Rotating frames, electrodynamics and Finsler's geometry

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We find a close analogy between electrodynamics and the non-relativistic treatment of the motion of a particle in rotating frames. We then pass to the special relativistic case and propose a Lagrangian containing a skewsymmetric field $w^{\mu\nu}$. We find that in the relativistic case the analogy previously mentioned is broken. Finally we show that the Hamilton principle as applied to classical particles in motion in rotating frames is equivalent to the choice of geodesics in a Finsler geometry.

Keywords: Rotating frames; Finsler spaces; classical fields

Existe una estrecha analogía entre la electrodinámica y el tratamiento no relativista del movimiento de partículas en sistemas rotantes. Esta analogía se rompe en el caso relativista, en el que es necesario proponer un lagrangiano que contiene un campo antisimétrico $w^{\mu\nu}$. Finalmente demostramos que el principio de Hamilton aplicado al movimiento de partículas en sistemas rotantes es equivalente a la existencia de geodésicas en un espacio de Finsler.

Descriptores: Sistemas rotantes; espacios de Finsler; campos clásicos

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1. Introduction

In general relativity it is possible to describe the motion of a particle under fictitious forces as due to the peculiar shape of the geodesics in a Riemannian space. In particular, for a frame that rotates in flat 4-space, we can show that the 3-space is not flat and that the time is not uniform. In this case the centrifugal and the Coriolis actions are accounted for by adopting the prescriptions $g_{00} = 1-w^2r^2/c^2$, $g_{02} = -wr^2/c$, $g_{11} = g_{22} = g_{33} = -1$.

This prescription is not the only way, as this paper attemps to show. There is a formal analogy between the Lorentz force and the fictitious forces in a rotating frame. We explore this analogy and propose a covariant Lagrangian that describes the fictitious forces in the context of special relativity. The motion of a particle in rotating frame is thus seen as the action of certain field A_μ in an inertial frame.

This new description leads us to the following proposition: Since Hamilton's principle is, in this case, equivalent to a geodesic condition, the motion of the particle, as registered by a rotating frame, appears as geodetical motion in a Finsler space, where $g_{\mu\nu} = \eta_{\mu\nu}(1+\phi/c^2)^2$, with $\phi = A_\mu A^\mu/2$.

2. Notation and conventions

Throughout this paper space-time points will be associated with contravariant components [4]

$$x^\mu = (x^0, \mathbf{r}) = (ct, x, y, z).$$

The differential line element is

$$dS^2 = dx_\mu dx^\mu = g_{\mu\nu} dx^\mu dx^\nu$$

with

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1; \quad g_{\mu\nu} = 0$$

for $\mu \neq \nu$.Thus: $x_\mu = g_{\mu\nu}x^\nu = (ct, -\mathbf{r})$, and

$$\partial^\mu = (\partial/c\partial t, -\nabla),$$
$$\partial_\mu = (\partial/c\partial t, \nabla).$$

Also $\delta^\mu_\nu = 1$for $\mu = \nu$, and zero for $\mu \neq \nu$. Summation convention is valid for repeated indices:

$$\mu, \nu \cdots = 0, 1, 2, 3; \quad i, j = 1, 2, 3.$$

4-velocity is defined by:

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \mathbf{v}), \; \gamma = (1 - \frac{v^2}{c^2})^{-1/2}, \; d\tau = \frac{dt}{\gamma}.$$

Finally:

$$\epsilon^{ijk0} = \epsilon^{ijk}, \quad \epsilon^{1230} = 1.$$

3. The non-relativistic case

In classical, non-relativistic mechanics the motion of a free particle as seen from a rotating frame [3] is described by

$$\mathbf{a} = -\mathbf{w} \times (\mathbf{w} \times \mathbf{r}) - 2\mathbf{w} \times \mathbf{v} - \frac{d\mathbf{w}}{dt} \times \mathbf{r} \tag{1}$$

or

$$\mathbf{F} = m\Big[-\mathbf{w} \times (\mathbf{w} \times \mathbf{r}) + \mathbf{v} \times (2\mathbf{w}) - \frac{\partial}{\partial t}(\mathbf{w} \times \mathbf{r})\Big], \tag{2}$$

where we take in consideration that $d\mathbf{w}/dt = \partial\mathbf{w}/\partial t$, as $\nabla\mathbf{w} = 0$ and $\partial\mathbf{r}/\partial t = 0$. With the definitions

$$\mathbf{A} = \mathbf{w} \times \mathbf{r}, \tag{3}$$

$$\phi = -\frac{\mathbf{A}\cdot\mathbf{A}}{2} = -\frac{A^2}{2}, \tag{4}$$

we have

$$\nabla\phi = \mathbf{w} \times (\mathbf{w} \times \mathbf{r}),$$

$$\nabla \times \mathbf{A} = 2\mathbf{w}.$$

Thus

$$\mathbf{F} = m\Big[-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\Big]. \tag{5}$$

Now, using the shorthand definitions

$$\mathbf{g} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} = -\mathbf{w} \times (\mathbf{w} \times \mathbf{r}) - \frac{\partial}{\partial t}(\mathbf{w} \times \mathbf{r}), \tag{6}$$

$$\mathbf{G} = \nabla \times \mathbf{A} = 2\mathbf{w}, \tag{7}$$

we have

$$\mathbf{F} = m\left[\mathbf{g} + \mathbf{v} \times \mathbf{G}\right]. \tag{8}$$

This approach formally reduces to the Lorentz force well known in electrodynamics. In our case, both $\mathbf{A}$ and ϕ, are expressed in terms of the "field" quantity $\mathbf{w}$. In fact ϕ and $\mathbf{A}$ are not independient fields due to (4).

Without any mention of inertial frames (except by the fact that $\mathbf{w}$ is measured relative to them) and working the whole problem in the rotating frame we may propose a corresponding Lagrangian, as seen from the rotating frame as

$$\begin{aligned} L &= \frac{1}{2}mv^2 + m\mathbf{v}\cdot(\mathbf{w} \times \mathbf{r}) + \frac{m}{2}(\mathbf{w} \times \mathbf{r})^2 \\ &= \frac{1}{2}mv^2 + m\mathbf{v}\cdot\mathbf{A} - m\phi. \end{aligned} \tag{9}$$

The potential ϕ describes centrifugal effects, $\partial\mathbf{A}/\partial t$ (corresponding in electrodynamics to Faraday effect) describes in our case inertial forces due to $\dot{\mathbf{w}}$. The term $\mathbf{v}\cdot\mathbf{A}$ is responsible for Coriolis force. Let's note that $\mathbf{w}$ depends just on time so that any variation will be "propagated" instantaneously. This behavior is that of an "action at a distance" theory.

Now, from (6) and (7) it follows that $\mathbf{g}$ and $\mathbf{G}$, the centrifugal and Coriolis fields, satisfy the equations

$$\nabla\cdot\mathbf{G} = 0, \tag{10}$$

$$\nabla \times \mathbf{g} + \frac{\partial\mathbf{G}}{\partial t} = 0, \tag{11}$$

and, from (3), (4), (6) and (7),

$$\nabla\cdot\mathbf{g} = 2w^2, \tag{12}$$

$$\nabla \times \mathbf{G} = 0. \tag{13}$$

Equation (10) corresponds in magnetism to the vanishing of magnetic flux, (11) corresponds to "the Faraday effect"; according to (12) the rotation is the "source" of centrifugal field, and (13) may be generalized to

$$\nabla \times \mathbf{G} - \frac{1}{\alpha^2}\frac{\partial\mathbf{g}}{\partial t} = 0, \tag{14}$$

α being a quantity with units of velocity. As $\partial\mathbf{g}/\partial t \neq 0$ in general, in our "action at a distance" theory we have $\alpha \to \infty$ in order to satisfy (13): rotation is propagated instantaneously.

Let's note, from (6), (7), (10), (11) that

$$\nabla^2\phi = -2w^2,$$

$$\nabla\cdot\mathbf{A} = 0.$$

Now, by using (9) the canonical momentum for a particle is

$$\begin{aligned} \mathbf{p} &= m\mathbf{v} + m\mathbf{A} \\ &= m\mathbf{v} + m\mathbf{w} \times \mathbf{r}, \end{aligned} \tag{15}$$

and the canonical energy can be calculated from (2) by scalar product with $\mathbf{v}$; we get

$$m\frac{d}{dt}\left(\frac{v^2}{2} + \phi\right) = -\mathbf{v}\cdot\frac{\partial\mathbf{A}}{\partial t}, \tag{16}$$

so that

$$E = \frac{mv^2}{2} + m\phi \tag{17}$$

as in the electromagnetic case. Energy, given by (17), is conserved if $\mathbf{A} = 0$ (*i.e.* if $\mathbf{w} = 0$).

4. The relativistic case

The preceding notes show that it is possible in the Newtonian approach to establish an analogy between inertial and Lorentz forces. However, as it will be clear later, this analogy is broken down in the relativistic version of the problem.

In the following lines we will write a covariant Lagrangian by assuming that inertial effects correspond to the existence, in an inertial frame, of a skewsymmetric field $w^{\mu\nu}$ satisfying the requirements of special relativity and without any appeal to general covariance.

We propose

$$L = mc(1 + \phi/c^2)\sqrt{u_\mu u^\mu} + mA_\mu u^\mu, \tag{18}$$

with

$$A_\mu = w_{\nu\mu}x^\nu, \tag{19}$$

$$\phi = \frac{A_\sigma A^\sigma}{2}. \tag{20}$$

Lagrange's equation of motion is

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^\sigma}\right) - \frac{\partial L}{\partial x^\sigma} = 0,$$

where τ is the proper time. This equation leads to

$$\frac{d}{d\tau}\left[m\left(1+\frac{\phi}{c^2}\right)u^\sigma\right] = m\phi^{\sigma\mu}u_\mu + m\partial^\sigma\phi, \tag{21}$$

where $\phi_{\sigma\mu}$ is defined by

$$\phi_{\sigma\mu} = \partial_\mu A_\sigma - \partial_\sigma A_\mu. \tag{22}$$

Equation (22) shows that it is possible to introduce a skewsymmetric tensor analogous to the electromagnetic field in special relativity.

The specific form of the Lagrangian in (18) has been chosen in such a way that the non-relativistic limit, [Eq. (5)], is obtained. In fact, from Eq. (21) for $v/c \ll 1$, and using the following conditions and definitions:

- $\phi^{ij} = -\epsilon^{ijk}G_k = \partial^i A^j - \partial^j A^i$ or $\mathbf{G} = \nabla\times\mathbf{A}$,
- $w^{i0} = 0$,
- $\phi^{i0} = \partial^i A^0 - \partial^0 A^i = -(\nabla A^0)_i - \frac{1}{c}\frac{\partial A^i}{\partial t} = -\frac{1}{c}\dot{A}^i$,
- $\phi = \frac{1}{2}[(A^0)^2 - (\mathbf{A})^2] = -\frac{\mathbf{A}}{2}$,

we obtain

$$\frac{d}{dt}(mv^i) = m(-\phi^{ij}v^j + \phi^{i0}u^0) - (\nabla\phi)_i$$

$$= m\epsilon^{ijk}v^j G_k - \frac{m}{c}\frac{\partial A^i}{\partial t}u^0 - m(\nabla\phi)_i,$$

or

$$\mathbf{F} = m\left[\mathbf{v}\times\mathbf{G} - \frac{\partial\mathbf{A}}{\partial t} - \nabla\phi\right] = m[\mathbf{g} + \mathbf{v}\times\mathbf{G}]$$

as in (8).

In order to arrive to this result we have assumed $w^{i0} = 0$. This condition may be expressed in covariant form as

$$w^{\mu\nu} = -\epsilon^{\mu\nu\sigma\rho}w_\sigma\frac{U_\rho}{c}, \tag{23}$$

where w_σ is the angular velocity 4-vector and U_ρ is the 4-velocity of the observer relative to the origin of the rotating frame. In standard notation, (23) is writen

$$w^{i0} = -\frac{(\mathbf{w}\times\mathbf{U})_i}{c},$$

$$w^{ij} = \frac{\epsilon^{ijk}}{c}\left(w^k U^0 - w^0 U^k\right).$$

In the simplest and usual case the observer is at rest in the origin so that

$$w^{i0} = 0,$$

$$w^{ij} = \epsilon^{ijk}w^k,$$

w^k being the dual of w^{ij} in 3-space; in an equivalent form, if the observer is at rest in the rotating frame (let's remember that we are in an inertial frame with a field A^μ): $U_\sigma = c\delta_{\sigma 0}$, such that

$$w^{\mu\nu} = -\epsilon^{\mu\nu\rho 0}w_\sigma,$$

thus $w^{i0} = 0$.

Now, from Eq. (18) the canonical 4-momentum is

$$p_\sigma = \frac{\partial L}{\partial u^\sigma} = m\left[\left(1+\frac{\phi}{c^2}\right)u_\sigma + A_\sigma\right],$$

whose space components are

$$\mathbf{p} = m\left(1+\frac{\phi}{c^2}\right)\gamma\mathbf{v} + m\mathbf{A}, \tag{24}$$

the non-relativistic limit coincides with (15), the term ϕ/c^2 corresponding to increase of mass due to potential energy.

And from $p^0 = E/c$:

$$E = \left(mc^2 + m\phi\right)\gamma,$$

with newtonian limit

$$E \simeq \left(mc^2 + m\phi\right)\left(1+\frac{v^2}{2c^2}\right)$$

$$\simeq mc^2 + \frac{mv^2}{2} + m\phi$$

as in (17) except by the rest mass.

Using (23), Eqs. (19) and (20) take the form

$$A_\sigma = w_{\nu\sigma}x^\nu = -\epsilon_{\nu\sigma\alpha\beta}w^\alpha U^\beta\frac{x^\nu}{c},$$

$$\phi = \frac{A_\sigma A^\sigma}{2} = \frac{1}{2c^2}\epsilon_{\nu\sigma\alpha\beta}\epsilon^{\gamma\sigma\phi\delta}w^\alpha w_\phi U^\beta U_\delta x^\nu x_\gamma.$$

5. Rotation and Finsler's spaces

According to Hamilton's principle [3]

$$\delta\int L\,d\tau = 0,$$

we may write, according to (18),

$$\delta\int\left[mc\left(1+\frac{\phi}{c^2}\right)\sqrt{dx_\nu\,dx^\nu} + mA_\nu\,dx^\nu\right]d\tau = 0,$$

and this equation may also be writen as a geodesic condition [1]:

$$\delta\int dS = 0$$

with

$$dS = \left(1+\frac{\phi}{c^2}\right)\sqrt{dx_\nu\,dx^\nu} + \frac{A_\nu}{c}dx^\nu \tag{25}$$

or

$$dS = \sqrt{g_{\mu\nu}\,dx^\mu\,dx^\nu} + \frac{A_\nu}{c}\,dx^\nu,$$

with $g_{\mu\nu} = (1+\phi/c^2)^2\,\eta_{\mu\nu}$, $\eta_{\mu\nu}$being the Lorentz metric.

We conclude that studying the motion of a particle in rotating frames by using special relativity and without any assistance of the ideas of general relativity, is equivalent to establishing a specific generalization of a Finsler's space with a line element given by (25).

6. Formal developments

1. Equation (21) may also be expressed in the form

$$K^\sigma = m\left[\phi^{\sigma\mu} + \frac{1}{c^2}\left(\partial^\sigma \phi\, u^\mu - \partial^\mu \phi\, u^\sigma\right)\right] u_\mu - \frac{m}{c^2}\phi\, \dot{u}^\sigma$$
$$= m\widehat{\phi}^{\sigma\mu}\, u_\mu - \frac{m}{c^2}\,\phi\, \dot{u}^\sigma, \qquad (26)$$

with

$$K^\sigma = \frac{d}{d\tau}\left(mu^\sigma\right), \qquad (27)$$

$$\widehat{\phi}^{\sigma\mu} = \phi^{\sigma\mu} + \frac{1}{c^2}\left(\partial^\sigma \phi u^\mu - \partial^\mu \phi u^\sigma\right) \qquad (28)$$

It is easy to prove that: $K^\sigma u_\sigma = 0$ as stated by special relativity.

Equation (26) shows that the analogy between inertial forces and Lorentz forces is broken down in the relativistic case, but maintains its validity in the newtonian limit where, according to (25), (26), (27):

$$K^\sigma = m\phi^{\sigma\mu}\, u_\mu\,.$$

We may in the general case (26) define $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{G}}$ as

$$\widehat{\phi}^{ij} = -\epsilon^{ijk}\widehat{G}_k\,,$$
$$\widehat{\phi}^{i0} = \frac{1}{c}\widehat{g}_i$$

with

$$\widehat{\mathbf{G}} = \nabla\times\mathbf{A} + \frac{1}{c^2}\nabla\phi\times\mathbf{v}\,,$$
$$\widehat{\mathbf{g}} = -\gamma\,\nabla\phi - \dot{\mathbf{A}} - \gamma\mathbf{v}\frac{\dot{\phi}}{c^2}$$

so that

$$\mathbf{K} = \gamma\mathbf{F} = m\gamma\left[\widehat{\mathbf{g}} + \mathbf{v}\times\widehat{\mathbf{G}}\right] - \frac{m}{c^2}\phi\dot{\mathbf{v}}$$
$$= m\gamma\left[\widehat{\mathbf{g}} + \mathbf{v}\times\widehat{\mathbf{G}}\right] - \frac{m}{c^2}\phi\left[\dot{\mathbf{v}}\gamma^2 + \frac{\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2}\gamma^4\right]$$

2. From (22) we get by derivation:

$$\partial^\nu\phi^{\sigma\mu} + \partial^\mu\phi^{\nu\sigma} + \partial^\sigma\phi^{\mu\nu} = 0;$$

in $3+1$ notation:

$$\nabla\cdot\mathbf{G} = 0$$
$$\nabla\times\mathbf{g} + \frac{\partial\mathbf{G}}{\partial t} = 0$$

7. Conclusions

The motion of a particle in a rotating frame can be described as a geodesic motion in a Finsler space, with $g_{\mu\nu} = \eta_{\mu\nu}(1+\phi/c^2)^2$. According to general relativity [1] the motion corresponds to a geodesic in a Riemannian space with $g_{00} = 1 - w^2r^2/c^2, g_{02} = -wr^2/c$, $g_{11} = g_{33} = -1$. In both approaches the Newtonian limits coincide.

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