Ermakov equation arising from electromagnetic fields propagating in 1D inhomogeneous media

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The Ermakov equation is derived from Maxwell's equations for inhomogeneous transparent media in one dimension. The general properties of this equation and its associated invariants are discussed. Numerical results are presented for refractive index changes, which take place in the order of fractions of the wavelength.

Keywords: Electromagnetism; wave propagation

La ecuación de Ermakov se obtiene de las ecuaciones de Maxwell para un medio inhomogéneo transparente en una dimensión. Se discuten las propiedades generales y los invariantes asociados de esta ecuación. Se presentan resultados numéricos para cambios en el índice de refracción en el rango de la longitud de onda.

Descriptores: Electromagnetismo; propagación de ondas

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1. Introduction

The propagation of plane electromagnetic waves in inhomogeneous media has been successfully described either when the permittivity variation takes place in a much larger or a much shorter distance than the wavelength scale. In the former limit, the amplitude derivatives are neglected on a wavelength scale leading to the Eikonal or ray equation [1]. In the latter, the usual procedure at a discrete boundary is to solve Maxwell's equations in two homogeneous media with constant permittivity say, ε_1 and ε_2. The wave solutions for each case are then joined at the interface by placing the appropriate boundary and continuity conditions.

The purpose in this paper is to consider the intermediate case where the refractive index or the permittivity variations take place on the wavelength scale. In order to simplify the problem, the description is restricted to one-dimensional propagation in a transparent medium whose permittivity gradient is orthogonal to the polarization. The artificial suppression of reflection for perpendicular incidence on absorbing media is a closely related problem, which has received considerable interest in the past [2, 3]. In the present treatment for purely dispersive media, the resulting nonlinear equation derived for the field amplitude is recognized as the Ermakov-Pinney equation, which appears in various fields of physics [4–6]. The general properties of this equation as well as the invariants, which arise from it, are discussed in some detail.

Numerical solutions are presented for a refractive index that varies spatially as a hyperbolic tangent function. Recasting these results in terms of two counter propagating waves allow for a more useful description in terms of the reflectivity as a function of the abruptness of the interface.

2. Derivation of the equation to be solved

The electric field equation arising from Maxwell's equations for non magnetic inhomogeneous media without free charges is given by

$$\nabla^2 \vec{E} - \frac{\varepsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = -\nabla(\vec{E}\cdot\nabla \ln \varepsilon), \tag{1}$$

where the velocity of light in vacuum is $c^2 = (\mu_0\varepsilon_0)^{-1}$ and ε is the relative space dependent permittivity. If the permittivity only varies in the z direction and the problem is restricted to plane waves at normal incidence to the constant permittivity planes, then $\partial\vec{E}/\partial y = 0$ and $\partial\vec{E}/\partial x = 0$. The electric field equations for the x or y polarization in this case, are

$$\frac{\partial^2 E_{x,y}}{\partial z^2} = \frac{\varepsilon}{c^2}\frac{\partial^2 E_{x,y}}{\partial t^2}, \tag{2}$$

whereas the field in the z direction, obeys the equation

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{\partial\left(E_z\dfrac{\partial \ln\varepsilon}{\partial z}\right)}{\partial z} = \frac{\varepsilon}{c^2}\frac{\partial^2 E_z}{\partial t^2}. \tag{3}$$

From the first of Maxwell's equations, $\partial E_z/\partial z = -E_z(\partial \ln\varepsilon/\partial z)$ thus, the second time derivative of E_z is equal to zero. The field in this direction is not a propagating field and is therefore not involved in the transmission of the electromagnetic wave. The equation for the $\hat{\imath}$ or $\hat{\jmath}$ component with a monochromatic dependence $\exp(-i\omega t)$ is then

$$\frac{\partial^2 E(z)}{\partial z^2} = -\varepsilon(z) k_0^2 E(z), \tag{4}$$

where we have substituted the wave vector magnitude in vacuum $k_0 = \omega/c$. This equation for infinite wave trains is non-autonomous due to the dependence of the permittivity on position. Allow for a solution of the form $E = A\exp(iq)$ where the amplitude A and phase q are real quantities. Equation (4) yields

$$\frac{\partial^2 A}{\partial z^2} - A\left(\frac{\partial q}{\partial z}\right)^2 + 2i\frac{\partial A}{\partial z}\frac{\partial q}{\partial z} + iA\frac{\partial^2 q}{\partial z^2} = -\varepsilon k_0^2 A. \tag{5}$$

Let the medium be transparent without absorption, the permittivity is therefore a purely real quantity and we may separate the real and imaginary parts of this equation:

$$\frac{\partial^2 A}{\partial z^2} - A\left(\frac{\partial q}{\partial z}\right)^2 = -\varepsilon k_0^2 A, \tag{6}$$

$$2i\frac{\partial A}{\partial z}\frac{\partial q}{\partial z} + iA\frac{\partial^2 q}{\partial z^2} = 0. \tag{7}$$

The latter equation may be rewritten as

$$\frac{1}{A}\left(2A\frac{\partial A}{\partial z}\frac{\partial q}{\partial z} + A^2\frac{\partial^2 q}{\partial z^2}\right) = \frac{1}{A}\frac{\partial\left(A^2\dfrac{\partial q}{\partial z}\right)}{\partial z} = 0, \tag{8}$$

and thus, provided that A is not zero, there exists a constant quantity given by

$$A^2\frac{\partial q}{\partial z} = Q, \tag{9}$$

Q is an exact invariant even for an arbitrary permittivity space dependence. Substitution of this result in Eq. (6) yields

$$\frac{\partial^2 A}{\partial z^2} - \frac{Q^2}{A^3} = -\varepsilon k_0^2 A. \tag{10}$$

This equation is the Ermakov equation and has received considerable interest [4]. In order to obtain a dimensionless equation, allow the invariant to be equal to $Q = k_0 A_0^2$ without loss of generality. The dimensionless Ermakov equation is then

$$\frac{1}{k_0^2}\frac{\partial^2 A_d}{\partial z^2} - \frac{1}{A_d^3} = -\varepsilon A_d, \tag{11}$$

where $A_d = A/A_0$ is now a dimensionless amplitude. The above results may also be derived in a rather compact way by proposing a solution with the form of an exponential integral function $E = A\exp[i\int(Q/A^2)\partial z]$, where Q is a real constant. The second derivative is then

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{A^3}\left(A^3\frac{\partial^2 A}{\partial z^2} - Q^2\right)\exp\left(i\int\frac{Q}{A^2}\partial z\right)$$
$$= \left(\frac{1}{A}\frac{\partial^2 A}{\partial z^2} - \frac{Q^2}{A^4}\right)E, \tag{12}$$

which upon substitution in Eq. (4) yields again Ermakov's equation and the permittivity ε must then be a real quantity. The use of a polar complex form gives a clear insight about the role played by the amplitude and phase in the wave equation. However, it is interesting to write the complex field E in cartesian coordinates, $E = E_R + iE_I$. The additive and multiplicative representations of complex numbers are related by $A\exp(iq) = (E_R^2 + E_I^2)^{1/2}\exp[\arctan(E_I/E_R)]$. The derivative of the exponential argument is $\partial q/\partial z = \partial\arctan(E_I/E_R)/\partial z = (E_I' E_R - E_I E_R')/(E_R^2 + E_I^2)$ and the amplitude-phase relationship is then

$$Q = A^2\left(\frac{\partial q}{\partial z}\right) = E_R\frac{\partial E_I}{\partial z} - E_I\frac{\partial E_R}{\partial z}, \tag{13}$$

and thus, the invariant Q is identical to the Wronskian formed by the real and imaginary parts of the field. From the theory of differential equations, we know that a non-null Wronskian implies that the functions E_R and E_I must be linearly independent. Therefore, if a solution $y_1(z)$ of a second-order linear differential equation $d^2y/dz^2 + P(z)dy/dz + Q(z)y = 0$ is known, then a linear independent solution $y_2(z)$ is obtained from the following integral [7]:

$$y_2(z) = y_1(z)W(a)\int_b^z \frac{\exp[-\int_a^{z_2} P(z_1)dz_1]}{[y_1(z_2)]^2}\,dz_2, \tag{14}$$

where a and b are arbitrary constants and W is the Wronskian. Allowing $P(z) = 0$ or by direct integration of Eq. (13) using the identity $E_R(\partial E_I/\partial z) - E_I(\partial E_R/\partial z) = E_R^2(\partial/\partial z)(E_I/E_R)$ yields

$$E_I(z) = E_R(z)\int_b^z \frac{Q}{E_R^2(z')}\,dz'. \tag{15}$$

This equation is the cartesian version of the invariant Q previously stated in polar coordinates. It is interesting to note the similitude of this result with the Kramers-Kronig relations, which couple the real and imaginary parts of the permittivity in Fourier space.[(a)] These relations are obtained by establishing a causal connection between the polarisation and the electric field, which impose a relationship between the absorption and dispersion of a material. In the present case, since we are dealing with a purely transparent medium, the real and imaginary parts of the field are not representing dispersive and absorptive processes but the in phase and out of phase coherent components of the field. Substitution of a cosine dependence with constant amplitude for the real part of the field in this equation yields the usual sine dependence for the imaginary part. However, in an inhomogeneous transparent region, the field amplitude is space dependent and thus the real and imaginary parts obey the above general relationship.

3. Solutions of the Ermakov equation

The simplest solution of Ermakov's equation is to consider a constant amplitude so that the second derivative of the amplitude is zero and $A_d = \varepsilon^{-1/4} = n^{-1/2}$, the phase is then $q = k_0 \int (1/A_d^2)\partial z = k_0 n z$ where n is the refractive index. These are the well-known results for a homogeneous medium.

There is, however, a second case, which is not so obvious. Namely that the amplitude is equal to

$$A_d = [(n^{-2} + C^4)^{1/2} + C^2 \cos(2q + \beta_0)]^{1/2}, \qquad (16)$$

where C and β_0 are integration constants. Several authors have exploited this result in order to construct a general solution of the non-linear equation from a particular one [5, 8]. The underlying justification stems from the fact that Ermakov's non-linear equation arises from a linear non-autonomous wave equation. Consider a particular solution of Eq. (4) in an inhomogeneous medium of the form $E_p = A_1 \exp(iq_1)$, where the amplitude and phase may both be spatially dependent. It is clear that the general solution, due to the linearity of the equation, is given by $E = A_1 \exp(iq_1) + A_2 \exp[-i(q_1 - \beta_0)]$ where A_2 is proportional to A_1. Let us now translate this theorem to the non-linear equation. To this end, we recast the general solution as a single exponential

$$E = [A_1^2 + A_2^2 + 2A_1A_2\cos(2q_1 - \beta_0)]^{1/2} \exp\left[i \arctan\left\{ \frac{\left[1 - \left(\frac{A_2}{A_1}\right)\cos\beta_0\right]\tan q_1 + \left(\frac{A_2}{A_1}\right)\sin\beta_0}{1 + \left(\frac{A_2}{A_1}\right)(\tan q_1 \sin\beta_0 + \cos\beta_0)} \right\}\right], \qquad (17)$$

and thus this amplitude is also a solution to the non-linear equation. In this case, the invariant of two counter propagating waves obtained from Eq. (9) and (17) is given, after some tedious algebra, by

$$Q = \left[A_2 \frac{\partial A_1}{\partial z} - A_1 \frac{\partial A_2}{\partial z}\right] \sin(2q - \beta_0) + (A_1^2 - A_2^2)\frac{\partial q}{\partial z}. \qquad (18)$$

Notice that, in general, the invariant of two counter propagating waves is not equal to the sum of the invariants of each wave. This issue will be discussed in some detail in a forthcoming communication. Let us return to the restriction where the amplitudes are constant, say $A_1 = A_0$ and $A_2 = B_0$. Comparing the Eqs. (16) and (17), $A_0^2 + B_0^2 = (n^{-2} + C^4)^{1/2}$, $2A_0B_0 = C^2$ and eliminating C:

$$A_0^2 - B_0^2 = n^{-1}. \qquad (19)$$

Consider a region of the medium in the vicinity of the plane z_0 where the initial conditions are defined. Allow this region to be homogeneous. The invariant at this plane z_0 is then $Q(z_0) = (A_0^2 - B_0^2)k_0 n(z_0)$ which is consistent with Eq. (18) provided that $Q(z_0) = k_0$. Thus, in the particular case of a homogeneous region, the invariant of the sum is equal to the sum of the invariants of each wave. The main conclusion from these results is that if the solution of the Ermakov equation oscillates as $[A_0^2 + B_0^2 + 2A_0B_0\cos(2q_1 - \beta_0)]^{1/2}$, such amplitude may be thought to stem from the propagation of two waves with amplitudes A_0 and B_0 travelling in opposite directions.

The second order derivative of the amplitude in the dimensionless Ermakov equation is multiplied by the inverse square of the wave vector magnitude, which may be recasted as the square of the wavelength. If the relative permittivity variation takes place in a range much larger than the wavelength the second derivative may be neglected. This approximation is equivalent to the slowly varying envelope approximation (SVA) often invoked in non-linear or quantum optics derivations. It is also used in order to obtain the Eikonal equation and it limits its validity. Neglecting the second derivative in Ermakov's equation yields an amplitude $A_d(z) = [n(z)]^{-1/2}$ and the phase is thus $q(z) = k_0 \int n(z)\,\partial z$. The reflected wave is always zero under these approximations since the transmittivity is given by [9], $\mathcal{T} = (n_2/n_1)(|A_2|^2/|A_1|^2) = 1$, and $\mathcal{R} + \mathcal{T} = 1$.

3.1. Numerical solutions

Propagation of electromagnetic waves may be tackled analytically for refractive index changes either in the slowly varying permittivity approximation or in the other extreme for abrupt interfaces. However, the case where the refractive index varies in the range of fractions of the wavelength cannot be dealt with either of these two mathematical approaches. It is then necessary to attempt solving the nonlinear equation without approximations. To this end, consider a hyperbolic tangent refractive index variation with an arbitrary slope. Allow for the refractive index to be written as

$$n(z) = n_{\min} + \frac{(n_{\max} - n_{\min})}{2}[1 + \tanh(\alpha z)], \qquad (20)$$

where $n_{\max}$ and $n_{\min}$ are the maximum and minimum refractive indices obtained in the limit where the refractive index is constant. The maximum slope of this function is exhibited at $z = 0$. A plot parameter $D = (2/\alpha)\mathrm{arctanh}[9/10]$ corresponds to the thickness over which the refractive index varies within 90% of its initial and final values as shown in Fig. 1.

The Ermakov differential equation to be numerically solved is then

$$(2\pi)^2\frac{\partial^2 A_d}{\partial z^2} - \frac{1}{A_d^3} = -\left(n_{\min} + \frac{(n_{\max} - n_{\min})}{2}\left\{1 + \tanh\left[\frac{2}{D}\operatorname{arctanh}\left(\frac{9}{10}\right)z\right]\right\}\right)^2 A_d. \tag{21}$$

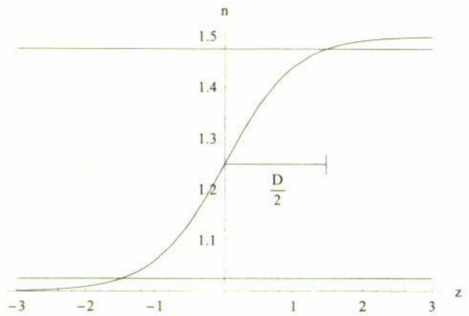

FIGURE 1. Refractive index change *versus* distance depicting the 90% variation criterion described by D.

The important issue here is to establish the appropriate conditions in order to solve this equation. The obvious choice would be to consider the incident field amplitude and its derivative at a given plane. However, such a proposal is inadequate since at any plane where the incident wave exists there is also a contribution from the reflected wave, which is so far unknown. This assertion is true even far away from the region where the refractive index varies considerably since we are dealing with infinite wave trains.

An alternative is to establish the conditions for the refracted wave, which in this one-dimensional case, is simply the transmitted wave. The assumption then required is that far from the interface region the transmitted wave is constant and that in this region there is no reflected wave. Thus the problem is like working backwards in time and obtaining the incident and reflected waves from the transmitted wave. Consider that the incident wave travels towards the positive z direction and the refractive index change takes place around $z = 0$. The initial conditions for the above nonlinear second order differential equation are then

$$\left.\frac{\partial A_d}{\partial z}\right|_{z=z_1} = 0,$$
$$A_d(z_1) = A_t = (n_{\max})^{-1/2}, \qquad z_1 \gg 0, \tag{22}$$

where A_t is the dimensionless transmitted wave amplitude far from the interface. Various solutions of Eq. (21), together with the conditions imposed by (22) are plotted in Fig. 2. In these graphs, oscillations reveal the existence of counter propagating waves according to the results obtained in the previous section. The larger the oscillation, the larger is the reflected wave amplitude. It may be seen that if the refractive index change takes place in the order of one or more wavelengths, the reflectivity is almost null. When this transition occurs in less than a wavelength there is a considerable increase in the reflectivity. Finally, when the transition thickness D_λ reaches a fiftieth of a wavelength or so, the reflectivity approaches a maximum, which barely increases for more abrupt index changes.

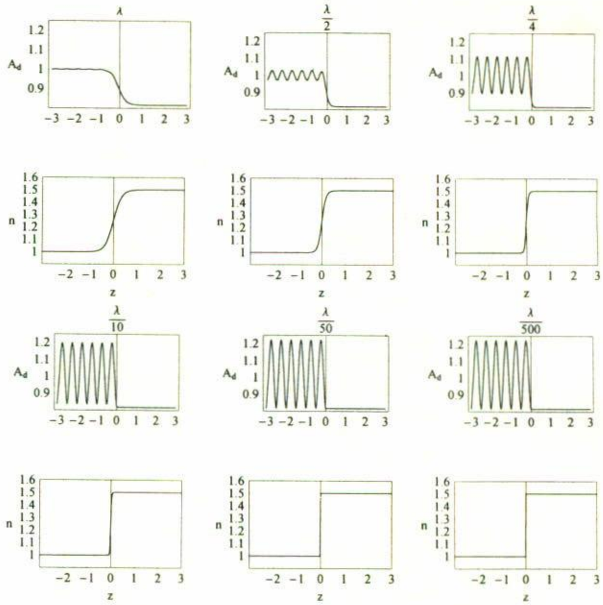

FIGURE 2. Numerical results showing the dimensionless amplitude of the electric field A_d as a function of distance for different refractive index thickness' D. Pairs of curves are presented depicting the field amplitude solutions with their corresponding refractive index variation. Notice that this amplitude does not distinguish between incident, reflected and transmitted waves but represents the actual overall field amplitude at any point.

3.1.1. *Reflectivity*

The conservation Eq. (19) may be rewritten in more familiar terms recalling that the transmitted amplitude is defined by the initial conditions Eq. (22). The usual form of the energy conservation equation $A_0^2 - B_0^2 = (n_{\max}/n_{\min})A_t^2$, is thus obtained. Since the fields are transverse at normal incidence, the continuity equation imposes that the tangential electric field must be continuous at the interface plane p, *i.e.*, $A(p) - B(p) = A_t(p)$. In order to depict this result, the numerical evaluation of the differential equation is plotted in Fig. 3 for a very steep refractive index change taking place in one thousand of a wavelength and another softer one taking place in one third of a wavelength. It may be seen that for the steep function, the above condition is fulfilled whereas for the softer interface, the minimum $A_0 - B_0$ is no longer equal to the transmitted amplitude. Nonetheless, in both cases, the amplitude function is continuous consistent with the continuity equations.

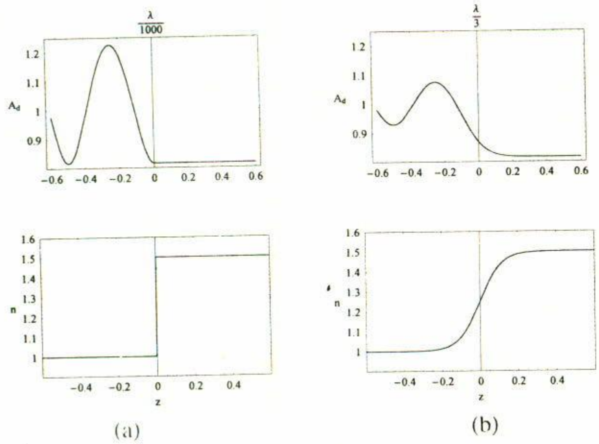

FIGURE 3. Dimensionless amplitude *versus* distance for a very steep refractive index change taking place in a thousand of a wavelength and a softer one taking place in one third. The minimum amplitude in the oscillating region for $z < 0$ is equal to the constant amplitude at $z > 0$ in the steep case whereas in the softer interface these quantities are not equal. In either case, the amplitude curve is continuous.

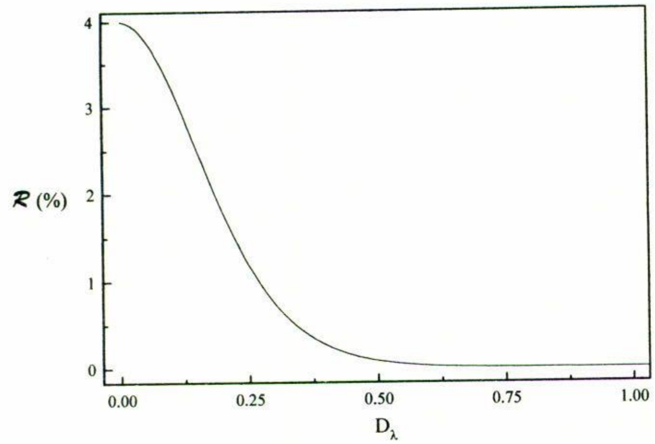

FIGURE 4. Reflectivity coefficient versus the thickness $D_\lambda \equiv D/\lambda$ in wavelength units over which the refractive index changes through 90% for a variation from $n = 1$ to $n = 1.5$.

The amplitudes of the incident and reflected waves A_0 and B_0 may be calculated from the numerical results of the maxima and minima in the oscillating region, far away from the interface, due to interference between them, since

$$A_0 = \frac{A_{d\max} + A_{d\min}}{2}, \qquad B_0 = \frac{A_{d\max} - A_{d\min}}{2}.$$

The amplitude reflection coefficient r, defined as the ratio of the reflected amplitude over the incident amplitude, has been obtained from the numerical solutions for various refractive indices with different spatial variations. The reflectivity $\mathcal{R}$ is defined as the square of the amplitude reflection coefficient. Figure 4 depicts a plot of the reflectivity *versus* the 90% refractive index variation thickness D_λ with data taken from 100 different solutions. The abrupt interface is obtained as

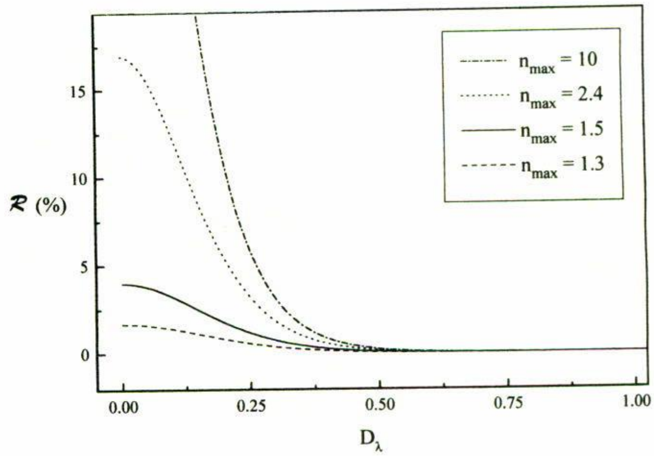

FIGURE 5. Reflectivity coefficient *versus* thickness D_λ for different refractive index steps.

TABLE I. Intensity reflectivity in percentage for various refractive index steps when their 90% variation takes place in a thickness D_λ from zero to one wavelength.

	Reflectivity $\mathcal{R}$ in %				
$n \backslash D$	$\lambda/1000$ (abrupt)	$\lambda/4$	$\lambda/2$	$3\lambda/4$	λ
1.3	1.70	0.57	0.049	0.0027	1.3×10^{-3}
1.5	4.01	1.17	0.085	0.0039	1.6×10^{-3}
2.4	16.9	3.17	0.16	0.0062	2.3×10^{-3}
10	66.9	5.76	0.22	0.0079	2.8×10^{-3}

D_λ tends to zero, $\mathcal{R}$ then tends to the expected value of $(n_{\max} - n_{\min})^2/(n_{\max} + n_{\min})^2$ obtained for normal incidence from the Fresnel equations. The reflectivity remains almost constant for thicknesses smaller than 0.05. For larger thickness D_λ it decreases monotonically and the reflectivity becomes negligible for refractive index variation thicknesses over one wavelength. These results suggest that highly efficient, broadband anti reflection coatings at normal incidence may be obtained with these profiles.

In order to evaluate the dependence of the reflectivity with the overall index change, plots of the reflectivity $\mathcal{R}$ *versus* D_λ are shown for different refractive indices in Fig. 5. It is clear that even for an enormously high refractive index as 10, the reflectivity becomes vanishingly small if the variation takes place in larger distances than the wavelength range. Table I shows the reflectivity for different refractive index steps with thickness variations taking place in fractions of wavelengths. In all cases, the reflectivity is of the order of a thousandth per cent when the refractive index 90% variation takes place in one wavelength.

4. Conclusions

The Ermakov equation governs the amplitude spatial evolution of a monochromatic electromagnetic wave propagating in a one-dimensional inhomogeneous transparent medium. The invariant of two counter-propagating waves that arises from this equation is, in a highly inhomogeneous region, different from the sum of the invariants of the two waves. Furthermore, in such a region, the invariant is not only proportional to the square of the wave amplitudes but also a function of the wave amplitude derivatives. The numerical solutions of the Ermakov equation allow for the description of propagation at normal incidence in a medium where the refractive index variation takes place in the order of a small fraction of the wavelength. This situation is often encountered in thin film growth if there is adsorption between layers. The numerical solutions have been interpreted in terms of counter propagating waves in order to calculate the reflection coefficient directly from the non-linear equation. A potentially interesting area of application is the fabrication of (non-interferometric) anti reflection coatings.

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[a] K-K relations do not apply in this case since we are dealing with a monochromatic wave and the permittivity is only specified as purely real at that single frequency.

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