About ambiguities appearing on the study of classical and quantum harmonic oscillator

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A family of nonequivalents Lagrangians and Hamiltonians are given for the one-dimensional harmonic oscillator. These Lagrangians are deduced using the constant of motion approach. The study is focused on one of these Lagrangians and Hamiltonians to analyze their implications on the quantization of the one-dimensional harmonic oscillator. The main feature is the incomnpatibilities of the units in the usual quantization approaches. Using the velocity quantization approach it is possible to get rid of this incompatibility units problem.

Keywords: Lagrangian; Hamiltonian; constant of motion; velocity quantization

Se encuentra una familia de lagrangianos y hamiltonianos no equivalentes del oscilador armónico en una dimensión. Estos Lagrangianos son deducidos utilizando el procedimiento de encontrar inicialmente una constante de movimiento del sistema. El estudio se centra en uno de estos Lagrangianos y su correspondiente hamiltoniano para analizar sus implicaciones en la cauntización del oscilador armónico unidimensional. El resultado fundamental es la incompatibilidad en el sistema de unidades en la cuantización normal. Usando la cuantización de la velocidad, es posible hacer a un lado este problema de incompatibilidad de las unidades.

Descriptores: Lagrangiano; hamiltoniano; constante de movimiento; cuantización de la velocidad

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1. Introduction

It is known that there is a practical approach to obtain nonequivalent Lagrangians for a given one-dimensional autonomous (forces are time independent) dynamical system [1–3]. This approach is based on the constant of motion associated to the system and can be used to obtain nonequivalent Hamiltonians which are not related each other through a "canonical transformation." This surprising result indicates that one may have an ambiguous description for the associated quantum counter part of the classical model (a review about this subject can be found in Ref. 9 and references there in). In addition to this ambiguousness, there is another problem which is much more important to take in consideration and which is related with the quantization. Using the Hamiltonian and Lagrangian approches to quantize nonequivalent systems may be meaningless due to noncompatibility with the units. The ambiguounsness and units problems for the harmonic oscillator are presented on this paper.

In this paper different constants of motion of the harmonic oscillator are used to obtain different nonequivalent Lagrangians through the above mentioned approach. The study and discussion about the quantization of the classical counter part is restricted to the Lagrangian and Hamiltonian obtained from the square of the total usual energy of the harmonic oscillator. Although, one restricts himself to the study of the harmonic oscillator, the analysis and results are expected to be also valid for any other system.

2. Nonequivalent Lagrangian and Hamiltonian

The Kobussen-Leubner-Lopez (KLL) expression for the Lagrangian associated to a one-dimensional autonomous system is given by

$$L(x,v)=v\int^v \frac{K(x,\xi)}{\xi^2}\,d\xi, \tag{1}$$

where v is the velocity, x is the coordinate, and $K(x,v)$ is a constant of motion of the system. A completed derivation and discussion of this expression is found in Ref. 1.

For the one dimensional harmonic oscillator, the equations of motion are given by the following autonomous dynamical system:

$$\frac{dx}{dt}=v \tag{2}$$

and

$$\frac{dv}{dt}=-\omega^2 x, \tag{3}$$

where ω represents the angular frequency of the oscillations. The energy

$$K_1=E=\frac{1}{2}mv^2+\frac{1}{2}m\omega^2x^2, \tag{4}$$

is the usual constant of motion of (2). However, any arbitrary function of this constant of motion is also a constant of motion. In particular, any power of Eq. (4) is a constant of motion, so a family of constants of motion can be given by

$$K_n(x,v)=E^n=\left(\frac{m}{2}\right)^n(v^2+\omega^2x^2)^n$$
$$=\left(\frac{m}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}v^{2k}(\omega^2x^2)^{n-k}, \tag{5}$$

where n is an integer number, and $\binom{n}{k} = n!/k!(n-k)!$ is the combinatorial coefficient. Using this family of constants of motion in Eq. (1), the following family of nonequivalent Lagrangians is obtained:

$$L_n(x,v) = \left(\frac{m}{2}\right) \sum_{k=0}^{n} \binom{n}{k} (\omega^2 x^2)^{n-k} \frac{v^{2k}}{2k-1}. \quad (6)$$

Clearly, for $n = 1$, the usual Lagrangian is gotten,

$$L = \frac{m}{2}(v^2 - \omega^2 x^2). \quad (7)$$

The generalized linear momentum is then given by

$$p_n(x,v) = \frac{\partial L_n}{\partial v} = \left(\frac{m}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} (\omega^2 x^2)^{n-k} \frac{2kv^{2k-1}}{2k-1}, \quad (8)$$

and whenever the inverse relation

$$v_n = v(x, p_n), \quad (9)$$

can be gotten from Eq. (8), the Hamiltonian of the system can be obtained if one makes the substitution of Eq. (9) into Eq. (5),

$$H_n(x,p_n) = K_n[x, v(x,p_n)]. \quad (10)$$

For the particular case $n = 2$, the constant of motion, Lagrangian, and generalized linear momentum can be obtained from Eq. (5), Eq. (6), and Eq. (8) as

$$K_2(x,v) = \frac{m^2}{4}(v^4 + 2\omega^2 x^2 v^2 + \omega^4 x^4), \quad (11)$$

$$L_2(x,v) = \frac{m^2}{4}\left(\frac{1}{3}v^4 + 2\omega^2 x^2 v^2 - \omega^4 x^4\right), \quad (12)$$

and

$$p = p_2(x,v) = m^2\left(\frac{1}{3}v^3 + \omega^2 x^2 v\right). \quad (13)$$

Solving Eq. (13) for v, the velocity can be written in terms of the position and the coordinate as

$$v(x,p) = \left[\frac{3p}{2m^2} - \sqrt{(\omega x)^6 + \left(\frac{3p}{2m^2}\right)^2}\,\right]^{1/3} + \left[\frac{3p}{2m^2} + \sqrt{(\omega x)^6 + \left(\frac{3p}{2m^2}\right)^2}\,\right]^{1/3}. \quad (14)$$

Now, substituting Eq. (14) in Eq. (11), the Hamiltonian is given by

$$H_2(x,p) = \frac{m^2}{4}\left\{\left[\frac{3p}{2m^2} - \sqrt{(\omega x)^6 + \left(\frac{3p}{2m^2}\right)^2}\,\right]^{1/3} + \left[\frac{3p}{2m^2} + \sqrt{(\omega x)^6 + \left(\frac{3p}{2m^2}\right)^2}\,\right]^{1/3}\right\}^4$$
$$+ \frac{m^2\omega^2 x^2}{2}\left\{\left[\frac{3p}{2m^2} - \sqrt{(\omega x)^6 + \left(\frac{3p}{2m^2}\right)^2}\,\right]^{1/3} + \left[\frac{3p}{2m^2} + \sqrt{(\omega x)^6 + \left(\frac{3p}{2m^2}\right)^2}\,\right]^{1/3}\right\}^2 + \frac{m^2\omega^4 x^4}{4}. \quad (15)$$

It is pointed out that this Hamiltonian associated to the harmonic oscillator can not be obtained through a canonical transformation of the usual Hamiltonian,

$$H_1(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \quad (16)$$

3. Classical time evolution for the case $n = 2$

Since the family of Lagrangians given by Eq. (6) brings about the same dynamical equations [Eqs. (2) and (3)], this describes the same dynamical behavior in the configuration space (x,v). However, it will show that the situation is somewhat different when the dynamics is seen in the phase space (x,p). To see this, the analysis will be restricted itself to the case $n = 2$ for simplicity.

Using the Hamilton's equation of motion,

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \quad (17)$$

and

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \quad (18)$$

the resulting dynamics of the particle in the phase space (x,p) can be studied. Using the Hamiltonian (11) in Eq. (17) and Eq. (18), one obtain the differential equations governing the behavior of the particle (these equations can be seen in the Appendix). These equations are solved through fourth order Runge-Kutta's method, and the solution is shown in the Fig. 1, where the dynamics of the particle due to the Hamiltonians H_2 and H_1 are presented. Clearly the dynamics must be different since the generalized momentum for the Hamiltonian H_2 is much more complex and has different units than that one related to H_1. Taking this in mind, the figure shows the phase-space (x,p) of the dynamical behavior of the particle for the Hamiltonians H_1 and H_2. It is pointed out that the units for the generalized linear momentum is different for these two Hamiltonians and that the evolution of $x(t)$ is the same for both Hamiltonians. Note that the point $(0,0)$ is a singular point on the phase space is a singular point of the Eqs. (42) and (43) (see Appendix), but this point is excluded since the constant of motion is different from zero. On the other hand, one has that $f_-(0,p) = 0$, but $(0,p)$ is not a sin-

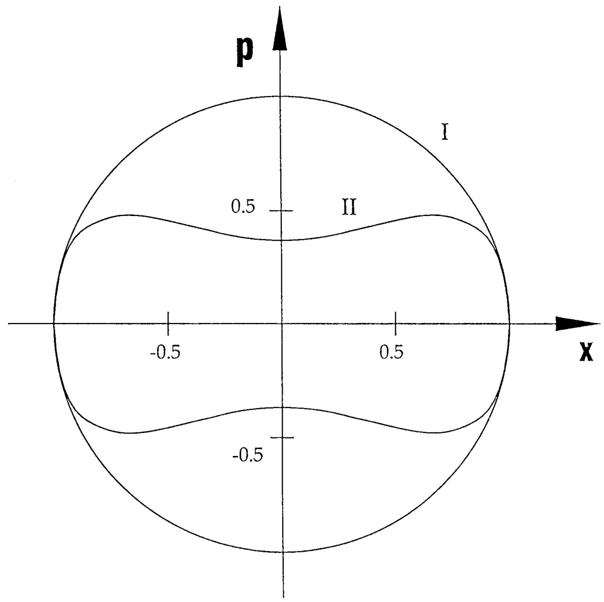

FIGURE 1. Phase-Space of the Harmonic Oscillator. Curve II shows the points $(x(t), p(t))$ which are solution of Eqs. (42) and (43) (shown in the appendix with $m = 1$ and $\omega = 1$). Curve I represents the solution of the Hamilton equations with the usual Hamiltonian [Eq. (16)]. The units of $p(t)$ on both cases are differents.

gular point of (42) or (43) either since (42) represents a regular function on $f_-(0,p)$, and (43) which contains terms of the form $x^5/f_-^{1/3}(x,p)$, $x^7/f_-^{1/3}(x,p)$, $x^5/f_-^{2/3}(x,p)$ and $x^7/f_-^{2/3}(x,p)$ satisfies the limits (46) and (47).

4. Discussion about quantization

One might think that since the nonequivalent Lagrangias (6) generate the same dynamics of the oscillators (12), the same should be happen when the quantization of the classical system would be made using the nonequivalent Hamiltonias. However, as a result of the above analysis, one must not expect that situation to be true since, even at the classical level, the dynamics generated by the nonequivalent Hamiltonians in the phase-space (x, p) is different from the original Hamiltonian.

There is one more remark that must be mentioned when one tries to quantize nonequivalent Hamiltonians. It is almost hopeless to find an Hermitian operator ($\widehat{H}_n$) associated to the classical nonequivalent Hamiltonian, says H_2, consisting of a finite number of terms which can be easy to handle. Therefore the quantization *a la* Schrönger [5],

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}_n(x,\hat{p})\Psi, \tag{19}$$

where Ψ is the wave function and $\hbar$ is the reduced Plank's constant, or *a la* Heisenberg [6],

$$i\hbar\frac{d\xi}{dt} = [\hat{\xi}, \widehat{H}_n], \tag{20}$$

where ξ represents the variables x or $\hat{p}$, may be unsolvable. In addition, these expressions have problems with the units since the left hand of Eq. (19) and Eq. (20) have units of "energy times something else." However, the units on their right side are "some power of energy times something else." One may overcome this problem by substituting the operator $(i\hbar\partial/\partial t)^n$ on the left hand side of Eq. (19) and Eq. (20). But one can not get rid of the previous difficulty mentioned above.

A similar problem appears if one tries to do the quantization *a la* Feynman [7],

$$K(b,a) = \lim_{\epsilon\to 0}\frac{1}{A^N}\int\dots\int\exp\left[\frac{iS(b,a)}{\hbar}\right]dx_1\dots dx_{N-1}, \tag{21}$$

where K is the quantum amplitude to go from the point x_a at the time t_a to the point x_b at the time t_b, A^N is a normalized factor, and $S(b,a)$ is defined as

$$S(b,a) = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt. \tag{22}$$

Of course there is the problem that using the nonequivalent Lagrangian (6) in Eq. (22) and this one in Eq. (21), the integration over the paths may not be easy at all. Moreover, the clear problem is that Eq. (22) has not the right units of action (energy $\times$ time) in general when using Lagrangian (6). This problem is not overcome by just having a denominator of the form $\hbar^n$ in the exponential of Eq. (21).

The same problem of units incompatibility arises when quantization *a la* Bohr-Sommerfeld [8] is done,

$$\oint p(x,H)\, dx = nh, \tag{23}$$

where n is a natural number, and the integration is done over a closed loop in the phase-space (x, p). Using Eq. (8) the units on the left and right side of Eq. (23) are completely different in general. This difficulty can not be overcome by just having a power of the Plank constant (h).

5. Quantization based on the velocity operator

Instead of quantizing the Hamiltonian associated to the system, one may quantize the constant of motion. This quantization can be gotten associating the known operator,

$$\hat{v} = -\frac{i\hbar}{m}\frac{\partial}{\partial x}, \tag{24}$$

to the velocity variable v (m is the mass of the particle) and the operator

$$\widehat{E} = i\hbar\frac{\partial}{\partial t}, \tag{25}$$

for the usual energy of the system. In addition, one constructs an operator associated to the constant of motion (K)

$$\widehat{K} = \widehat{K}(x, \hat{v}). \tag{26}$$

In this way, the associated Schrödinger equation to the harmonic oscillator characterized by the constant of motion (5) could be given by

$$\left(i\hbar\frac{\partial}{\partial t}\right)^n\Psi = \widehat{K}(x,\hat{v})\Psi. \qquad (27)$$

Or even more general, if the constant of motion is of the form $K(x,v) = G(E)$, where G is an arbitrary function of the energy (4), the quantization may be of the form

$$G\left(i\hbar\frac{\partial}{\partial t}\right)\Psi = \widehat{K}(x,\hat{v})\Psi. \qquad (28)$$

Of course, this approach leaves invariant the normal nonrelativistic quantum mechanics in the Schrödinger squeme, and it does not need the concept of Langrangian and Hamiltonian.

On the other hand, one could also quantize the loops resulting in the space (x,v) in the following way

$$\oint v(x,K)\,dx = \frac{nh}{m}, \qquad (29)$$

where n is a natural number. Eq. (27) and Eq. (29) seem to be free from units ambiguities like those appearing in the Lagrangian and Hamiltonian formalism. One may apply this last approach (Eq. (29) to the harmonic oscillator characterized by the constant of motion (11) to see if the result is reasonable. This application will be given below.

A) Given the constant value K_2 for the constant of motion (11), the velocity can be written in terms of the position and this constant as

$$v(x,K_2) = \begin{cases} \sqrt{-\omega^2x^2 + 4\dfrac{\sqrt{K_2}}{m}}, \\ -\sqrt{-\omega^2x^2 + 4\dfrac{\sqrt{K_2}}{m}}, \end{cases} \qquad (30)$$

where the two cases correspond to the upper and lower region in the plane (x,v). Therefore, the integral (29) can be written as

$$\oint v(x,K_2)\,dx = 2\int_{x_-}^{x_+}\sqrt{4\frac{\sqrt{K_2}}{m} - \omega^2x^2}\,dx, \qquad (31)$$

where x_- and x_+ are the points such that $v(x_\pm,K_2)=0$,

$$x_+ = -x_- = \frac{1}{\omega}\sqrt{\frac{2\sqrt{K_2}}{m}}. \qquad (32)$$

Integration of Eq. (31) and Eq. (23) bring about the relation

$$\oint v(x,K_2)\,dx = \frac{4\pi\sqrt{K_2}}{m\omega} = \frac{nh}{m}. \qquad (33)$$

Then, the allowed values for the constant of motion are

$$K_{2,n} = \left(\frac{1}{2}n\omega\hbar\right)^2, \qquad (34)$$

which is the result expected since the constant of motion K_2 is associated to the same dynamical equation [Eq. (27)] and the same dynamics in the space (x,v).

B) On the other hand, one could try to solve (27) to find the spectrum of the system. By proposing a solution of the form

$$\Psi(x,t) = \exp(i\alpha t)\psi(x), \qquad (35)$$

and using canonical quantization on Eq. (11), that is for example:

$$\widehat{x^2v^2} = \frac{x^2\hat{v}^2 + \hat{v}^2x^2 + x\hat{v}x + \hat{v}x^2\hat{v} + x\hat{v}x\hat{v} + \hat{v}x\hat{v}x}{6},$$

it follows the eigenvalue problem

$$\widehat{K}_2\psi = (\hbar\alpha)^2\psi, \qquad (36)$$

where the operator $\widehat{K}_2$ is given by

$$\widehat{K}_2 = \frac{m^2}{4}\hat{v}^4 + \frac{\omega^2m^2}{12}\left(x^2\hat{v}^2 + \hat{v}^2x^2 + x\hat{v}x + \hat{v}x^2\hat{v} + x\hat{v}x\hat{v} + \hat{v}x\hat{v}x\right) + \frac{m^2\omega^4}{4}x^4, \qquad (37)$$

which can be written, using the conmutation relation

$$[x,\hat{v}] = i\frac{\hbar}{m}, \qquad (38)$$

as

$$\widehat{K}_2 = \frac{m^2}{4}\hat{v}^4 + \frac{\omega^2m^2}{12}\left[6x^2\hat{v}^2 - i\frac{12\hbar}{m}x\hat{v} + 3\left(\frac{i\hbar}{m}\right)^2\right] + \frac{m^2\omega^4}{4}x^4. \qquad (39)$$

It is clear that the spectrum of this operator is different from that of the harmonic oscillator square $(\widehat{K_1}\circ\widehat{K_1})$ since this last one corresponds to the operator

$$\widehat{K}_2^* = \widehat{K_1}\circ\widehat{K_1} = \frac{m^2}{4}\hat{v}^4 + \frac{\omega^2m^2}{4}\left[2x^2\hat{v}^2 - i\frac{4\hbar}{m}x\hat{v} + 2\left(\frac{i\hbar}{m}\right)^2\right] + \frac{m^2\omega^4}{4}x^4, \qquad (40)$$

where the canonical quantization of $\widehat{K_1}\circ\widehat{K_1}$ has been performed (K_1 given by Eq.(4)), and the conmutation (38) has been used. If one changes the operator $\widehat{K}_2$ for $\widehat{K}_2^*$ in Eq. (36) and takes the known eigenfunction of the harmonic oscillator [5], it follows

$$(\hbar\alpha_n)^2 = \hbar^2\omega^2\left(n+\frac{1}{2}\right)^2, \qquad (41)$$

which is the square of the harmonic oscillator energies, $\alpha_n = E_n/\hbar$ (of course, the spectrum is unbounded due to the negative energies which come from the fact of having a second order time differentiation). So, it is clear that (39) leads us to different eigenvalues to those of Eq. (41), and it is unnecessary to find them.

6. Conclussions

Using different constants of motion for the harmonic oscillator and an integral expression for the Lagrangian, nonequivalent Lagrangians and Hamiltonians associated to this system were found. The Lagrangians bring about the same classical dynamical equations of motion, therefore, the same behavior in the (x, v) space, but he Hamiltnonians may bring about different dynamical behavior in the phase-space (x, p) because the variable p can be a very complicated function of "x and v." Quantization of the harmonic oscillator with nonequivalent Lagrangians and Hamiltonians leads into units problems, in addition to the known complication to look for a reasonable operator associated to Hamiltonians.

Quantization of the velocity variable (v) instead of the generalized linear momentum (p) seem to be free of incompatibility with units, and preliminary results indicate that this approach has sense. Although only the one-dimensional problem has been studied here, the mathematical advantage of using the quantization of v and the constant of motion $K(x, v)$ is that for higher dimensional dynamical systems this constant of motion always exists. However, the same can not be said about the Lagrangian and the Hamiltonian [10].

Appendix

Using the Hamiltonian (15) with $m = 1$ and $\omega = 1$ in Eq. (17) and Eq. (18), it follows

$$\frac{dx}{dt} = \frac{f_+^{2/3}(x,p) - f_-^{2/3}(x,p)}{2\sqrt{x^6 + \frac{9p^2}{4}}}\Big[f_-^{2/3}(x,p) + f_+^{2/3}(x,p) + 2f_-^{1/3}(x,p)f_+^{1/3}(x,p) + x^2\Big]. \tag{42}$$

and

$$\begin{aligned}\frac{dp}{dt} = -x^3 &+ \frac{x^7}{\sqrt{\frac{9p^2}{4} + x^6}\; f_-^{1/3}(x,p)} + \frac{2x^5 f_-^{1/3}(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}} + x f_-^{2/3}(x,p) + \frac{x^7 f_-^{1/3}(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}\; f_+^{2/3}(x,p)} \\ &+ \frac{x^5 f_-(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}\; f_+^{2/3}(x,p)} + \frac{x^7}{\sqrt{\frac{9p^6}{4} + x^6}\; f_+^{1/3}(x,p)} + \frac{3x^5 f_-^{2/3}(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}\; f_+^{1/3}(x,p)} - \frac{2x^5 f_+^{1/3}(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}} \\ &- \frac{x^7 f_+^{1/3}(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}\; f_-^{2/3}(x,p)} - 2x^3 + x f_+^{2/3}(x,p) - \frac{3x^5 f_+^{2/3}(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}\; f_-^{1/3}(x,p)} - \frac{x^5 f_+(x,p)}{\sqrt{\frac{9p^2}{4} + x^6}\; f_-^{2/3}(x,p)},\end{aligned} \tag{43}$$

where the functions f_+ and f_+ are defined as

$$f_+(x,p) = \frac{3p}{2} + \sqrt{\frac{9p^2}{4} + x^6} \tag{44}$$

and

$$f_-(x,p) = \frac{3p}{2} - \sqrt{\frac{9p^2}{4} + x^6}. \tag{45}$$

One has the following limits:

$$\lim_{x\to 0} \frac{x^\beta}{f_-^{1/3}} = 2^{1/3}\left(\frac{3p}{2}\right)^{1/3} \lim_{x\to 0} x^{\beta-2} = 0, \quad \text{if} \quad \beta > 2, \tag{46}$$

$$\lim_{x\to 0} \frac{x^\beta}{f_+^{2/3}} = 2^{2/3}\left(\frac{3p}{2}\right) \lim_{x\to 0} x^{\beta-4} = 0, \quad \text{if} \quad \beta > 4. \tag{47}$$

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