Two-dimensional lattice with electromagnetic gauge field and fermions included

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Recibido el 22 de septiembre de 1999; aceptado el 5 de junio de 2000

In this work we deal with the inclusion of fermions on a 2-dimensional lattice with electromagnetic field as the gauge field. We have improved a previous work by introducing explicitly the Dirac gamma matrices in the fermionic determinant and by testing gauge invariance of others terms that appear in the integrals. We find that gauge invariance is preserved, which in turn means that no Itô terms are needed in the effective continuum action.

Keywords: Lattice theory; gauge theory; gauge invariance

En este trabajo tratamos la inclusión de fermiones en una red de dos dimensiones con campo electromagnético como campo de norma. Hemos mejorado un trabajo anterior introduciendo explícitamente las matrices gamma de Dirac en el determinante fermiónico y comprobando la invariancia de norma de otros términos que aparecen en las integrales. Encontramos que la invariancia de norma se preserva, lo cual a la vez significa que no son necesarios los términos de Itô en la acción efectiva continua.

Descriptores: Teoría de redes; teoría de norma; invariancia de norma

PACS: 03.70+k

1. Introduction

In a previous paper [1] we have reported the invariance under local gauge transformations of the partition function associated with the electromagnetic field in the presence of fermions. This was done for the $d = 2$ and $d = 3$ cases, d being the lattice space-time dimension. This accomplishes the work [2] where it is shown gauge invariance for a lattice with electromagnetic field, but in the absence of fermions. The fact, which is actually expected because the flatness of the $U(1)$ manifold, has also to do with the nonsymmetric Itô terms [3, 4] in the effective action, which should be added to the action to maintain gauge invariance.

The next task was supposed to be the four-dimensional case, but the computational job turned out very hard, going up to some weeks of continuous work using Mathematica program plus other simplifications in the fermionic determinant. Such extreme work is typical when dealing with lattice gauge theories [5, 6] so it should not be surprising.

Anyway we have opted for the easier case $d = 2$, but employing explicitly the Dirac matrices in the determinant, instead of using some simplification introduced in Ref. 1. This renders a resulting determinant with 8 rows and 8 columns, that is, twice the size of that one if such matrices were ignored, and this involves more calculations. Of course, the larger the dimension d the larger the computational work.

We have organized this paper as follows: In Sec. 2 we write the partition function Z, expressing then the corresponding determinant. Section 3 is devoted to check the gauge invariance of Z. Finally, in Sec. 4 the conclusions are presented.

2. The fermionic determinant

For a review of lattice theories see for instance [7–10]. Here we will concentrate in some results for a lattice, just to know the set of variables that have to be managed and which of them are relevant.

The problem of fermions inclusion on a lattice is not so trivial since it brings renormalization difficulties [9]. Therefore one has to test effectively gauge invariance of the corresponding action in the continuum limit $a \to 0$; the corresponding action is written as

$$S = \sum_{n,m} \overline{\Psi}\Big\{K \sum_{\mu}[(\mathbf{I} + \gamma_\mu)U^*_{\mu n}\delta^d_{n+\mu,m} + (\mathbf{I} - \gamma_\mu)U_{\mu m}\delta^d_{n-\mu,m}] + \delta^d_{n,m}\Big\}\Psi_m + \frac{(Ma)^{d-4}}{g^2}\sum_{\mu,\nu}[\mathbf{I} - \mathrm{Re}\,(U_{\mu n}U_{\nu,n+\mu}U^*_{\mu,n+\nu}U^*_{\nu n})], \quad (1)$$

where $\mathbf{I}$ is the unit matrix.

We shall follow the notation of Ref. 1, with the corrected expression for Eq. (4), where $U_{\mu n} = e^{\mathrm{i}\theta_{\mu n}}$. The associated partition function takes the form [7]

$$Z = \prod_{l'} \int_{-\pi}^{\pi} d\theta_{l'} \prod_{P'} \exp^{-\frac{(Ma)^{d-4}}{g^2}(1-\cos\theta_{P'})} \times \int_{-\pi}^{\pi} d\theta_1 (\det A_{nm}) \exp^{-\frac{(Ma)^{d-4}}{g^2}\sum_{P=1}^{2d-1}(1-\cos\theta_P)}. \quad (2)$$

We have chosen to integrate over the link variable $\theta_{22} = \theta_1$. Here $l' \neq l = 1$ and P' represent the set of plaquettes that does not share the $l = 1$ link. The figure for the plaquette and the definitions of the plaquette variables α_i, $i = 1, \ldots, 4$, change with respect to Ref. 1 as follows (see Fig. 1):

$$\begin{aligned} \alpha_1 &= \theta_{11} + \theta_{22} - \theta_{14} - \theta_{21} = \theta_{22} + \beta_1, \\ \alpha_2 &= \theta_{12} + \theta_{21} - \theta_{13} - \theta_{22} = -\theta_{22} + \beta_2, \\ \alpha_3 &= \theta_{13} + \theta_{24} - \theta_{12} - \theta_{23}, \\ \alpha_4 &= \theta_{14} + \theta_{23} - \theta_{11} - \theta_{24}, \end{aligned} \quad (3)$$

with

$$\beta_1 \equiv \theta_{11} - \theta_{14} - \theta_{21}, \quad (4)$$

and

$$\beta_2 \equiv \theta_{12} + \theta_{21} - \theta_{13} \quad (5)$$

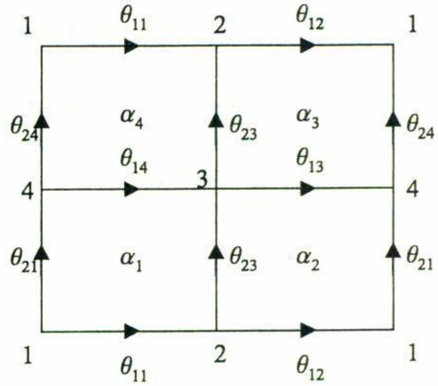

FIGURE 1. Two-dimensional lattice with plaquettes $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

The convention that we adopt for the gamma matrices is

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (6)$$

which fulfiles

$$\begin{aligned} \{\gamma_1, \gamma_2\} &= 0, \\ \gamma_1^2 &= 1, \\ \gamma_2^2 &= 1. \end{aligned} \quad (7)$$

The determinant obtained from (2) is

$$\Delta = \begin{vmatrix} 1 & 0 & 2Ke^{-i\theta_{11}} & 0 & 0 & 0 & c_{17} & c_{18} \\ 0 & 1 & 0 & 2Ke^{i\theta_{12}} & 0 & 0 & c_{27} & c_{28} \\ 2Ke^{-i\theta_{12}} & 0 & 1 & 0 & c_{35} & c_{36} & 0 & 0 \\ 0 & 2Ke^{i\theta_{11}} & 0 & 1 & c_{45} & c_{46} & 0 & 0 \\ 0 & 0 & c_{53} & c_{54} & 1 & 0 & 2Ke^{-i\theta_{13}} & 0 \\ 0 & 0 & c_{63} & c_{64} & 0 & 1 & 0 & 2Ke^{i\theta_{14}} \\ c_{71} & c_{72} & 0 & 0 & 2Ke^{-i\theta_{14}} & 0 & 1 & 0 \\ c_{81} & c_{82} & 0 & 0 & 0 & 2Ke^{i\theta_{13}} & 0 & 1 \end{vmatrix}, \quad (8)$$

where

$$\begin{aligned} &c_{17} = K(e^{-i\theta_{21}} + e^{i\theta_{24}}), && c_{18} = K(e^{-i\theta_{21}} - e^{i\theta_{24}}), && c_{27} = K(e^{-i\theta_{21}} - e^{i\theta_{24}}), && c_{28} = K(e^{-i\theta_{21}} + e^{i\theta_{24}}), \\ &c_{35} = K(e^{-i\theta_{22}} + e^{i\theta_{23}}), && c_{36} = K(e^{-i\theta_{22}} - e^{i\theta_{23}}), && c_{45} = K(e^{-i\theta_{22}} - e^{i\theta_{23}}), && c_{46} = K(e^{-i\theta_{22}} + e^{i\theta_{23}}), \\ &c_{53} = K(e^{-i\theta_{23}} + e^{i\theta_{22}}), && c_{54} = K(e^{-i\theta_{23}} - e^{i\theta_{22}}), && c_{63} = K(e^{-i\theta_{23}} - e^{i\theta_{22}}), && c_{64} = K(e^{-i\theta_{23}} + e^{i\theta_{22}}), \\ &c_{71} = K(e^{-i\theta_{24}} + e^{i\theta_{21}}), && c_{72} = K(e^{-i\theta_{24}} - e^{i\theta_{21}}), && c_{81} = K(e^{-i\theta_{24}} - e^{i\theta_{21}}), && c_{82} = K(e^{-i\theta_{24}} + e^{i\theta_{21}}). \end{aligned} \quad (9)$$

The terms of ($\det A_{\mathrm{nm}}$) which contribute to θ_{22} are displayed as

$$\det A_{\mathrm{nm}} = 1 + 64K^2 + 768K^6 + aK^2 + bK^4 + cK^6 + dK^8, \quad (10)$$

with

$$\mathrm{a} = -8\cos\phi_0, \quad (11)$$

$$\mathrm{b} = -8(\cos\phi_1 - \cos\phi_2 + \cos\phi_3 - \cos\phi_4 + \cos\phi_9 + \cos\phi_{10} + \cos\phi_{11} + \cos\phi_{12}) + 16(\cos\phi_5 + \cos\phi_6 + \cos\phi_7 + \cos\phi_8) + 32(\cos\phi_{13} + \cos\phi_{14}),$$

$$\begin{aligned}
c &= 128(-\cos\phi_0 + \cos\phi_2 + \cos\phi_4 + \cos\phi_9 + \cos\phi_{10} - \cos\phi_{13} + \cos\phi_{14}) \\
&\quad - 32(\cos\phi_{15} + \cos\phi_{16} + \cos\phi_{17} + \cos\phi_{18} + \cos\phi_{19} + \cos\phi_{20} + \cos\phi_{21} + \cos\phi_{22}), \qquad (12)\\
d &= -256(\cos\phi_1 - \cos\phi_2 + \cos\phi_3 - \cos\phi_4 + \cos\phi_9 + \cos\phi_{10} + \cos\phi_{11} + \cos\phi_{12}) + 128(\cos\phi_{13} + \cos\phi_{14}) \\
&\quad + 64(\cos\phi_{19} + cos\phi_{20} + \cos\phi_{21} + \cos\phi_{22} - \cos\phi_{23} - \cos\phi_{24} - \cos\phi_{25} - \cos\phi_{26}) \\
&\quad + 32(\cos\phi_{27} + \cos\phi_{28} + \cos\phi_{29} + \cos\phi_{30} + \cos\phi_{31} + \cos\phi_{32} + \cos\phi_{33} + \cos\phi_{34}) + 512\cos\phi_{35},
\end{aligned}$$

and

$$\begin{aligned}
&\phi_0 = \theta_{22} + \theta_{23}, && \phi_1 = \theta_{11} + \theta_{13} - \theta_{21} + \theta_{22}, && \phi_2 = -\theta_{12} + \theta_{13} - \theta_{21} + \theta_{22},\\
&\phi_3 = -\theta_{12} - \theta_{14} - \theta_{21} + \theta_{22}, && \phi_4 = \theta_{11} - \theta_{14} - \theta_{21} + \theta_{22}, && \phi_5 = \theta_{11} + \theta_{12} + \theta_{22} + \theta_{23},\\
&\phi_6 = -\theta_{11} - \theta_{12} + \theta_{22} + \theta_{23}, && \phi_7 = \theta_{13} + \theta_{14} + \theta_{22} + \theta_{23}, && \phi_8 = -\theta_{13} - \theta_{14} + \theta_{23} + \theta_{22},\\
&\phi_9 = \theta_{11} + \theta_{13} + \theta_{22} + \theta_{24}, && \phi_{10} = -\theta_{12} + \theta_{13} + \theta_{22} + \theta_{24}, && \phi_{11} = \theta_{11} - \theta_{14} + \theta_{22} + \theta_{24},\\
&\phi_{12} = -\theta_{12} - \theta_{14} + \theta_{22} + \theta_{24}, && \phi_{13} = \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24}, && \phi_{14} = -\theta_{21} + \theta_{22} + \theta_{23} - \theta_{24},
\end{aligned}$$

$$\begin{aligned}
&\phi_{15} = \theta_{11} + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{22} + \theta_{23}, && \phi_{16} = -\theta_{11} - \theta_{12} + \theta_{13} + \theta_{14} + \theta_{22} + \theta_{23},\\
&\phi_{17} = \theta_{11} + \theta_{12} - \theta_{13} - \theta_{14} + \theta_{22} + \theta_{23}, && \phi_{18} = -\theta_{11} - \theta_{12} - \theta_{13} - \theta_{14} + \theta_{22} + \theta_{23},\\
&\phi_{19} = \theta_{11} + \theta_{12} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24}, && \phi_{20} = -\theta_{11} - \theta_{12} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24},\\
&\phi_{21} = \theta_{13} + \theta_{14} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24}, && \phi_{22} = -\theta_{13} - \theta_{14} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24},\\
&\phi_{23} = -\theta_{11} - \theta_{12} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24}, && \phi_{24} = \theta_{11} + \theta_{12} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24},\\
&\phi_{25} = -\theta_{13} - \theta_{14} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24}, && \phi_{26} = \theta_{13} + \theta_{14} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24},\\
&\phi_{27} = \theta_{11} + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24}, && \phi_{28} = -\theta_{11} - \theta_{12} + \theta_{13} + \theta_{14} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24},\\
&\phi_{29} = \theta_{11} + \theta_{12} - \theta_{13} - \theta_{14} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24}, && \phi_{30} = -\theta_{11} - \theta_{12} - \theta_{13} - \theta_{14} + \theta_{21} + \theta_{22} + \theta_{23} + \theta_{24},\\
&\phi_{31} = -\theta_{11} - \theta_{12} - \theta_{13} - \theta_{14} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24}, && \phi_{32} = \theta_{11} + \theta_{12} - \theta_{13} - \theta_{14} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24},\\
&\phi_{33} = -\theta_{11} - \theta_{12} + \theta_{13} + \theta_{14} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24}, && \phi_{34} = \theta_{11} + \theta_{12} + \theta_{13} + \theta_{14} - \theta_{21} + \theta_{22} + \theta_{23} - \theta_{24},\\
&\phi_{35} = \theta_{11} - \theta_{12} + \theta_{13} - \theta_{14} - \theta_{21} + \theta_{22} - \theta_{23} + \theta_{24}.
\end{aligned}$$

Although there exist so many terms in Eq. (13), we shall see in next section that they can be managed without difficulty to give us the invariance that we are seeking.

3. Testing gauge invariance

Now we proceed to verify the gauge invariance of Eq. (13), and therefore of the partition function (2).

As we can see from (3), α_3 and α_4 do not contribute to the integrand over θ_1, since θ_{22} does not appear there, but α_1 and α_2 do contain θ_1. Now, for $d = 2$, (2) becomes

$$Z_2 = \int_{-\pi}^{\pi} d\theta_1 (\det A_{nm}) \exp^{-\frac{K'}{2}(\alpha_1^2+\alpha_2^2)}, \qquad (13)$$

where we have omitted the first integral there and K' is defined in Ref. 1. Furthermore, we made the approximations

$$\begin{aligned}
\cos\theta_{P_1} &= \cos\alpha_1 \simeq 1 - \frac{1}{2}\alpha_1^2,\\
\cos\theta_{P_2} &= \cos\alpha_2 \simeq 1 - \frac{1}{2}\alpha_2^2. \qquad (14)
\end{aligned}$$

Rearranging (14) and taking the continuum $\lim a \to 0$, one has

$$Z_2 = \int_{-\infty}^{\infty} d\theta_1' (\det A_{nm}) \exp^{-K'[2\theta_1'^2 + \frac{1}{2}(\beta_1+\beta_2)^2]}, \qquad (15)$$

with

$$\theta_1' = \theta_1 - \frac{1}{2}(\beta_1 + \beta_2). \qquad (16)$$

In this work we further improve with respect to Ref. 1 by testing the gauge invariance of the remaining terms which even if they do not contribute to the integral (14) they, however, arise directly from (13). In fact, let us consider the first term different from constant of (10), which is Eq. (11):

$$\begin{aligned}
\cos(\theta_{22} + \theta_{23}) &= \cos(\alpha_1 + \delta_0),\\
\delta_0 &= \theta_{14} + \theta_{21} - \theta_{11}. \qquad (17)
\end{aligned}$$

We require local gauge invariance of θ_P (or α) under

$$\theta_{\mu n} \to \theta_{\mu n} + g[\Lambda(n+\mu) - \Lambda(n)], \tag{18}$$

where $\Lambda(n)$ is an arbitrary function defined on the lattice sites n. Then each $\theta_{\mu n}$ of Fig. 1 changes as

$$\begin{aligned}
\theta_{11} &\to \theta_{11} + g[\Lambda(2) - \Lambda(1)], \\
\theta_{12} &\to \theta_{12} + g[\Lambda(1) - \Lambda(2)], \\
\theta_{13} &\to \theta_{13} + g[\Lambda(4) - \Lambda(3)], \\
\theta_{14} &\to \theta_{14} + g[\Lambda(3) - \Lambda(4)], \\
\theta_{21} &\to \theta_{21} + g[\Lambda(4) - \Lambda(1)], \\
\theta_{22} &\to \theta_{22} + g[\Lambda(3) - \Lambda(2)], \\
\theta_{23} &\to \theta_{23} + g[\Lambda(2) - \Lambda(3)], \\
\theta_{24} &\to \theta_{24} + g[\Lambda(1) - \Lambda(4)].
\end{aligned} \tag{19}$$

From here one confirms easily that δ_0 is invariant under (19). $I^0_{z_2}$ becomes in turn:

$$I^0_{z_2} = \exp\Big\{[-\frac{1}{2}K'(\beta_1+\beta_2)^2][-N(\beta_1+\beta_2)\sin\delta_0 + (2N - M - \frac{1}{4}(\beta_1+\beta_2)^2)\cos\delta_0]\Big\}, \tag{20}$$

where

$$N = \int_0^\infty d\theta_1'\, e^{-2K'\theta_1'^2} = \frac{1}{2}\sqrt{\frac{\pi}{2K'}}, \tag{21}$$

and

$$M = \int_0^\infty d\theta_1'\, e^{-2K'\theta_1'^2}\theta_1'^2 = \frac{1}{4}\sqrt{\frac{\pi}{(2K')^3}}. \tag{22}$$

Since $\beta_1 + \beta_2$ results to be gauge invariant, as well as δ_0, one concludes that $I^0_{Z_2}$ also satisfies such property.

Similarly

$$\cos(\theta_{22} - \theta_{21} + \theta_{13} + \theta_{11}) = \cos(\alpha_1 + \delta_a), \tag{23}$$

with $\delta_a = \alpha_4 - \theta_{23} - \theta_{24}$ invariant under (19), as well as

$$I^1_{z_2} = \exp^{2[-\frac{K'}{2}(\beta_1+\beta_2)^2][N-M+\frac{1}{2}N(\beta_1+\beta_2)^2]}. \tag{24}$$

Proceeding in the same way we obtain that the other terms

$$\cos(\theta_{22} - \theta_{21} - \theta_{14} - \theta_{12}) = \cos(\alpha_1 - \delta_3),$$
$$\delta_3 = \theta_{11} + \theta_{12}, \tag{25}$$

etc., are gauge invariant.

Notice that

$$\cos(\theta_{22} + \theta_{23}) = \cos\theta_{22}\cos\theta_{23} - \sin\theta_{22}\sin\theta_{23} \tag{26}$$

is not convenient to be integrated directly for our purposes, since neither θ_{23} nor the remaining integral, which is proportional to $(\beta_1 - \beta_2)^2$, are gauge invariant; nevertheless the whole term [right hand side of Eq. (18)] has the required invariance as we saw already. So we have shown that all the terms (13) that contribute to (2) are gauge invariant and then the partition function satisfies the same property. One also realizes that since $\cos\alpha_1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha_1)^{2n}$ and $\sin\alpha_1 = \sum_{n=o}^{\infty}\frac{(-1)^n}{(2n+1)!}(\alpha_1)^{2n+1}$, these expressions are gauge invariant in general.

4. Conclusions

We have found that the effective continuum action for fermions on a lattice with electromagnetic field is gauge invariant. This result, which also means that the Itô terms are not necessary in the continuum action, could be expected since we have in first instance a flat manifold, but the insertion of fermions leave out such expectations. Besides, we have decided to perform the calculations explicitly at least for the 2-dimensional case. For larger dimensions one must invest a lot of computational work. In fact, we have intended to perform the calculation of the 16×16 determinant, which is the $d = 4$ case following Ref. 1 but with negative results. The computational time employed was about 2–3 weeks, continuously, using a work station and a Mathematica program. Since our purpose is simpler, just to show the mentioned invariance, and because it was hoped to confirm it, we preferred to follow the way outlined in this paper.

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