# Interfaces of relativistic membranes and Neumann's triangle

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We examine, from a geometrical point of view, the dynamics of relativistic extended objects joined at some interface. Using simple variational techniques, we obtain the equations of motion for these objects, together with a set of dynamical boundary conditions, that express the feedback of the motion of the interface on the joining membranes. These conditions reduce, in a particular limit, to a relativistic dynamical generalization of Neumann's triangle. For simplicity, we restrict our attention to Dirac-Nambu-Goto extended objects.

*Keywords:* Relativistic extended objects; membranes; variational methods

Consideramos, desde un punto de vista geométrico, la dinámica de objetos extendidos relativistas juntos en una interfase. Usando técnicas variacionales sencillas, obtenemos las ecuaciones de movimiento para estos objetos y también un conjunto de condiciones de frontera dinámicas que expresan el efecto del movimiento de la interfase sobre las membranas que se unen. Estas condiciones se reducen, en un límite particular, a una generalización relativista del triángulo de Neumann. Por sencillez, consideramos solamente objetos extendidos de Dirac-Nambu-Goto.

*Descriptores:* Objetos extendidos relativistas; membranas; métodos variacionales

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## 1. Introduction

In a phenomenological description, to lowest order, a relativistic extended object is described by an action functional proportional to the area of its worldsheet, $m$. This functional is known as the Dirac-Nambu-Goto [DNG] action. The dynamics of DNG extended objects has been studied extensively [1–3].

In a recent publication , we have developed a geometrical framework for the description of the dynamics of extended objects with loaded edges [4]. Our analysis is based on the key observation that the worldsheet of each edge can be considered as a hypersurface in the worldsheet of the parent extended object, which coincides with its timelike boundary [5]. The dynamics of the system is determined by a set of equations of motion for the parent extended object, the same one would obtain by neglecting the boundary, and a set of equations of motion for the edges. The description is completed by a set of boundary conditions, expressed as constraints on the geometry of the parent worldsheet at the edges that contain the needed information about the dynamical feedback that the edges have on the parent object spanning them.

In this paper, we extend such geometrical framework to a system composed of an arbitrary number of DNG extended objects joined at some interface, which we treat as a shared edge (this general type of system was envisaged by Carter in Ref. 6). The interface is considered as some extended object itself, with a dynamics of its own, embedded as a hypersurface in the higher dimensional extended objects, and it coincides with a component of the timelike boundary of such objects. For a defect of co-dimension one, when $D = 2$, this could represent several strings meeting at a point or at a point mass. When $D > 2$, the interface might be the physical interface along which two membranes touch such as a phase boundary. The former case has been studied in the context of the stringy description of hadrons, *e.g.*, where the hadron is modeled as three open strings whose edges are quarks, and that meet at some point, in the so-called Y-model for hadrons [7, 3]. The dynamics of systems where topological defects of different dimensionality are joined together is also relevant in the cosmological context [1]. An appropriate sequence of phase transitions can produce hybrid topological defects, such as domain walls bounded by a string, or a network of $Z_N$-strings with monopoles vertices where the strings join. These hybrid topological defects have properties radically different from simple non-composite topological defects.

We use simple variational techniques to derive the equations of motion for this system. We find that the bulk equations of motion of the joining extended objects are not affected by the presence of the interface, *i.e.* they are extremal. At the interface, we obtain a set of coupled equations which express the coupling of the interface to the joining objects. In the limit of a vanishing tension for the interface, these equations reduce to a relativistic generalization of the well-known Neumann triangle, which features, for example, in the clustering of soap bubbles [9, 10].

## 2. Interface geometry

Consider $\mathcal{N}$, $(D-1)$-dimensional, relativistic extended objects decribed by $(D-1)$-dimensional spacelike surfaces, which meet along some interface. (The case of many interfaces follows naturally, but implies an additional notational burden.) Their trajectories in spacetime, or worldsheets, are oriented timelike surface $m_{(\mathcal{I})}$ ($\mathcal{I}=1,\cdots,\mathcal{N}$), of dimension $D$, embedded in a fixed $N$-dimensional background spacetime $\{M, g_{\mu\nu}\}$.

The worldsheet $m_{(\mathcal{I})}$ can be represented in parametric form by the embedding $x^\mu = X^\mu_{(\mathcal{I})}(\xi^a_{(\mathcal{I})})$, where $x^\mu$ are local coordinates on $M$, $X^\mu_{(\mathcal{I})}$ embedding functions, and $\xi^a_{(\mathcal{I})}$ are local coordinates on $m_{(\mathcal{I})}$. ($\mu,\nu,\cdots=0,1,\cdots,N-1$, and $a,b,\cdots=0,1,\cdots,D-1$. For the labels $\mathcal{I},\mathcal{J},\cdots$ the summation convention is suspended.) The interface where the $\mathcal{N}$ sheets meet, $int$, can be represented by the embedding as a hypersurface into the $\mathcal{I}^{\text{th}}$ sheet, as $\xi^a_{(\mathcal{I})} = \chi^a_{(\mathcal{I})}(u^A)$, where $u^A$ are local coordinates on $int$, and $\chi^a_{(\mathcal{I})}$ embedding functions. ($A,B,\ldots=0,1,\cdots,D-2$.). The embeddings of the worldsheets agree on the interface. An alternative way to describe the interface is via its direct embedding in spacetime, through map composition, $x^\mu = \chi^\mu_{(\mathcal{I})}(u^A)$, where $\chi^\mu_{(\mathcal{I})} = (\partial X^\mu_{(\mathcal{I})}/\partial\xi^a_{(\mathcal{I})})\chi^a_{(\mathcal{I})}$.

For the sake of simplicity, we consider the case of $\mathcal{N}$ extended objects described by a DNG action, joined at a single interface, an extended object of one lower dimension, described by a DNG action as well. The generalization to arbitrary phenomenological actions can nonetheless be carried through along the lines illustrated in this paper, but it is considerably more involved. The DNG action depends only on the intrinsic geometry of the worldsheet. The intrinsic metric on each $m_{(\mathcal{I})}$ is defined by

$$\gamma_{ab(\mathcal{I})} = e^\mu_{a(\mathcal{I})} e^\nu_{b(\mathcal{I})} g_{\mu\nu} = g(e_{a(\mathcal{I})}, e_{b(\mathcal{I})}), \tag{1}$$

where we introduce, on each sheet, the $D$ tangent vectors, $e_{a(\mathcal{I})} := (\partial x^\mu_{(\mathcal{I})}/\partial\xi^a_{(\mathcal{I})})\partial_\mu$. Note that we can define the $i^{\text{th}}$ unit space-like normal to the worldsheet $m_{(\mathcal{I})}$, $n^{\mu i}_{(\mathcal{I})}$ ($i,j,\cdots = 1,\cdots,N-D$), with, up to a local $O(N-D)$ rotation, $g(e_{a(\mathcal{I})}\, n^i_{(\mathcal{I})}) = 0$, and normalized with $g(n^i_{(\mathcal{I})}, n^j_{(\mathcal{I})}) = \delta^{ij}$. (We use a signature for the space-time metric with only one minus sign). The spacetime vectors $\{n^i_{(\mathcal{I})}, e_{a(\mathcal{I})}\}$ form a basis for spacetime adapted to $m_{(\mathcal{I})}$. We will make use of this basis below. The intrinsic geometry of the interface is the same, independently of which sheet metric induces it, and thus it will be independent of the label, so that the intrinsic metric on $int$ is given by

$$h_{AB} = \gamma_{(\mathcal{I})}(\epsilon_{A(\mathcal{I})}, \epsilon_{B(\mathcal{I})}), \tag{2}$$

where we have introduced the $D-1$ tangent vectors to $int$, $\epsilon_{A(\mathcal{I})} := (\partial\chi^a/\partial u^A)\partial_a$. Also in this case, the unit normal vector to $int$ in $m_{(\mathcal{I})}$, $\eta^a_{(\mathcal{I})}$, is defined, up to a sign, with $\gamma_{(\mathcal{I})}(\epsilon_{A(\mathcal{I})}, \eta_{(\mathcal{I})}) = 0$, and normalized with $\gamma_{(\mathcal{I})}(\eta_{(\mathcal{I})}, \eta_{(\mathcal{I})}) = 1$. The surface vectors $\{\epsilon_{A(\mathcal{I})}, \eta_{(\mathcal{I})}\}$ form a basis adapted to the interface.

## 3. Action and first variation

The total action we consider is

$$S = S_0 + S_{int}, \tag{3}$$

where we separate the "bulk" and interface parts,

$$S_0[X_{(\mathcal{I})}, \chi^a_{(\mathcal{I})}] = -\sum_{(\mathcal{I})} \mu_{(\mathcal{I})} \int_{m_{(\mathcal{I})}} d^D\xi_{(\mathcal{I})} \sqrt{-\gamma_{(\mathcal{I})}}, \tag{4}$$

$$S_{int}[\chi^a_{(\mathcal{I})}, X_{(\mathcal{I})}] = -\mu_{int} \int_{int} d^{D-1}u\sqrt{-h}. \tag{5}$$

Here $\gamma_{(\mathcal{I})}$ denotes the determinant of the intrinsic metric on the $\mathcal{I}^{\text{th}}$ sheet (1), and $h$ the determinant of the metric induced on the interface, (2). We allow for different tensions for the $\mathcal{I}^{\text{th}}$ extended object, which we denote with $\mu_{(\mathcal{I})}$, and $\mu_{int}$ is the interface tension.

In order to derive the equations of motion for this system, we proceed in a standard way, and we consider first an arbitrary infinitesimal displacement of each sheet [11],

$$X^\mu_{(\mathcal{I})} \to X^\mu_{(\mathcal{I})} + \delta X^\mu_{(\mathcal{I})}. \tag{6}$$

These displacements are independent, but subject to the condition of agreeing on the interface. Moreover, the displacements are assumed to vanish on two spacelike hypersurfaces on each sheet, that play the role of initial and final configurations. We decompose the $\mathcal{I}^{\text{th}}$ displacement with respect to the spacetime basis, $\{e_{a(\mathcal{I})}, n_{i(\mathcal{I})}\}$, adapted to that sheet:

$$\delta X^\mu_{(\mathcal{I})} = \Phi^a_{(\mathcal{I})} e^\mu_{a(\mathcal{I})} + \Phi^i_{(\mathcal{I})} n^\mu_{i(\mathcal{I})}, \tag{7}$$

where the components $\Phi^a$ and $\Phi^i$ transform, respectively, as a vector and as a multiplet of $N-D$ scalars on $m_{(\mathcal{I})}$. This infinitesimal displacement induces the variation of the intrinsic metric on each sheet according to

$$\delta_{X_{(\mathcal{I})}}\gamma_{ab(\mathcal{I})} = 2K_{ab^i(\mathcal{I})}\Phi_{i(\mathcal{I})} + \nabla_{a(\mathcal{I})}\Phi_{b(\mathcal{I})} + \nabla_{b(\mathcal{I})}\Phi_{a(\mathcal{I})}, \tag{8}$$

where $\nabla_{a(\mathcal{I})}$ is the covariant derivative on $m_{(\mathcal{I})}$, compatible with $\gamma_{ab(\mathcal{I})}$. We denote with $K_{ab^i(\mathcal{I})}$ the extrinsic curvature of $m_{(\mathcal{I})}$ along the $i^{\text{th}}$ unit normal,

$$K_{ab^i(\mathcal{I})} = -g(n^i_{(\mathcal{I})}, D_{a(\mathcal{I})} e_{b(\mathcal{I})}), \tag{9}$$

and $D_{a(\mathcal{I})} = e^\mu_{a(\mathcal{I})} D_\mu$ is the spacetime gradient along the tangential vectors.

Let us consider first the variation of the bulk part of the action, $S_0$. We find,

$$\delta_X S_0 = -\sum_{(\mathcal{I})} \mu_{(\mathcal{I})} \int_{m_{(\mathcal{I})}} d^D\xi_{(\mathcal{I})}\sqrt{-\gamma_{(\mathcal{I})}} K^i_{(\mathcal{I})}\Phi_{i(\mathcal{I})} - \sum_{(\mathcal{I})} \mu_{(\mathcal{I})} \int_{int} d^{D-1}u\sqrt{-h}\eta_{a(\mathcal{I})}\Phi^a{}_{(\mathcal{I})}. \quad (10)$$

We have used the fact that, on each sheet, the intrinsic metric varies according to (8), we have defined the trace of the $i^{\text{th}}$ extrinsic curvature with $K^i_{(\mathcal{I})} = \gamma^{ab}_{(\mathcal{I})} K^i_{ab(\mathcal{I})}$, and we have used Stoke's theorem to obtain a surface term. Recall that $\eta_{a(\mathcal{I})}$ is the inward normal to $int$ into the $\mathcal{I}^{\text{th}}$ sheet. In this derivation, we have tacitly assumed that $int$ is the only boundary for the $\mathcal{N}$ sheets. If there are additional boundaries, the surface term will be augmented by the appropriate contributions.

Since the displacements agree at the interface, without loss of generality, at $int$, we can express the $\mathcal{I}^{\text{th}}$ displacement with respect to a single one, say the one corresponding to the sheet $\mathcal{J}$. Thus, at the interface, we have that

$$\Phi^a_{(\mathcal{I})} = \Phi^i_{(\mathcal{J})} g(n_{i(\mathcal{J})}, e^a_{(\mathcal{I})}) + \Phi^b_{(\mathcal{J})} g(e_{b(\mathcal{J})}, e^a_{(\mathcal{I})}). \quad (11)$$

Moreover, the part of the displacement tangential to $m_{(\mathcal{J})}$, $\Phi^a_{(\mathcal{J})}$, evaluated at $int$, can be decomposed in a part tangential to the interface, and a part perpendicular to it,

$$\Phi^a_{(\mathcal{J})} = \Phi^A \epsilon^a_{A(\mathcal{J})} + (\Phi^b_{(\mathcal{J})}\eta_{b(\mathcal{J})})\eta^a_{(\mathcal{J})}. \quad (12)$$

Note that the part tangential to the interface carries no label. Inserting in the surface term in (10), one finds that the part tangential to the interface does not contribute, and we obtain the interface contribution from the bulk action,

$$\delta_X S_0|_{int} = -\sum_{(\mathcal{I})} \mu_{(\mathcal{I})} \int_i d^{D-1}u\sqrt{-h} \times \left[\Phi^i_{(\mathcal{J})} g(n_{i(\mathcal{J})}, \eta_{(\mathcal{I})}) + (\Phi^b_{(\mathcal{J})}\eta_{b(\mathcal{J})}) g(\eta_{(\mathcal{J})}, \eta_{(\mathcal{I})})\right], \quad (13)$$

where we have defined the push-forward, $\eta^\mu_{(\mathcal{I})} := e^\mu_a \eta^a_{(\mathcal{I})}$.

Let us turn now to the variation of the interface action, $S_{int}$, under the displacements of the interface itself. Since at the interface all of the displacements must agree, we can choose to elect the displacement of the sheet $m_{(\mathcal{J})}$, as we did above, as the independent one. As a preliminary step, consider the variation of the interface intrinsic metric under $\delta x^\mu_{(\mathcal{J})}$, given by [4]

$$\delta_X h_{AB} = \left[2K^i_{ab}(\mathcal{J})\Phi_{i(\mathcal{J})} + \nabla_{a(\mathcal{J})}\Phi_{b(\mathcal{J})} + \nabla_{b(\mathcal{J})}\Phi_{a(\mathcal{J})}\right] \epsilon^a_{A(\mathcal{J})}\epsilon^b_{B(\mathcal{J})}. \quad (14)$$

Therefore, one finds,

$$\delta_X S_{int} = -\mu_{int} \int_{int} d^{D-1}u\sqrt{-h}\mathcal{H}^{ab}_{(\mathcal{J})} \times \left[K^i_{ab(\mathcal{J})}\Phi_{i(\mathcal{J})} + \nabla_{a(\mathcal{J})}\Phi_{b(\mathcal{J})}\right], \quad (15)$$

where we have defined the projector from sheet $\mathcal{J}$ onto the interface, $\mathcal{H}^{ab}_{(\mathcal{J})} := h^{AB}\epsilon^a_{A(\mathcal{J})}\epsilon^b_{B(\mathcal{J})}$. Now, using the decomposition (12), we have that

$$\mathcal{H}^{ab}_{(\mathcal{J})}\nabla_{a(\mathcal{J})}\Phi_{b(\mathcal{J})} = \mathcal{D}_A\Phi^A + k_{(\mathcal{J})}(\Phi^b_{(\mathcal{J})}\eta_{b(\mathcal{J})}), \quad (16)$$

where $\mathcal{D}_A$ denotes the covariant derivative on the interface, compatible with $h_{AB}$, and we have defined the mean extrinsic curvature of the interface as embedded in $m_{(\mathcal{J})}$ with $k_{(\mathcal{J})} = \nabla_{a(\mathcal{J})}\eta^a_{(\mathcal{J})}$. Inserting now in (15), since the total divergence $\mathcal{D}_A\Phi^A$ does not contribute if we assume that $int$ is a smooth boundary, we get,

$$\delta_X S_{int} = -\mu_{int} \int_{int} d^{D-1}u\sqrt{-h} \times \left[\mathcal{H}^{ab}_{(\mathcal{J})} K^i_{ab(\mathcal{J})}\Phi_{i(\mathcal{J})} + k_{(\mathcal{J})}\nabla_{a(\mathcal{J})}\Phi^a_{(\mathcal{J})}\right]. \quad (17)$$

## 4. Equations of motion

Setting the independent parts of the displacements equal to zero we find the following equations of motion. Each $\mathcal{I}^{\text{th}}$ sheet is extremal,

$$K^i_{(\mathcal{I})} = 0. \quad (18)$$

The bulk equations of motion are not affected directly in their form by the presence of an interface. The interface contributions give

$$\mu_{int}\mathcal{H}^{ab}_{(\mathcal{J})} K^i_{ab(\mathcal{J})}|_{int} + \sum_{(\mathcal{I})} \mu_{(\mathcal{I})} g(n_{i(\mathcal{J})}, \eta_{(\mathcal{I})})|_{int} = 0, \quad (19)$$

$$\mu_{int} k_{(\mathcal{J})} + \sum_{(\mathcal{I})} \mu_{(\mathcal{I})} g(\eta_{(\mathcal{J})}, \eta_{(\mathcal{I})})|_{int} = 0. \quad (20)$$

This set of coupled equations express the interchange of momentum between the bulk degrees of freedom and the interface degrees of freedom.

In the degenerate case of a single sheet, (19) and (20) reduce to the boundary conditions $\mathcal{H}^{ab}K^i_{ab} = 0$, and to the edge equations of motion $\mu_{int}k = -\mu$, respectively, previously derived in [4]. When we have more than one sheet, however, the distinction between interface equations of motion and the boundary conditions on the conjoining sheets dissolves.

When we have only two joining sheets, and if we assume $\mu_{(1)} = \mu_{(2)} = \mu$, (20) takes the form

$$\mu_{int}k_{(1)} + \mu[1 + g(\eta_{(1)}, \eta_{(2)})] = 0,$$
$$\mu_{int}k_{(2)} + \mu[1 + g(\eta_{(1)}, \eta_{(2)})] = 0,$$

from which it follows that the mean extrinsic curvature of the interface as embedded in one parent sheet or in the other must be equal:

$$k = k_{(1)} = k_{(2)}. \quad (21)$$

When we have three joining sheets, with equal tension, from (20) one obtains in the same way that the angles $g(\eta_{(\mathcal{J})}, \eta_{(\mathcal{I})})$ can be expressed algebraically with respect to $k_{(1)}, k_{(2)}, k_{(3)}$. For more than three joining sheets, there are not enough equations to eliminate the dynamical angles between the sheets in favour of the interface geometric scalars.

We can also express (19) and (20) in an alternative way, by defining the spacetime vector field along the interface:

$$V = \sum_{(\mathcal{I})} \mu_{(\mathcal{I})}\eta_{(\mathcal{I})} + \mu_{int}[k_{(\mathcal{J})}\eta^a_{(\mathcal{J})} + \mathcal{H}^{cd}_{(\mathcal{J})} K^i_{cd(\mathcal{J})} n_{i(\mathcal{J})}]. \quad (22)$$

Then (19) and (20) can be cast in the form

$$g(e^a_{(\mathcal{J})}, V) = 0 \quad g(n^i_{(\mathcal{J})}, V) = 0, \quad (23)$$

or, alternatively, with,

$$V = 0. \quad (24)$$

Note that we have gone from $N - D + 1$ equations to $N$ equations. The extra $D - 1$ equations vanish identically and correspond to reparameterizations of the interface.

Although the tensions are dimensionful quantities, we can envisage approximations in which one dominates. If the interface tension can be ignored, the vanishing of $V$ reduces to

$$\sum_{(\mathcal{I})} \mu_{(\mathcal{I})}\eta_{(\mathcal{I})} = 0, \quad (25)$$

which can be interpreted as the conservation of momentum at the interface.

In particular, if $\mathcal{N} = 2$, we require $\mu_{(1)} = \mu_{(2)}$ and $\eta_{(1)} = -\eta_{(2)}$—no discontinuity is possible, unless we let the interface turn null. If, however, $\mathcal{N} = 3$, we have

$$\mu_{(1)}\eta_{(1)} + \mu_{(2)}\eta_{(2)} + \mu_{(3)}\eta_{(3)} = 0. \quad (26)$$

Thus, the three normals lie in a two dimensional plane. If, in addition, the tensions coincide, the angle between normals is $120^0$. This is, in a relativistic context, the analogue of the Neumann triangle which features in the solution of the Plateau problem [9, 10]. In general, at a minimizing junction, the normals to the $\mathcal{N}$ sheets always lie in a $\mathcal{N} - 1$- plane, dividing it into $\mathcal{N} - 1$ equal regions regardless of the global details of the problem. Recall that the celebrated problem of minimizing the tree length connecting four points involves identical vertices regardless of the positions of the points. Note, in addition, that the angle is completely independent of the background geometry.

In general, when the line tension is not zero, these angles will be dynamical variables themselves. In particular, they will not be conserved by the evolution.

The other limit of interest is when the bulk tensions may be neglected. In this case, one recovers the extremal dynamics of the interface. The easiest way to see this is to appeal to the direct embedding of $int$ in spacetime.

## 5. Discussion

In this paper we have analyzed, using a geometrical approach, the issue of the appropriate equations of motion for a system of relativistic objects joined at some interface. We have restricted our attention to the case of simple DNG objects. However the basic variational techniques we have employed generalize to higher order, curvature dependent, extended objects, with the complication of more involved variational formulas. Work on this is in progress.

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