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An analytical expression for the singularities developed by an aberrated wavefront

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Since the expressions of the components of the aberration represent a map from $\mathcal{R}^3$ to $\mathcal{R}^3$, in this work we apply the theory of singularities of differentiable maps to obtain an analytical expression of the caustic of the propagation of an aberrated wavefront.

Keywords: Wavefronts; caustics; singularities of wavefronts

Puesto que las expresiones de las componentes de la aberración representan un mapeo de $\mathcal{R}^3$ a $\mathcal{R}^3$, en este trabajo aplicamos la teoría de singularidades de mapeos diferenciables para obtener una expresión analítica de la cáustica de la propagación de un frente de onda aberrado.

Descriptores: Frentes de onda; caústica; singularidades de frentes de onda

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1. Introduction

The computation of the singularities developed by the evolution of a wavefront in a medium has many important and interesting applications in physics. For example: in astrophysics to perform some measurements and predictions it is important to know the singularities of the past light cone (a three-surface in the spacetime) of the observer [1] and in optics the singularities are very closely related to the defects of the optical systems. Actually, the *caustic test* [2] is based in the knowledge of the caustic. To compute the circle of least confusion of a rotationally symmetric mirror, when the point source is located on the optical axis, it is necessary to compute the caustic of the evolution of an aberrated wavefront [3].

Since the caustic set is interpreted as the location of focusing regions, where the cross-sectional area of the bundle of the light rays collapses to zero, which leads to an increase in intensity; then it is important to know the caustic because it is the place where the geometric optics limit is not applicable. From a mathematical point of view the study of the singularities of the propagation of electromagnetic wavefronts, in the high-frequency limit has importance because they have been an illustration of Arnold's theory of Lagrangian and Legendre maps [4–6].

In the present paper, under certain approximations and by using a simple procedure, we obtain an analytical expression for the singularities developed by the evolution of a wavefront which has passed throughout an optical system which produces aberrations on it. Thus in Sec. 2 we first review the concepts of aberration of a ray, wave aberration function and the derivation of the expressions for the ray aberration components. Furthermore, we give the definitions of critical and caustic sets of a three-dimensional map and we apply it to our problem. Although our result can be applied to any kind of aberration, in Sec. 3, we apply it to compute the caustic when the wavefront is affected by spherical aberration.

2. Computation of the caustic of an aberrated wavefront

In optics one starts with the wave equation

$$\nabla^2\phi(x,y,z) = \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2}, \tag{1}$$

where n is the index of refraction and c is the speed of light in vacuum; whose solution can be written as

$$\phi(x,y,z,t) = e^{a(x,y,z)}e^{ik[s(x,y,z)-ct]}, \tag{2}$$

where $a(x,y,z)$ and $s(x,y,z)$ are real functions. Substituting Eq. (2) into Eq. (1), one obtains that the real part of the resulting equation is given by

$$(\nabla S)^2 - n^2 = \frac{\lambda^2}{4\pi^2}[\nabla^2 a + (\nabla a)^2]. \tag{3}$$

In the limit of high-frequency, *i.e.*, $\lambda \to 0$ one obtains the basic eikonal equation of geometrical optics

$$(\nabla S)^2 = n^2. \tag{4}$$

Therefore surfaces of constant S are surfaces of constant optical phase, and thus they define the wavefronts. Furthermore, the ray trayectories are normal to the wavefronts.

Consider an initial spherical wavefront of radius R_0 at $t=0$, given parametrically by $x_0 = R_0\cos\varphi\sin\theta$, $y_0 = R_0\sin\varphi\sin\theta$, and $z_0 = R_0\cos\theta$. To find the form of the wavefront when it is contracting, in the empty space with velocity c, after a period of time t, we define the following function $f(x_0,y_0,z_0) = x_0^2+y_0^2+z_0^2-R_0^2$ and we construct the unit normal vector field to the surface $f(x_0,y_0,z_0)=0$. An easy computation shows that the cartesian components of the unit normal vector field are given by $n_{x_0} = x_0/R_0$, $n_{y_0} = y_0/R_0$ and $n_{z_0} = z_0/R_0$. Therefore, the form of the

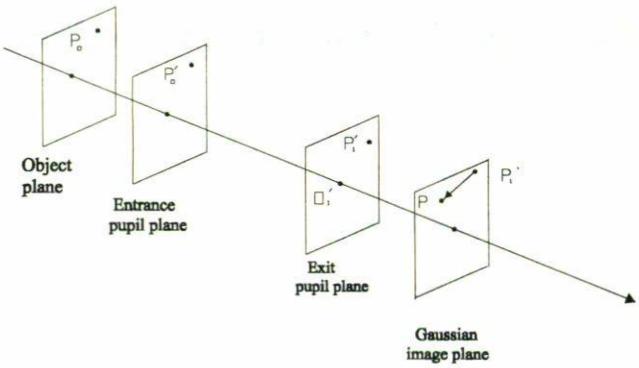


FIGURE 1. Symbolic representation of the optical system by four planes representing the objet plane, the pupil planes and the image plane. The vector from P_1 to P is the aberration of the ray.

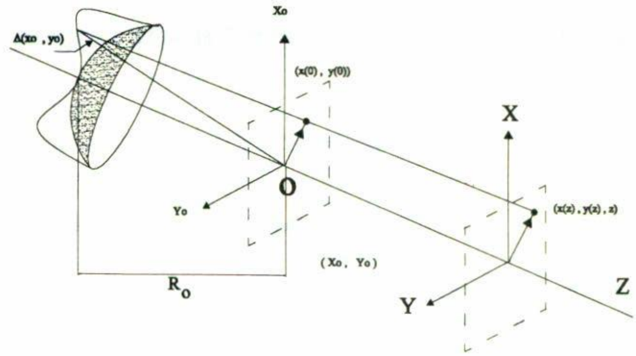


FIGURE 2. The ideal wavefront of radius R_0 and the real wavefront. $\Delta(x_0, y_0)$ represents the deviation of the real wavefront from the spherical one, which is measured along the normal to the reference sphere. (x_0, y_0, z) are the coordinates of the domain space and (X, Y, Z) are the coordinates of the target space. The coordinates of the points on the real wavefront are $[x_0, y_0, z_0(x_0, y_0)]$ and the coordinates of the Gaussian image point of P_0, *i.e.*, the coordinates of the point P_1 are $(0, 0, 0)$.

wavefront will be given by

$$x(t) = x_0 - ctn_{x_0} = (R_0 - ct)\cos\varphi\sin\theta,$$

$$y(t) = y_0 - ctn_{y_0} = (R_0 - ct)\sin\varphi\sin\theta$$

and

$$z(t) = z_0 - ctn_{z_0} = (R_0 - ct)\cos\theta$$

Observe that at $t = R_0/c$ the wavefront collapses to a point. And after that it reappears on the other side of the sphere. So we get an eversion of the sphere. The collapsing point is referred to as the singularity developed by the spherical wavefront and is non-generic in the sense that under a small deformation of the initial spherical wavefront it decomposes into more complicated singularities, which have been locally classified [4–6]. Since $(z - z_0)/n_{z_0} = -ct$ then the wavefront can be written as

$$x = x_0 + (z - z_0)\frac{n_{x_0}}{n_{z_0}} = z\cos\varphi\tan\theta,$$

$$y = y_0 + (z - z_0)\frac{n_{y_0}}{n_{z_0}} = z\sin\varphi\tan\theta,$$

$$z = z. \tag{5}$$

An interesting deformation of the initial spherical wavefront is that obtained by performing the following transformation:

$$R_0 \to R_0 + \Delta(\theta, \varphi), \tag{6}$$

where $\Delta(\theta, \varphi)$ is a regular function on the (θ, φ) variables.

Now we give the definitions of aberration of a ray, wave aberration function and the derivation of the expressions for the ray aberration components (for a detailed explanation see Ref. 7). Consider a rotationally symmetric optical system, let P_0 be an object point and let P_0', P_1' and P be the points in which a ray from P_0 intersects the plane of the entrance pupil, the exit pupil and the Gaussian image plane respectively (see Fig. 1). If P_1 is the Gaussian image of P_0, then the vector from P_1 to P is called the aberration of the ray, or simply the ray aberration. Let W be the wavefront through the centre O_1 of the exit pupil, associated with the image-forming pencil which reaches the image space from P_0. In the absence of aberrations, W coincides with a spherical wavefront which is centered on P_1 and which passes through O_1'. The spherical wavefront is called the Gaussian reference sphere or the ideal wavefront, W is called the real wavefront or aberrated wavefront and the deviation of the real wavefront from the spherical one measured along the normal to the reference sphere is the wave aberration function. If we choose the coordinate system shown in Fig. 2 and we denote the wave aberration function by $\Delta(x_0, y_0)$, then the real wavefront is given by

$$g(x_0, y_0, z_0) = x_0^2 + y_0^2 + z_0^2 - [R_0 + \Delta(x_0, y_0)]^2 = 0; \tag{7}$$

as in the spherical case, explained earlier, as the real wavefront evolves after a period of time t in vacuum space with velocity c it will be given by

$$x(t) = x_0 - ctn_{x_0},$$

$$y(t) = y_0 - ctn_{y_0},$$

$$z(t) = z_0 - ctn_{z_0}, \tag{8}$$

where

$$(n_{x_0}, n_{y_0}, n_{z_0}) = \frac{\left(\dfrac{\partial g}{\partial x_0}, \dfrac{\partial g}{\partial y_0}, \dfrac{\partial g}{\partial z_0}\right)}{\mid \nabla g \mid}. \tag{9}$$

From Eqs. (7)–(9) we obtain that

$$\frac{x - x_0}{x_0 - (R_0 + \Delta)\dfrac{\partial \Delta}{\partial x_0}} = \frac{y - y_0}{y_0 - (R_0 + \Delta)\dfrac{\partial \Delta}{\partial y_0}} = \frac{z - z_0}{z_0}, \tag{10}$$

where

$$z_0 = \pm\sqrt{(R_0+\Delta)^2 - x_0^2 - y_0^2}. \quad (11)$$

Therefore, the real wavefront parameterized by z is given by

$$x = x_0 + \left(\frac{z-z_0}{z_0}\right)\left\{x_0 - [R_0 + \Delta(x_0,y_0)]\frac{\partial\Delta(x_0,y_0)}{\partial x_0}\right\},$$
$$y = y_0 + \left(\frac{z-z_0}{z_0}\right)\left\{y_0 - [R_0 + \Delta(x_0,y_0)]\frac{\partial\Delta(x_0,y_0)}{\partial y_0}\right\},$$
$$z = z. \quad (12)$$

[Observe that these equations are equivalent to those obtained from the transformation (6)]. In the Gaussian image plane ($z=0$), Eqs. (12) reduce to the important equations

$$x(z=0) = [R_0 + \Delta(x_0,y_0)]\frac{\partial\Delta(x_0,y_0)}{\partial x_0},$$
$$y(z=0) = [R_0 + \Delta(x_0,y_0)]\frac{\partial\Delta(x_0,y_0)}{\partial y_0}. \quad (13)$$

These equations are the components of the transverse aberration of the ray, *i.e.*, $[x(z=0), y(z=0)]$ are the deviations from the Gaussian image point located at $(0,0,0)$. In what follows we will refer to $[x(x_0,y_0,z), y(x_0,y_0,z)]$ as the components of the transverse aberration for a fixed valued of z. It is important to remark that Eqs. (12) are exact; however, in most applications it is assumed that

$$|\Delta| \ll R_0, \quad z_0 \cong -R_0,$$

and

$$\frac{z}{|z_0|} \cong \frac{z}{R_0} \ll 1. \quad (14)$$

Under these approximations Eqs. (12) reduce to

$$X(x_0,y_0,z) = -\frac{zx_0}{R_0} + R_0\frac{\partial\Delta(x_0,y_0)}{\partial x_0},$$
$$Y(x_0,y_0,z) = -\frac{zy_0}{R_0} + R_0\frac{\partial\Delta(x_0,y_0)}{\partial y_0},$$
$$Z(x_0,y_0,z) = z. \quad (15)$$

Observe that from a mathematical point of view, these equations represent a map between two three-dimensional spaces, (x_0,y_0,z) are the coordinates of the domain space and (X,Y,Z) are the coordinates of the target space (see Fig. 2).

Equivalently, Eqs. (15) can be seen as a one-parameter family of two-dimensional maps, with parameter z, each member of the family has the following interpretation: it maps points on the real wavefront with coordinates $(x_0, y_0, z_0(x_0,y_0))$ to points to the plane $Z=$ constant in the target space. To compute the singularites developed by the evolution of the real wavefront we need to introduce the definitions of critical and caustic sets of a map between three-dimensional spaces.

If $g: \mathcal{M} \to \mathcal{N}$ is a map between two differentiable manifolds, all the points in $\mathcal{M}$ such that g is not one-to-one are referred to as its critical set and the image of the critical set is referred to as the caustic set of g [4–6]. If $\mathcal{M}$ and $\mathcal{N}$ are three-dimensional manifolds with local coordinates (x_1,x_2,x_3) and (y_1,y_2,y_3) respectively, then locally g is given by $y_i = g_i(x_j)$, where $i,j = 1,2,3$. Therefore, the critical set is obtained from the following condition

$$J \equiv \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = 0. \quad (16)$$

Applying these definitions to the spherical case we find that the critical set is given by

$$(R_0 - ct)^2 \sin\theta = 0. \quad (17)$$

To obtain this equation from Eq. (16), we took

$$y_1 = (R_0 - ct)\cos\varphi\sin\theta,$$
$$y_2 = (R_0 - ct)\sin\varphi\sin\theta,$$
$$y_3 = (R_0 - ct)\cos\theta,$$
$$x_1 = ct, x_2 = \theta$$

and

$$x_3 = \varphi$$

The image of the critical set is given by the only point (0, 0, 0). This means that if initially we have an spherical wavefront of radius R_0 then as the time evolves the wavefront remains spherical and collapses to one point at $t = R_0/c$. For generic initial two-wavefronts (as those obtained from Eqs. (12) or (15)), locally they develop singularities of cusp ridge and swallowtail types [4–6]. Now we will compute the critical and caustic sets of the map given by Eqs. (15).

A straightforward computation shows that the critical set of the map given by Eqs. (15), is

$$z = \frac{R_0^2}{2}\left[\frac{\partial^2\Delta}{\partial x_0^2} + \frac{\partial^2\Delta}{\partial y_0^2} \pm \sqrt{\left(\frac{\partial^2\Delta}{\partial x_0^2} - \frac{\partial^2\Delta}{\partial y_0^2}\right)^2 + 4\left(\frac{\partial^2\Delta}{\partial x_0\partial y_0}\right)^2}\,\right]. \quad (18)$$

The image of the critical set, that is, the caustic set which is obtained by substituting Eq. (18) into Eqs. (15) is given by

$$X_c = -R_0\left\{\frac{1}{2}\left[\frac{\partial^2\Delta}{\partial x_0^2} + \frac{\partial^2\Delta}{\partial y_0^2} \pm \sqrt{\left(\frac{\partial^2\Delta}{\partial x_0^2} - \frac{\partial^2\Delta}{\partial y_0^2}\right)^2 + 4\left(\frac{\partial^2\Delta}{\partial x_0\partial y_0}\right)^2}\,\right]x_0 - \frac{\partial\Delta}{\partial x_0}\right\},$$

$$Y_c = -R_0 \left\{ \frac{1}{2} \left[\frac{\partial^2 \Delta}{\partial x_0^2} + \frac{\partial^2 \Delta}{\partial y_0^2} \pm \sqrt{\left(\frac{\partial^2 \Delta}{\partial x_0^2} - \frac{\partial^2 \Delta}{\partial y_0^2} \right)^2 + 4 \left(\frac{\partial^2 \Delta}{\partial x_0 \partial y_0} \right)^2} \right] y_0 - \frac{\partial \Delta}{\partial y_0} \right\},$$

$$Z_c = \frac{R_0^2}{2} \left[\frac{\partial^2 \Delta}{\partial x_0^2} + \frac{\partial^2 \Delta}{\partial y_0^2} \pm \sqrt{\left(\frac{\partial^2 \Delta}{\partial x_0^2} - \frac{\partial^2 \Delta}{\partial y_0^2} \right)^2 + 4 \left(\frac{\partial^2 \Delta}{\partial x_0 \partial y_0} \right)^2} \right]. \quad (19)$$

This is the analytical expression of the singularities developed by the evolution of the aberrated wavefront given by Eqs. (15). Observe that under the approximations (14) the computation of the caustic is very simple, but the result is very general in the sense that it can be applied to any wavefront that suffers from any kind of aberration.

3. Example: spherical aberration

When the real wavefront suffers from Seidel's spherical aberration [8], the wavefront aberration function is given by

$$\Delta(x_0, y_0) = C_1 (x_0^2 + y_0^2)^2, \quad (20)$$

where C_1 is a constant. For this case the critical set is given by

$$z = z_\pm = 4 C_1 R_0^2 (x_0^2 + y_0^2)(2 \pm 1). \quad (21)$$

The image of the critical points $z = z_+$, which is obtained from Eqs. (19) using Eqs. (20) and (21), is given by

$$X_{c+} = -8 C_1 R_0 (x_0^2 + y_0^2) x_0,$$
$$Y_{c+} = -8 C_1 R_0 (x_0^2 + y_0^2) y_0,$$
$$Z_{c+} = 12 C_1 R_0^2 (x_0^2 + y_0^2), \quad (22)$$

in the same way one finds that the image of $z = z_-$ is given by

$$X_{c-} = 0, \quad Y_{c-} = 0, \quad Z_{c-} = 4 C_1 R_0^2 (x_0^2 + y_0^2). \quad (23)$$

The part of the caustic given by Eqs. (23) is a segment of line (see Fig. 4). By contrast, the part of the caustic given by Eqs. (22) is a revolution surface (see Fig. 3), actually the non-parametric form of this surface is given by

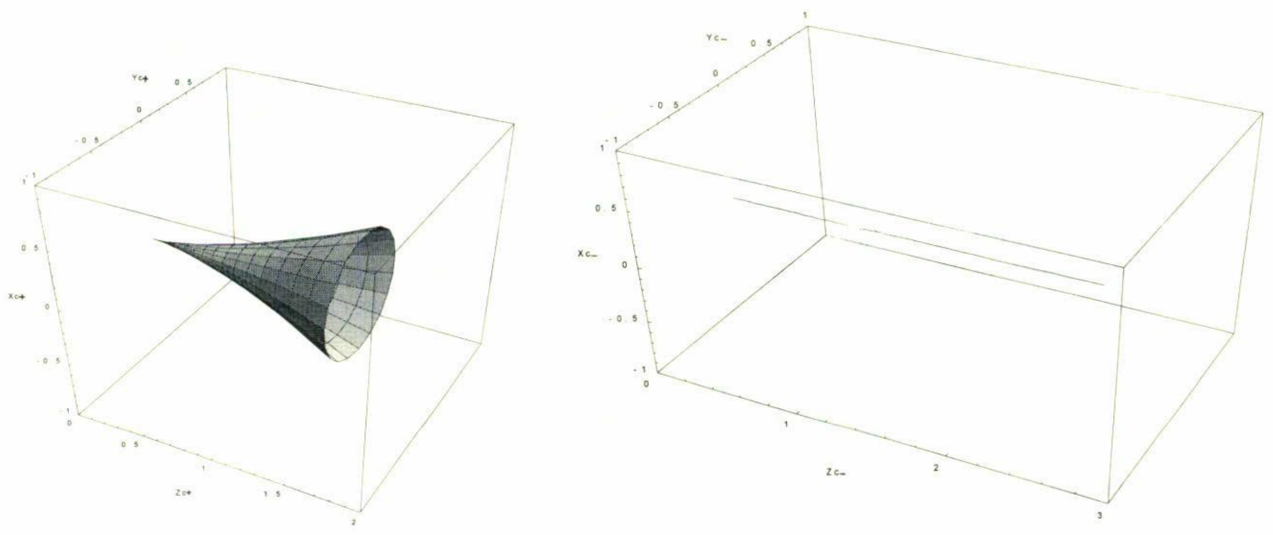
$$X_{c+}^2 + Y_{c+}^2 = \frac{Z_{c+}^3}{27 C_1 R_0^4}, \quad (24)$$

which is a singularity of cusp type. If we take $x_0 = R_0 \sin\theta \cos\varphi$ and $y_0 = R_0 \sin\theta \cos\varphi$, with $0 \leq \varphi \leq 2\pi$ and $0 \leq \theta \leq \theta_0 \equiv \arcsin(a/R_0)$, where a is the radius of the exit pupil, then Eqs. (22) can be rewritten in the following form

$$X_{c+} = -8 C_1 R_0^4 \sin^3\theta \cos\varphi,$$
$$Y_{c+} = -8 C_1 R_0^4 \sin^3\theta \sin\varphi,$$
$$Z_{c+} = 12 C_0 R_0^4 \sin^2\theta, \quad (25)$$

and Eqs. (23) can be written as

$$X_{c-} = 0, \quad Y_{c-} = 0, \quad Z_{c-} = 4 C_1 R_0^4 \sin^2\theta. \quad (26)$$



(a) (b)

FIGURE 3. The part of the caustic given by (a) Eqs. (25) and (b) Eqs. (26), when $R_0 = 0$, $C_1 = 0.5$ and $a = 0.5$.

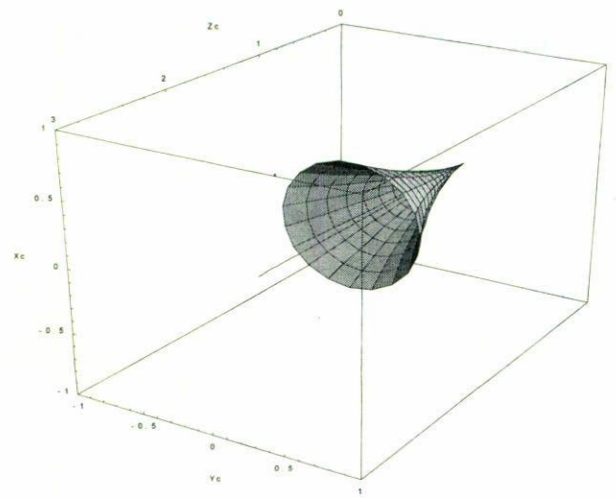


FIGURE 4. The caustic of the real wavefront when it suffers from spherical aberration *i.e.*, the superposition of Figures 3a and 3b.

In Figs. 3a and 3b we plot the surface of revolution given by Eqs. (25) and the segment of line given by Eqs. (26) respectively, when $C1 = 0.1, R_0 = 5$ and $a = 0.5$. In Fig. 4 we present the superposition of Figs. 3a and 3b (the caustic set).

4. Conclusions

In this work we have obtained an analytical expression for the caustic (the singularities developed) of (by) the aberrated wavefront. From Eqs. (19) we see that the wave aberration function, $\Delta(x_0, y_0)$, encode all the information about the singularities developed by the wavefront. It is important to remark that even though the exact expressions for the aberrated wavefront, Eqs. (12), are not too complicated, the expression of the critical and caustic sets cannot be written in a compact form by using cartesian coordinates.

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