tr}X_\mu X^\mu)^2 + 2(\mathrm{Tr}X_\mu X_\nu)^2 - 6\mathrm{Tr}\,(X_\mu X^\mu)^2\right],$$

$$L_1 = \frac{1}{32g^2} = \frac{g_\rho^2F_\pi^4}{8m_\rho^4}, L_2 = 2L_1, L_3 = -6L_1. \quad (6)$$

To obtain g_ρ ($g = g_\rho$) we have to compute the $\rho \to 2\pi$ decay width. From the Lagrangian we obtain for it

$$\mathcal{L}_{\rho\to 2\pi} = \frac{ig_\rho}{4}\mathrm{Tr}\,(\rho_\mu[\pi, \partial^\mu\pi]) = -ig_\rho\left[\rho_\mu^0\pi^+\partial^\mu\pi^- + \rho_\mu^+\pi^-\partial^\mu\pi^0 - \rho_\mu^-\pi^+\partial^\mu\pi^0\right],$$

$$\Gamma(\rho\to 2\pi) = \frac{g_\rho^2}{6\pi}\frac{p^3}{m_\rho^2} \quad (7)$$

where $p = \frac{1}{2}\sqrt{m_\rho^2 - 4m_\pi^2}$ is the pion momenta. From the experimental values [18] $m_\rho = (768.5 \pm 0.6)$ MeV and $\Gamma(\rho\to 2\pi) = (150.7 \pm 1.1)$ MeV we can obtain the Chiral Coefficients. However these values have to be corrected by a factor of $(57/69)^2 \simeq 0.8$ because the g_ρ in Eq. (6) has to be measured in the soft pions region and this is not the case for the $\rho \to 2\pi$ decay where the pions are not soft [15]. Other resonance contributions and the results of other models are shown for comparison in Table II.

3. Gasser-Leutwyler method

This method of deal with Vector Mesons fields was proposed by Gasser and Leutwyler in Ref. 2. The idea is to use the antisymmetric tensor field $F(V)_{\mu\nu} \equiv \lambda^i F_{\mu\nu}^i$ as a fundamental quantity instead of the vector field V_μ. A 'covariant derivative' for the $F(V)_{\mu\nu}$-tensor field can be constructed as

$$D_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} - [A_\alpha, F_{\mu\nu}] \to \Sigma_V D_\alpha F_{\mu\nu}\Sigma_V^\dagger, \quad (8)$$

where A_μ is defined in the Appendix. Now we can write a Chiral invariant Lagrangian

$$\mathcal{L}_{G-L} = -\frac{1}{4}\mathrm{Tr}\,\left[(D^\lambda F_{\lambda\mu})(D_\nu F^{\nu\mu})\right] + \frac{1}{8}m_\rho^2\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{ig_\rho F_\pi^2}{2m_\rho}\mathrm{Tr}\,\left[F_{\mu\nu}\xi^\dagger\left(X^\mu X_\mu^\dagger\right)\xi\right] = \frac{1}{4}F^{\alpha\beta}D_{\alpha\beta,\mu\nu}^{-1}F^{\mu\nu} + F_{\mu\nu}^iJ_i^{\mu\nu} + \mathcal{O}(E^6), \quad (9)$$

with

$$J_{\mu\nu}^i = -\frac{ig_\rho F_\pi^2}{2m_\rho}\mathrm{Tr}\,\left[\lambda^i\xi^\dagger X_\mu X_\nu\xi\right],$$

$$D_{\alpha\beta,\mu\nu}^{-1} = m_\rho^2 I_{\alpha\beta,\mu\nu} + \Delta_{\alpha\beta,\mu\nu}, \quad (10)$$

TABLE II. Resonances contribution to E^4-coefficients. Scalar and Tensor [15] contributions are given too. Observe that they are smaller and tend to cancel between themselves. Experimental values and predictions of other models are shown for comparison. For the No-Local Lagrangian Model (NLL) we take the case of $A = 1$. Notice that all the models are consistent with the experimental values (unfortunately the experimental errors are large).

Source	L_1	L_2	L_3
Vector Oct. (1^3s_1: $\rho(770)$, etc.)	0.68	1.36	-4.08
Vector* Oct. (2^3s_1: $\rho(1450)$, etc.)	0.01	0.02	-0.06
Scalar Oct. (1^3p_0: $a_0(980)$, etc.)	-0.18	0	0.53
Scalar Sin. (1^3p_0: $f_0(980)$?, $f_0(1370)$?)	0.18	0	
Tensor Oct. (1^3p_2: $a_2(1320)$, etc)	< 0.43	< 0.47	< -1.63
Tensor Sin. (1^3p_2: $f_2(1270)$?, $f_2'(1430)$?)	<-0.11	< 0.32	
Tensor Oct.* (2^3p_2: $K_2(1980)$?, etc.)	< 0.4	< 0.43	< -1.53
Total Resonance contribution	1 ± 0.4	2.6 ± 0.7	-6.8 ± 2.3
Q. M. [10]	0.8	1.6	-3
ENJL [11]	0.85	1.7	-4.2
NLL. [12]	1.29	2.58	-6.15
Exp. [2]	0.9 ± 0.3	1.7 ± 0.7	-4.4 ± 2.5
Exp. [5]	0.4 ± 0.3	1.35 ± 0.3	-3.5 ± 1.1

where $I_{\alpha\beta,\mu\nu} = (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})/2$, and $\Delta_{\mu\nu,\alpha\beta} = g_{\alpha\mu}\partial_\beta\partial_\nu - g_{\alpha\nu}\partial_\beta\partial_\mu + g_{\beta\nu}\partial_\alpha\partial_\mu - g_{\beta\mu}\partial_\alpha\partial_\mu$. Using the fact that $I_{\alpha\beta,\mu\nu}I^{\mu\nu,\rho\phi} = g_\alpha{}^\rho g_\beta{}^\phi$ we expand the propagator in powers of $1/m_\rho^2$ to have

$$D^{\mu\nu}_{\alpha\beta}D^{-1}_{\mu\nu,\rho\phi} = I_{\alpha\beta,\rho\phi},$$

$$D_{\alpha\beta,\mu\nu} = \frac{1}{m_\rho^2}I_{\alpha\beta,\mu\nu} - \frac{1}{m_\rho^4}\Delta_{\alpha\beta,\mu\nu} + \cdots . \quad (11)$$

The Path integral on $F_{\mu\nu}$ is done in the usual way to obtain

$$\begin{aligned}\mathcal{L}^{\text{eff}} &= -J^i_{\alpha\beta}D^{\alpha\beta,\mu\nu}J^i_{\mu\nu} + \mathcal{O}(E^6),\\ &= -\frac{1}{m_\rho^2}J^i_{\mu\nu}I^{\mu\nu,\alpha\beta}J^i_{\alpha\beta} + \mathcal{O}(E^6),\\ &= \mathcal{L}^{\text{eff}}_4 + \mathcal{O}(E^6).\end{aligned} \quad (12)$$

This is the same expression of the VMD-HS case of Eq. (6), however we have to show that the g_ρ is the same in both versions. The $\rho \to 2\pi$ decay Lagrangian is given by the last term of Eq. (9). The decay width can be obtained (the calculation is different, because the fundamental vector field is the tensor) in agreement with Eq. (7). Therefore the result for the Chiral Coefficients is the same.

4. VMD in the Yang-Mills version

Finally we want to consider the Yang-Mills or gauged model [8]. In this version of VMD the Chiral Symmetry is promoted to be a local one, with the resonance fields [$\rho(770)$ and $a_1(1270)$, or $L_\mu = \frac{1}{2}(\rho_\mu + a_\mu)$, $R_\mu = \frac{1}{2}(\rho_\mu - a_\mu)$] as its gauge fields. The covariant derivative of U can be constructed as $D_\mu U = \partial_\mu U - igL_\mu U + igUR_\mu$ and we can write the Lagrangian for this model [8]:

$$\begin{aligned}\mathcal{L} = \frac{\tilde{F}_\pi^2}{4}\text{Tr}(D_\mu U)(D_\mu U)^\dagger - \frac{1}{4}\text{Tr}\left(F^{L\,2}_{\mu\nu} + F^{R\,2}_{\mu\nu}\right)\\ + \frac{M^2}{2}\text{Tr}\left(L_\mu^2 + R_\mu^2\right) + \frac{\tilde{B}}{2}\text{Tr}\{L^\mu, UR_\mu U^\dagger\}.\end{aligned} \quad (13)$$

The constants $\tilde{F}_\pi$, M, g and $\tilde{B}$ are the unrenormalized constants and their relations with the physical ones will be obtained later. In this model the Weinberg relation $m_a^2 \simeq 2m_\rho^2$ is not necessarily valid and in this way it is possible to solve consistency problems [19] for the case of $\tilde{B} \neq 0$. The Lagrangian can be rewritten as

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 + J^i_\mu L^\mu_i + h^i_\mu R^\mu_i - \frac{1}{4}\text{Tr}\left(F(L)^2_{\mu\nu} + F(R)^2_{\mu\nu}\right)\\ + \frac{m^2}{2}\text{Tr}\left(L_\mu^2 + R_\mu^2\right) + B\,\text{Tr}\left(L_\mu UR^\mu U^\dagger\right),\end{aligned}$$

$$\begin{aligned}&\mathcal{L}_0 = \frac{\tilde{F}_\pi^2}{4}\text{Tr}\left(X_\mu X^{\mu\dagger}\right), \quad J^i_\mu = i\frac{g\tilde{F}_\pi^2}{2}\text{Tr}\left(\lambda^i X_\mu\right),\\ &h^i_\mu = -i\frac{g\tilde{F}_\pi^2}{2}\text{Tr}\left(\lambda^i Y_\mu\right), \quad m^2 = M^2 + \frac{g^2\tilde{F}_\pi^2}{2},\\ &B = \tilde{B} - \frac{g^2\tilde{F}_\pi^2}{2},\\ &m_\rho^2 = m^2 + B, \qquad m_a^2 = m^2 - B\end{aligned} \quad (14)$$

where m_ρ and m_a are the physical masses. The Effective Lagrangian can be obtained by doing the integral on the Gauge

Fields. Let us start with the L_μ field. We can write the Lagrangian as

$$\mathcal{L} = \tilde{\mathcal{L}}_0 + \tilde{J}^i_\mu L^\mu_i + m^2 L^2_\mu + f_4(L_\mu, R_\mu),$$
$$\tilde{\mathcal{L}}_0 = \mathcal{L}_0 + h^i_\mu R^\mu_i + \frac{m^2}{2}\mathrm{Tr}R^2_\mu,$$
$$\tilde{J}^i_\mu = J^i_\mu + B\,\mathrm{Tr}\left(\lambda^i U R_\mu U^\dagger\right),$$
$$f_4(L_\mu, R_\mu) = -\frac{1}{4}\mathrm{Tr}\left(F(L)^2_{\mu\nu} + F(R)^2_{\mu\nu}\right). \quad (15)$$

The integral over L_μ can be done at tree level and to order four derivatives to obtain

$$\mathcal{L}'_{\text{eff}} = \tilde{\mathcal{L}}_0 - \frac{1}{4m^2}(\tilde{J}^i_\mu)^2 + f_4\Big(-\frac{1}{2m^2}\tilde{J}_\mu, R_\mu\Big),$$
$$= \mathcal{L}'_0 + J^{i\mu}_R R_{i\mu} + R D^{-1}_R R + f_4\Big(-\frac{1}{2m^2}\tilde{J}_\mu, R_\mu\Big), \quad (16)$$

where in order to do the remaining R_μ integral we wrote $\mathcal{L}'_{\text{eff}}$ in the given form, with

$$\mathcal{L}'_0 = \mathcal{L}_0 - \frac{1}{4m^2}J^2_{i\mu},$$
$$J^{i\mu}_R = h^{i\mu} - \frac{B}{2m^2}\mathrm{Tr}\left(J^\mu U \lambda^i U^\dagger\right),$$
$$\left(D^{ij}_R\right)^{-1} = \left(m^2 - \frac{B^2}{m^2}\right)\delta^{ij} = \frac{m_a^2 m_\rho^2}{m^2}\delta^{ij}. \quad (17)$$

Doing the R_μ integral we obtain

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 - \frac{1}{4m^2}J^2_{i\mu} - \frac{1}{4}J^{i\mu}_R D^{ij}_R J^{jR}_R + f_4\Big(-\frac{\tilde{J}_\mu}{2m^2}, -\frac{1}{2}D_R J_{R\mu}\Big). \quad (18)$$

After some computation we can find that

$$J^{i\mu}_R = -i\frac{g\tilde{F}^2_\pi}{2}\left(1 + \frac{B}{m^2}\right)\mathrm{Tr}\left(\lambda^i Y^\mu\right),$$
$$-\frac{1}{4m^2}J^2_{i\mu} = -\frac{g^2\tilde{F}^2_\pi}{8m^2}\mathrm{Tr}\Big(X_\mu X^{\mu\dagger}\Big),$$
$$-\frac{1}{4}J^i_R \mu D^{ij}_R J^j_{R\mu} = -\frac{g^2\tilde{F}^4_\pi}{8}\frac{m_\rho^2}{m^2 m_a^2}\mathrm{Tr}\left(X^\mu X^\dagger_\mu\right). \quad (19)$$

In this case new E^2-terms are obtained and $\tilde{F}_\pi$ becomes renormalized. The E^2 total contribution is

$$\mathcal{L}^{\text{eff}}_2 = \frac{\tilde{F}^2_\pi}{4}\Big(1 - \frac{g^2\tilde{F}^2_\pi}{m_a^2}\Big)\mathrm{Tr}\left(X^\mu X^\dagger_\mu\right) = \frac{F^2_\pi}{4}\mathrm{Tr}\left(X^\mu X^\dagger_\mu\right) \quad (20)$$

where the physical pion-decay constant is

$$F^2_\pi = \tilde{F}^2_\pi\left(1 - g^2\tilde{F}^2_\pi/m_a^2\right).$$

The E^4-Lagrangian is

$$\mathcal{L}^{\text{eff}}_4 = f_4\Big(-\frac{1}{2m^2}J_\mu, -\frac{1}{2}D_R J_R\Big). \quad (21)$$

It can be shown

$$F^L_{\mu\nu}(-\frac{J}{2m^2}) = -i\frac{g\tilde{F}^2_\pi}{2m_a^2}\Big(1 - \frac{g^2\tilde{F}^2_\pi}{2m_a^2}\Big)X_{\mu\nu},$$
$$F^R_{\mu\nu}(-\frac{1}{2}D_R J_R) = i\frac{g\tilde{F}^2_\pi}{2m_a^2}\Big(1 - \frac{g^2\tilde{F}^2_\pi}{2m_a^2}\Big)Y_{\mu\nu}. \quad (22)$$

And finally we obtain $\mathcal{L}^{\text{eff}}_4$

$$\mathcal{L}^{\text{eff}}_4 = \frac{g^2\tilde{F}^4_\pi}{8m_a^2}\Big(1 - \frac{g^2\tilde{F}^2_\pi}{2m_a^2}\Big)^2\mathrm{Tr}\left(X^2_{\mu\nu}\right). \quad (23)$$

The coefficient is related to the $\rho \to 2\pi$ decay constant g_ρ by the equation, as will be show

$$\frac{g_\rho}{m_\rho^2} = \frac{g\tilde{F}^2_\pi}{m_a^2 F^2_\pi}\Big(1 - \frac{g^2\tilde{F}^2_\pi}{2m_a^2}\Big). \quad (24)$$

Thus we obtain again the same result of Eq. (6)

$$\mathcal{L}_{\text{eff}} = \frac{F^2_\pi}{4}\mathrm{Tr}\left(X^\mu X^\dagger_\mu\right) + \mathcal{L}^{\text{eff}}_4 + \mathcal{O}(E^6). \quad (25)$$

However we still have to prove Eq. (24). The Lagrangian for the $\rho \to 2\pi$ decay can be written as

$$\mathcal{L}(\rho \to 2\pi) = \frac{1}{2}\mathrm{Tr}\left(J_\mu L^\mu + h_\mu R^\mu\right) - \frac{1}{4}\mathrm{Tr}\left[F(L_\alpha)^2_{\mu\nu} + F(R_\alpha)^2_{\mu\nu}\right] + \frac{m^2}{2}\mathrm{Tr}\left(L^2_\mu + R^2_\mu\right) + B\mathrm{Tr}\left(L_\mu U R^\mu U^\dagger\right) \quad (26)$$

with $J^\mu = ig\tilde{F}^2_\pi X^\mu$, $h^\mu = -ig\tilde{F}^2_\pi Y^\mu$. Now we can integrate out the a_μ fields. To do that we have to replace $a_\mu \to -(J_\mu - h_\mu)/2m_a^2$. Thus we get for the tensor fields, keeping only the terms that contribute to the decay

$$F(L,R)^2_{\mu\nu} = \mp\frac{ig\tilde{F}^2_\pi}{4m_a^2} \times\Big\{\rho_{\mu\nu}(X^{\mu\nu}+Y^{\mu\nu}) \mp \frac{g^2\tilde{F}^2_\pi}{4m_a^2}\rho_{\mu\nu}[X^\mu+Y^\mu, X^\nu+Y^\nu]\Big\}. \quad (27)$$

Then

$$\mathcal{L}(\rho \to 2\pi) = \frac{1}{4}\mathrm{Tr}\left[\rho_\mu(J^\mu + h^\mu)\right] - i\frac{g}{2}\left(\frac{g\tilde{F}^2_\pi}{4m_a^2}\right)^2\mathrm{Tr}(\rho_{\mu\nu}[X^\mu + Y^\mu, X^\nu + Y^\nu])$$
$$+ \frac{B}{4}\Big(-\frac{ig\tilde{F}^2_\pi}{2m_a^2}\Big)\mathrm{Tr}\left[\rho_\mu U^\dagger(X^\mu + Y^\mu)U - \rho_\mu U(X^\mu + Y^\mu)U^\dagger\right] = \frac{ig\tilde{F}^2_\pi}{4F^2_\pi}\Big(1 + \frac{2B}{m_a^2} - \frac{1}{2}\frac{g^2\tilde{F}^2_\pi}{m_a^4}m_\rho^2\Big)\mathrm{Tr}\left(\rho_\mu[\partial^\mu\phi, \phi]\right), \quad (28)$$

TABLE III. Chiral Transformations, and definitions of the relevant Fields. $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \quad \forall Z_\mu$

Item	$SU(3)_L \times SU(3)_R$ Tranf.	Definitions
U	$\Sigma_L U \Sigma_R^\dagger$	$U = \exp(i\pi/F_\pi), \pi = \pi^i\lambda^i;\ T_i = \lambda_i/2,\ \mathrm{Tr}\ (T^iT^j) = \delta^{ij}/2$
χ	$\Sigma_L \chi \Sigma_R^\dagger$	$\chi \simeq 2B\mathrm{diag.}(m_u, m_d, m_s) \simeq \mathrm{diag.}(m_\pi, m_\pi, 2m_K - m_\pi)$
ξ	$\xi \to \Sigma_L \xi \Sigma_V^\dagger = \Sigma_V \xi \Sigma_R^\dagger$	$U = \xi\xi$
u_μ	$u_\mu \to \Sigma_V u_\mu \Sigma_V^\dagger$	$u_\mu = i\left(\xi^\dagger \partial_\mu \xi - \partial_\mu \xi \xi^\dagger\right) = i\xi^\dagger(\partial_\mu U)\xi^\dagger$
Γ_μ	$\Gamma_\mu \to \Sigma_V \Gamma_\mu \Sigma_V^\dagger + \Sigma_V \partial_\mu \Sigma_V^\dagger$	$\Gamma_\mu = \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi + \partial_\mu \xi \xi^\dagger\right)$
$\chi_\pm$	$\chi_\pm \to \Sigma_V \chi_\pm \Sigma_V^\dagger$	$\chi_\pm = \xi^\dagger \chi \xi^\dagger \pm \xi \chi^\dagger \xi = \pm\chi^\pm$
X_μ	$X_\mu \to \Sigma_L X_\mu \Sigma_L^\dagger$	$X_\mu = (\partial_\mu U)U^\dagger = \xi(A_\mu + B_\mu)\xi^\dagger,\ X_\mu^\dagger = -X_\mu$
Y_μ	$Y_\mu \to \Sigma_R X_\mu \Sigma_R^\dagger$	$Y_\mu = U^\dagger(\partial_\mu U) = U^\dagger X_\mu U,\ Y_\mu^\dagger = -Y_\mu$
$X_{\mu\nu}$	$\to \Sigma_L X_{\mu\nu} \Sigma_L^\dagger$	$X_{\mu\nu} \equiv \partial_\mu X_\nu - \partial_\nu X_\mu = [X_\mu, X_\nu]$
$Y_{\mu\nu}$	$\to \Sigma_R X_{\mu\nu} \Sigma_R^\dagger$	$Y_{\mu\nu} \equiv \partial_\mu Y_\nu - \partial_\nu Y_\mu = -U^\dagger X_{\mu\nu} U$
A_μ	$\Sigma_L[A_\mu - \xi\Sigma_V^\dagger \partial_\mu \Sigma_V \xi^\dagger]\Sigma_L^\dagger =$ $\Sigma_V A_\mu \Sigma_V^\dagger + \partial_\mu \Sigma_V \Sigma_V^\dagger$	$A_\mu = \partial_\mu \xi \xi^\dagger$
B_μ	$\Sigma_R[B_\mu + \xi^\dagger \Sigma_V^\dagger \partial_\mu \Sigma_V \xi]\Sigma_R^\dagger =$ $\Sigma_V B_\mu \Sigma_V^\dagger + \Sigma_V \partial_\mu \Sigma_V^\dagger$	$B_\mu = \xi^\dagger \partial_\mu \xi$
Z_μ	$\Sigma_Z Z_\mu \Sigma_Z + i\left(\partial_\mu \Sigma_Z\right)\Sigma_Z^\dagger$	$\Sigma_Z = \exp(iT\cdot\theta_Z),\ Z = L, R, V,\ Z_\mu = Z_\mu^i T^i$
$F_{\mu\nu}(Z)$	$\Sigma_Z F_{\mu\nu}(Z)\Sigma_Z^\dagger$	$F_{\mu\nu}(Z) = \partial_\mu Z_\nu - \partial_\nu Z_\mu + ig_Z[Z_\mu, Z_\nu],\ Z = L, R, V, A$
$f_\pm^{\mu\nu}$	$\Sigma_V f_\pm^{\mu\nu} \Sigma_V^\dagger$	$f_\pm^{\mu\nu} \equiv \xi^\dagger F(L)^{\mu\nu}\xi \pm \xi F(R)^{\mu\nu}\xi^\dagger$

where we have used the equation of motion for ρ, at tree level $\partial^\mu \rho_{\mu\nu} + m_\rho^2 \rho_\nu = 0$ and dropped the terms proportional to the pion mass. As we can see there are three contributions to the $\rho \to 2\pi$ decay. They correspond to the three Feynman diagrams possible contributions: $\rho \to \pi - \pi$, $\rho \to (a_1 \to \pi) - \pi$ and $\rho \to (a_1 \to \pi) - (a_1 \to \pi)$. Using the expression for B from Eq. (14) we get

$$\mathcal{L}(\rho \to 2\pi) = \frac{igF_0^2}{4F_\pi^2}\frac{m_\rho^2}{m_a^2}\left(1 - \frac{1}{2}\frac{g^2F_0^2}{m_a^2}\right)\mathrm{Tr}\left(\rho_\mu[\partial^\mu\phi, \phi]\right)$$
$$\equiv \frac{ig_\rho}{4}\mathrm{Tr}\left(\rho_\mu[\partial^\mu\phi, \phi]\right). \tag{29}$$

To obtain Eq. (24) as we wanted to prove.

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Appendix A: Miscellaneous relations

For any $A, B \in SU(N)$ we have that

$$\sum_{i=1}^{N^2-1} \mathrm{Tr}\left(\lambda^i A\right)\cdot\mathrm{Tr}\left(\lambda^i B\right) = 2\mathrm{Tr}\left(AB\right) - \frac{2}{N}\mathrm{Tr}(A)\cdot\mathrm{Tr}(B),$$
$$\sum_{i=1}^{N^2-1} \mathrm{Tr}(\lambda^i A \lambda^i B) = 2\mathrm{Tr}(A)\cdot\mathrm{Tr}(B) - \frac{2}{N}\mathrm{Tr}(AB), \tag{30}$$
$$\mathrm{Tr}(ABAB) = -2\mathrm{Tr}(A^2B^2) + \frac{1}{2}(\mathrm{Tr}A^2)(\mathrm{Tr}B^2) + (\mathrm{Tr}AB)^2,$$

where the last one is valid only for any A, $B \in SU(3)$.

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