Alternative Hamiltonians in classical mechanics and geometrical optics

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It is shown that for a given conservative system in classical mechanics, there is an infinite number of Hamiltonians, functions that determine the evolution of the system, which may correspond to different parametrizations of the evolution curves in phase space. Furthermore, for a given reparametrization of the time evolution of the system, an infinite number of alternative Hamiltonians can be found. It is also shown that analogous results hold in the case of the geometrical optics of isotropic media.

Keywords: Hamiltonian mechanics; geometrical optics

Se muestra que para un sistema conservativo en mecánica clásica dado existe un número infinito de hamiltonianas, funciones que determinan la evolución del sistema, las cuales pueden corresponder a diferentes parametrizaciones de las curvas de evolución en el espacio fase. Además, para una reparametrización dada de la evolución temporal del sistema, puede hallarse un número infinito de hamiltonianas alternativas. Se muestra también que en la óptica geométrica de medios isótropos se cumplen resultados análogos.

Descriptores: Mecánica hamiltoniana; óptica geométrica

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1. Introduction

In the framework of Hamiltonian mechanics, the time evolution of a conservative system is determined by a single function H, the Hamiltonian of the system, defined on the corresponding phase space in such a way that, if $q_1, \ldots q_n, p_1, \ldots, p_n$ is a set of canonical coordinates, the evolution curves are given by the solution of the differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{1}$$

The function H is conserved, in the sense that the point with coordinates $q_i(t)$, $p_i(t)$, representing the state of the system at time t, remains on a hypersurface given by $H = E$, where E is a constant.

In many cases, the Hamiltonian can be taken as the total energy; however, as we shall show, for any given value of E, the Hamiltonian function H can be replaced by *any* other (well behaved) function h with the only condition that the hypersurface $H = E$ coincides with the hypersurface $h = \varepsilon$, for some appropriate value of the constant ε. The curves determined by an alternative Hamiltonian, h, may not be parametrized by the time. In other words, the curves lying on the hypersurface $H = E$ given by Eqs. (1) coincide with the curves on $h = \varepsilon$ defined by

$$\frac{dq_i}{d\tau} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{\partial h}{\partial q_i}, \tag{2}$$

where the parameter τ may be different from t. Furthermore, for any reparametrization of the curves defined by Eqs. (1) on the hypersurface $H = E$, one can give an infinite number of alternative Hamiltonians, h (which may depend parametrically on E), such that the solutions of Eqs. (1) on $H = E$ coincide with the solutions on $h = \varepsilon$ of Eqs. (2). The possibility of using different Hamiltonians for a given problem allows us to relate it with other problems, while in some cases it is useful to employ parameters different from the time.

The Hamiltonian formalism is also applicable to the geometrical optics, but in this latter case all possible states lie on just one hypersurface of phase space, *i.e.*, only one value of E is admissible (an analogous situation is encountered in the covariant Hamiltonian formulation for a particle in relativistic mechanics).

In Sec. 2 we establish the main results and give several examples. In Sec. 3 we show that in the case of geometrical optics and of relativistic mechanics, where all possible states lie on a single hypersurface of phase space, alternative Hamiltonians can be found in a phase space with two of the original coordinates suppressed.

2. Alternative Hamiltonians and parametrizations

Let us consider a curve $q_i = q_i(t)$, $p_i = p_i(t)$ on the hypersurface $H = E$, that satisfies Eqs. (1). If this curve is also given by Eqs. (2), in terms of a possibly different parameter τ, then, using the chain rule,

$$\frac{dq_i}{d\tau} = \frac{dq_i}{dt}\frac{dt}{d\tau} = \frac{\partial H}{\partial p_i}\frac{dt}{d\tau}, \qquad \frac{dp_i}{d\tau} = \frac{dp_i}{dt}\frac{dt}{d\tau} = -\frac{\partial H}{\partial q_i}\frac{dt}{d\tau},$$

and comparing with Eqs. (2) we find that

$$\frac{\partial h}{\partial p_i} = \frac{\partial H}{\partial p_i}\frac{dt}{d\tau}, \qquad \frac{\partial h}{\partial q_i} = \frac{\partial H}{\partial q_i}\frac{dt}{d\tau}, \tag{3}$$

on the hypersurface $H = E$ or, equivalently,

$$dh = \frac{dt}{d\tau} dH, \tag{4}$$

on $H = E$. Equations (3) imply that the gradient of H is proportional to the gradient of h and therefore the hypersurface $H = E$ coincides, at least locally, with a hypersurface $h = \varepsilon$.

Conversely, if the hypersurface $H = E$ coincides with the hypersurface $h = \varepsilon$, the gradients of H and h must be proportional to each other on that hypersurface, thus,

$$dh = f dH \tag{5}$$

on $H = E$ (or $h = \varepsilon$), for some function f, which implies that the solutions of Eqs. (1) and (2), with the same initial conditions on $H = E$, are related by a change of parameter with

$$dt = f d\tau. \tag{6}$$

Thus, we have the following result:

Proposition 1. Let H be the Hamiltonian of a conservative system and let E be an admissible value of H. The function h is an alternative Hamiltonian for the system [in the sense that the curves given by Eqs. (2) coincide with those given by Eqs. (1)] if and only if the hypersurface $H = E$ coincides with a hypersurface $h = \varepsilon$.

2.1. Examples

Two of the mechanical systems in two dimensions that admit a solution by separation of variables in more than one coordinate system (called superintegrable systems) examined in Ref. 1 correspond to the potentials

$$V_1 = \frac{\omega^2}{2}(x^2 + y^2) + \frac{k_1^2 - 1/4}{x^2} + \frac{k_2^2 - 1/4}{y^2} \tag{7}$$

and

$$V_3 = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{k_1^2 - 1/4}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}+x)} + \frac{k_2^2 - 1/4}{\sqrt{x^2+y^2}(\sqrt{x^2+y^2}-x)} \tag{8}$$

(which reduce to those of the isotropic harmonic oscillator and the Kepler problem in Cartesian coordinates, respectively, when $k_1^2 = k_2^2 = 1/4$). Letting

$$H = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{\omega^2}{2}(x^2 + y^2) + \frac{k_1^2 - 1/4}{x^2} + \frac{k_2^2 - 1/4}{y^2}, \tag{9}$$

the condition $H = E$ can be rewritten as

$$\frac{1}{2M}\frac{p_x^2 + p_y^2}{x^2 + y^2} - \frac{E}{x^2 + y^2} + \frac{k_1^2 - 1/4}{x^2(x^2 + y^2)} + \frac{k_2^2 - 1/4}{y^2(x^2 + y^2)} = -\frac{\omega^2}{2},$$

which is of the form $h =$ constant, with

$$h = \frac{1}{2M}\frac{p_x^2 + p_y^2}{x^2 + y^2} - \frac{E}{x^2 + y^2} + \frac{k_1^2 - 1/4}{x^2(x^2 + y^2)} + \frac{k_2^2 - 1/4}{y^2(x^2 + y^2)}. \tag{10}$$

Hence, according to Proposition 1, the function h given by Eq. (10) is an alternative Hamiltonian for the system described by the Hamiltonian (9).

The coordinate transformation given by

$$x' = \tfrac{1}{2}(x^2 - y^2), \qquad y' = xy,$$
$$p_x = xp'_x + yp'_y, \qquad p_y = xp'_y - yp'_x, \tag{11}$$

is canonical. Noting that $x' + iy' = \frac{1}{2}(x + iy)^2$ and $p_x + ip_y = (x - iy)(p'_x + ip'_y)$, one finds that Eq. (10) is equivalent to

$$h = \frac{1}{2M}({p'_x}^2 + {p'_y}^2) - \frac{E}{2\sqrt{x'^2 + y'^2}} + \frac{k_1^2 - 1/4}{2\sqrt{x'^2 + y'^2}(\sqrt{x'^2 + y'^2} + x')} + \frac{k_2^2 - 1/4}{2\sqrt{x'^2 + y'^2}(\sqrt{x'^2 + y'^2} - x')}, \tag{12}$$

which is the Hamiltonian for a particle in a potential of the form (8). Thus, the Hamiltonian (10), which reproduces the evolution determined by the Hamiltonian (9), is essentially the Hamiltonian corresponding to the potential (8) [note that the coordinate transformation (11) is not bijective].

From Eqs. (9) and (10) we find that

$$h = (x^2 + y^2)^{-1}(H - E) - \frac{\omega^2}{2}, \tag{13}$$

hence, $dh = (x^2 + y^2)^{-1} dH + (H - E)\, d(x^2 + y^2)^{-1}$ and, on the hypersurface $H = E$, $dh = (x^2 + y^2)^{-1} dH$; comparing with Eq. (4) we see that the parameter, τ, associated with the new Hamiltonian h is related to the time by $dt = (x^2 + y^2)^{-1} d\tau$. (In the case of the Kepler problem, the parameter τ is related to the eccentric anomaly [2].)

Considering now the two-dimensional Kepler problem with Hamiltonian

$$H = \frac{1}{2M}(p_x^2 + p_y^2) - \frac{k}{\sqrt{x^2+y^2}}, \tag{14}$$

the condition $H = 0$ can be written as

$$(x^2+y^2)(p_x^2+p_y^2)^2 = (2Mk)^2, \tag{15}$$

which is of the form $h =$ constant, with

$$h = (x^2+y^2)(p_x^2+p_y^2)^2, \tag{16}$$

therefore, h is an alternative Hamiltonian for the two-dimensional Kepler problem with zero energy.

The coordinate transformation given by

$$u_1 + iu_2 = \frac{1}{\sqrt{2M}\,(p_x - ip_y)},$$

$$P_1 + iP_2 = \sqrt{2M}\,(x-iy)(p_x+ip_y)^2, \tag{17}$$

is canonical and is such that

$$P_1^2 + P_2^2 = (2M)(x^2+y^2)(p_x^2+p_y^2)^2,$$

hence,

$$h = \frac{1}{2M}(P_1^2 + P_2^2), \tag{18}$$

which is the Hamiltonian for a free particle in two dimensions.

From Eqs. (14) and (16) it follows that

$$h = (2M)^2(x^2+y^2)\left(H + \frac{k}{\sqrt{x^2+y^2}}\right)^2$$

$$= (2M)^2\left[(x^2+y^2)H^2 + 2k\sqrt{x^2+y^2}\,H + k^2\right], \tag{19}$$

therefore, on $H = 0$, $dh = 8M^2k\sqrt{x^2+y^2}\,dH$, which implies that the parameters t and τ, associated with H and h, respectively, are related by [see Eq. (6)]

$$dt = 8M^2k\sqrt{x^2+y^2}\,d\tau. \tag{20}$$

On the other hand, making use of Eqs. (17) one finds that

$$x + iy = \sqrt{2M}\,(P_1 - iP_2)(u_1+iu_2)^2, \tag{21}$$

hence, from Eqs. (18) and (20) we obtain

$$dt = 32M^4k^2(u_1^2+u_2^2)\,d\tau. \tag{22}$$

Since the Hamiltonian (18) corresponds to a free particle with energy $(2Mk)^2$ [see Eqs. (15) and (16)], by means of a rotation if necessary, we can assume that $u_1(\tau) = a\tau$, $u_2(\tau) = b$, where b is an arbitrary constant and $a^2 = 8Mk^2$. Then $P_1 = Ma$, $P_2 = 0$ and from Eq. (21) it follows that $x+iy=\sqrt{2M}\,Ma(a\tau+ib)^2$, *i.e.*, $x=\sqrt{2M}\,Ma(a^2\tau^2-b^2)$, $y = (2M)^{3/2}a^2b\tau$, which are parametric equations of a parabola. According to Eq. (22), the parameter τ is related to t by $t = 32M^4k^2(\frac{8}{3}Mk^2\tau^3 + b^2\tau)$ (*cf.* Ref. 3, Secs. 3–8).

2.2. Choice of parametrization

If we want to replace the time, t, by another parameter, τ, with $dt = f d\tau$, where f is a given function, one can give an alternative Hamiltonian h whose integral curves are parametrized by τ. In fact, if we let

$$h = f(H-E) + \varepsilon, \tag{23}$$

where E and ε are constants, then $H = E$ if and only if $h = \varepsilon$, and $dh = f\,dH + (H-E)\,df$; hence, on the hypersurface $H = E$, $dh = f\,dH$, which leads to Eq. (6) as desired.

The alternative Hamiltonian (23) is by no means unique, for instance,

$$h' = g(H-E)^k + f(H-E) + \varepsilon, \tag{24}$$

where g is an arbitrary function and k is an integer greater than 1, is such that $h' = \varepsilon$ if and only if $H = E$ and $dh' = f\,dH$ on $H = E$. Hence, we have:

Proposition 2. Let H be the Hamiltonian of a conservative system and let f be an arbitrary function. For each admissible value E of H there exists an infinite number of alternative Hamiltonians for the system corresponding to a parameter τ such that $dt = f\,d\tau$ for the evolution curves on $H = E$.

Note that Eqs. (13) and (19) are of the form (23) and (24), respectively, and that by comparing the corresponding expressions one can identify the function f.

3. Geometrical optics and reduced phase space

In the geometrical optics approximation, the propagation of light in an isotropic medium can be described by the Hamilton equations (1), with

$$H = \frac{c}{2n^2}g^{ij}p_ip_j, \tag{25}$$

where n is the refractive index of the medium, c is the speed of light in vacuum, (g^{ij}) is the inverse of the matrix (g_{ij}) formed by the components of the metric tensor in the coordinate system employed and there is summation over repeated indices (see, *e.g.*, Ref. 4 and the references cited therein). The speed of light in the medium is equal to c/n provided that $g^{ij}p_ip_j = n^2$, *i.e.*, $H = c/2$; thus, in the present case only one value of H is admissible and all the possible states lie on a single hypersurface of phase space. If F is *any* (differentiable, real-valued) function of one variable such that $F'(c/2) \neq 0$, then $h = F(H)$, with H given by Eq. (25), is an alternative Hamiltonian to H for the light rays. Then, on $H = c/2$,

$$dh = F'(c/2)\,dH,$$

hence, owing to Eqs. (5) and (6), the parameter τ associated with h is determined by $dt = F'(c/2)\,d\tau$.

The evolution of the light rays can be parametrized, say, by the Cartesian coordinate z; then, from Eqs. (1) and (25) we have $dz/dt = \partial H/\partial p_z = (c/n^2)p_z$, therefore, taking $f = n^2/(cp_z)$ [see Eq. (6)], from Eqs. (23) and (25), with $E = c/2$ and $\varepsilon = 0$, we obtain the Hamiltonian

$$h = \frac{1}{2p_z}\left(p_x^2 + p_y^2 + p_z^2 - n^2\right) = \frac{1}{2p_z}\left(p_z - \sqrt{n^2 - p_x^2 - p_y^2}\right)\left(p_z + \sqrt{n^2 - p_x^2 - p_y^2}\right), \tag{26}$$

which gives the evolution of the light rays, parametrized by z. Since on the hypersurface $H = c/2$, the relation $p_x^2 + p_y^2 + p_z^2 = n^2$ holds, assuming $p_z \geq 0$, *i.e.*, $p_z = \sqrt{n^2 - p_x^2 - p_y^2}$, from Eqs. (2) and (26) we obtain, *e.g.*,

$$\begin{aligned}
\left.\frac{dx}{dz}\right|_{p_z=\sqrt{n^2-p_x^2-p_y^2}} &= \frac{1}{2p_z}\left[\left(p_z - \sqrt{n^2 - p_x^2 - p_y^2}\right)\frac{\partial}{\partial p_x}\left(p_z + \sqrt{n^2 - p_x^2 - p_y^2}\right)\right.\\
&\qquad \left.\left. + \left(p_z + \sqrt{n^2 - p_x^2 - p_y^2}\right)\frac{\partial}{\partial p_x}\left(p_z - \sqrt{n^2 - p_x^2 - p_y^2}\right)\right]\right|_{p_z=\sqrt{n^2-p_x^2-p_y^2}}\\
&= \left.\frac{\partial}{\partial p_x}\left(p_z - \sqrt{n^2 - p_x^2 - p_y^2}\right)\right|_{p_z=\sqrt{n^2-p_x^2-p_y^2}}\\
&= \left.\frac{\partial}{\partial p_x}\left(-\sqrt{n^2 - p_x^2 - p_y^2}\right)\right|_{p_z=\sqrt{n^2-p_x^2-p_y^2}},
\end{aligned} \tag{27}$$

and, similarly,

$$\left.\frac{dp_x}{dz}\right|_{p_z=\sqrt{n^2-p_x^2-p_y^2}} = -\left.\frac{\partial}{\partial x}\left(-\sqrt{n^2 - p_x^2 - p_y^2}\right)\right|_{p_z=\sqrt{n^2-p_x^2-p_y^2}}. \tag{28}$$

Thus, in the reduced phase space with canonical coordinates x, y, p_x, p_y, the function $-\sqrt{n^2 - p_x^2 - p_y^2}$ is a Hamiltonian that gives the evolution of the light rays parametrized by z (alternative derivations are given in Refs. 5–7).

Another parametrization of the light rays, already considered in Ref. 7, corresponds to the use of the arc length, s. In this case we have

$$\begin{aligned}
f^{-1} &= \frac{ds}{dt}\\
&= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\\
&= \frac{c}{n^2}\sqrt{p_x^2 + p_y^2 + p_z^2},
\end{aligned}$$

and from Eqs. (23) and (25), taking $\varepsilon = 0$,

$$\begin{aligned}
h &= \frac{n^2}{c\sqrt{p_x^2 + p_y^2 + p_z^2}}\left(\frac{c}{2n^2}(p_x^2 + p_y^2 + p_z^2) - \frac{c}{2}\right)\\
&= \frac{n}{\sqrt{p_x^2 + p_y^2 + p_z^2}}\left(\frac{p_x^2 + p_y^2 + p_z^2}{2n} - \frac{n}{2}\right).
\end{aligned} \tag{29}$$

Apart from the factor $n/\sqrt{p_x^2 + p_y^2 + p_z^2}$, which is equal to 1 on the hypersurface $H = c/2$, the Hamiltonian (29) coincides with one of the Hamiltonians found in Ref. 7 [Eq. (3.7)].

Finally, in the case of a free particle in flat space-time, a (Lorentz-covariant) Hamiltonian that gives the evolution of the particle parametrized by its proper time, τ, is given by

$$\begin{aligned}
h &= \frac{1}{2M}\eta^{\alpha\beta}p_\alpha p_\beta\\
&= \frac{1}{2M}[(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2],
\end{aligned} \tag{30}$$

where M is the rest mass of the particle and $(\eta^{\alpha\beta}) = \mathrm{diag}\,(1, -1, -1, -1)$. In order for τ to be the proper time of the particle, the four-momentum p_α must satisfy the condition $\eta^{\alpha\beta}p_\alpha p_\beta = M^2c^2$ and, therefore, the only admissible value of h is $\varepsilon = Mc^2/2$. If F is any real-valued function of one variable such that $F'(Mc^2/2) \neq 0$, $H = F(h)$, with h given by Eq. (30), is an alternative Hamiltonian to h.

If t denotes the time measured in an inertial frame, making use of Eqs. (2), (6) and (30) we have

$$f = \frac{dt}{d\tau} = \frac{1}{c}\frac{dx^0}{d\tau} = \frac{1}{c}\frac{\partial h}{\partial p_0} = \frac{p_0}{Mc},$$

hence, from Eq. (23), taking $E = 0$, we find that

$$\begin{aligned}
H &= f^{-1}(h - \varepsilon)\\
&= \frac{c}{2p_0}\left(\eta^{\alpha\beta}p_\alpha p_\beta - M^2c^2\right)\\
&= \frac{c}{2p_0}\left(p_0 - \sqrt{\mathbf{p}^2 + M^2c^2}\right)\left(p_0 + \sqrt{\mathbf{p}^2 + M^2c^2}\right),
\end{aligned} \tag{31}$$

where $\mathbf{p} \equiv (p^1, p^2, p^3) = -(p_1, p_2, p_3)$, is an alternative Hamiltonian for a free particle [*cf.* Eq. (26)]. Thus, proceeding as in the previous example, assuming $p_0 \geq 0$, from Eqs. (1) it follows that in the reduced phase space with canonical coordinates x^1, x^2, x^3, p_1, p_2, p_3, the function $-c\sqrt{\mathbf{p}^2 + M^2c^2}$ is a Hamiltonian for a free particle. The sign of this Hamiltonian must be reversed if one employs $(p^1, p^2, p^3) = -(p_1, p_2, p_3)$ as the canonical coordinates conjugate to (x^1, x^2, x^3).

4. Conclusions

As we have shown, in the case of a conservative system in classical mechanics or of an isotropic medium in geometrical optics, the evolution is essentially determined by a family of hypersurfaces in phase space, and each of these hypersurfaces can be seen as a level surface of an infinite number of functions which act as Hamiltonians. Among other things, the results presented here provide a general framework that allows us to obtain the Hamiltonians found in Refs. 6–8 by means of other procedures.

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