Supersymmetry and the constants of motion of the two-dimensional isotropic harmonic oscillator

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It is shown that the constants of motion of the two-dimensional isotropic harmonic oscillator not related to the rotational invariance of the Hamiltonian can be derived using the ideas of supersymmetric quantum mechanics.

Keywords: Supersymmetric quantum mechanics; hidden symmetries

Se muestra que las constantes de movimiento del oscilador armónico bidimensional isótropo no relacionadas con la invarianza rotacional del hamiltoniano, pueden derivarse usando las ideas de la mecánica cuántica supersimétrica.

Descriptores: Mecánica cuántica supersimétrica; simetrías ocultas

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1. Introduction

The hydrogen atom and the isotropic harmonic oscillator are perhaps the best known examples of systems with hidden symmetry, *i.e.*, the Hamiltonian is invariant under a group of transformations which is larger than the rotation group associated with the “obvious” rotational invariance of the Hamiltonian. The existence of this larger group corresponds to that of constants of motion in addition to the angular momentum. In Ref. 1 it has been shown that the hidden symmetry of the two-dimensional hydrogen atom can be deduced using the ideas of supersymmetric quantum mechanics. The aim of this paper is to present a similar derivation for the two-dimensional isotropic harmonic oscillator (TIHO). As we shall show below, despite the well-known connection between the hydrogen atom and the isotropic harmonic oscillator, there are some interesting differences in the derivation of the extra constants of motion which are related to the fact that in the case of the hydrogen atom these extra constants of motion are the components of a vector (the so-called Runge–Lenz vector) while in the case of the isotropic harmonic oscillator the extra constants of motion are the components of a second-rank tensor.

2. Supersymmetric factorization and shift operators

The rotational invariance of the Hamiltonian of the TIHO allows one to separate the Schrödinger equation in polar coordinates and the radial part of the wavefunction is determined by an effective potential that is the sum of the true (harmonic oscillator) potential and a centrifugal potential that depends on the angular momentum quantum number m. In the case of the hydrogen atom, the effective potentials for two neighboring values of m are supersymmetric partners of each other [1]; however, in the case of the TIHO, following an analogous procedure, one obtains operators that connect radial wavefunctions with neighboring values of m but with values of the energy that differ by one unit of $\hbar\omega$ (by contrast, for the hydrogen atom there is no energy shift). Then by composing two of these shifting operators, with appropriate values of the parameters, one obtains operators that connect radial wavefunctions with the same energy and values of m differing by 2. Since a constant of motion (that does not depend explicitly on the time) maps an eigenstate of the Hamiltonian into another with the same energy, one may suspect that the composition mentioned above corresponds to a constant of motion. As shown below, one indeed obtains two constants of motion in this manner and the fact that the value of m is shifted by two units is related to the fact that these constants of motion are components of a tensor (and not a vector as in the case of the hydrogen atom).

Following Ref. 1, we shall summarize the relevant information about supersymmetric quantum mechanics that will be required in what follows (for a more complete discussion see Refs. 2–8). If u_0 is the wavefunction of the ground state of a particle of mass M in a one-dimensional potential $V(x)$, *i.e.*,

$$Hu_0(x) = -\frac{\hbar^2}{2M}u_0''(x) + V(x)u_0(x) = E_0u_0(x), \quad (1)$$

then, making $E_0 = 0$ by redefining the energy, the potential $V(x)$ is given by

$$V(x) = \frac{\hbar^2}{2M}\frac{u_0''(x)}{u_0(x)}, \tag{2}$$

and letting

$$A \equiv \hbar(2M)^{-1/2}\left(\frac{d}{dx} + \frac{u_0'}{u_0}\right),$$
$$A^\dagger \equiv \hbar(2M)^{-1/2}\left(-\frac{d}{dx} + \frac{u_0'}{u_0}\right), \tag{3}$$

one finds that

$$AA^\dagger = -\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V(x) = H. \tag{4}$$

The operator

$$H_\mathrm{S} \equiv A^\dagger A = H + \frac{\hbar^2}{M}\left[\left(\frac{u_0'}{u_0}\right)^2 - \frac{u_0''}{u_0}\right], \tag{5}$$

is called the supersymmetric partner of H. The operators H and H_S have the same eigenvalues, with the exception of the ground state energy E_0, which is missing in the spectrum of H_S. The operator $A^\dagger$ maps any eigenstate of H, different from the ground state, into an eigenstate of H_S, with the same energy. (Note that $A^\dagger u_0 = 0$.) In a similar way, A maps any eigenstate of H_S into an eigenstate of H with the same eigenvalue.

The time-independent Schrödinger equation for a particle in two dimensions subjected to a central potential $V(r)$, in polar coordinates has the form

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right] + V(r)\psi = E\psi, \tag{6}$$

hence, writing $\psi(r,\theta) = r^{-1/2}u(r)\mathrm{e}^{im\theta}$, $(m = 0, \pm 1, \pm 2, \ldots)$ one finds that u satisfies the one-dimensional Schrödinger equation

$$H_m u \equiv -\frac{\hbar^2}{2M}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2M}\frac{m^2 - 1/4}{r^2}\right]u = E\psi. \tag{7}$$

In the case of the TIHO, $V(r) = M\omega^2r^2/2$ and the effective potential in Eq. (7) is given by

$$V_m(r) = \frac{1}{2}M\omega^2r^2 + \frac{\hbar^2}{2M}\frac{m^2 - 1/4}{r^2}. \tag{8}$$

The "ground state" of the one-dimensional Schrödinger equation with the potential (8) is

$$u_0 = Nr^{m+1/2}\exp(-M\omega^2r^2/2\hbar), \tag{9}$$

where N is a normalization constant; hence, from (3) we find that

$$A_m = \hbar(2M)^{-1/2}\left(\frac{d}{dr} + \frac{m + 1/2}{r} - \frac{M\omega r}{\hbar}\right),$$
$$A_m^\dagger = \hbar(2M)^{-1/2}\left(-\frac{d}{dr} + \frac{m + 1/2}{r} - \frac{M\omega r}{\hbar}\right), \tag{10}$$

and a straightforward computation gives

$$A_mA_m^\dagger = H_m - (m+1)\hbar\omega,$$
$$A_m^\dagger A_m = H_{m+1} - m\hbar\omega. \tag{11}$$

Thus, except for constant terms, $A_mA_m^\dagger$ and $A_m^\dagger A_m$ are the Hamiltonians H_m and H_{m+1} respectively; however, the fact that the constant terms in Eq. (11) differ by $\hbar\omega$ implies that the operators $A_m^\dagger$ and A_m transform eigenstates of H_m into eigenstates of H_{m+1} and conversely, with energies that differ by $\hbar\omega$. For instance, if u is an eigenfunction of H_m with eigenvalue E, $H_mu = Eu$, then, according to (11),

$$\begin{aligned} H_{m+1}(A_m^\dagger u) &= (A_m^\dagger A_m + m\hbar\omega)A_m^\dagger u \\ &= A_m^\dagger\big[H_m - (m+1)\hbar\omega\big]u + m\hbar\omega A^\dagger u \\ &= (E - \hbar\omega)A_m^\dagger u, \end{aligned}$$

showing that $A_m^\dagger u$ is an eigenfunction of H_{m+1} with eigenvalue E, provided that $A_m^\dagger u \neq 0$. In an analogous way one finds that if u is an eigenfunction of H_{m+1} with eigenvalue E, then A_mu is an eigenfunction of H_m with eigenvalue $E + \hbar\omega$; hence, loosely speaking, $A_m^\dagger$ increases the value of m by 1 and decreases the value of the energy by $\hbar\omega$ while A_m has the opposite effects.

Denoting the operators A_m and $A_m^\dagger$, defined by Eq. (10), as $A_m(\omega)$ and $A^\dagger(\omega)$, respectively, one finds that $A_m^\dagger(-\omega)$ increases the value of m and also increases the energy by $\hbar\omega$, while $A_m(-\omega)$ decreases the value of m and of the energy. Hence, the composition

$$Q_m^\dagger \equiv A_{m+1}^\dagger(-\omega)A_m^\dagger(\omega) = A_{m+1}^\dagger(\omega)A_m^\dagger(-\omega) \tag{12}$$

must increase the value of m by 2, without changing the energy; therefore, it may correspond to an operator that commutes with the Hamiltonian of the TIHO. In effect, making use of Eqs. (10) and (12), one obtains

$$Q_m^\dagger = \frac{\hbar^2}{2M}\left[\frac{d^2}{dr^2} - \frac{2(m+1)}{r}\frac{d}{dr} + \frac{(m+1/2)(m+5/2)}{r^2} - \frac{M^2\omega^2r^2}{\hbar^2}\right]. \tag{13}$$

In order to find the effect of $Q_m^\dagger$ on the complete wavefunction, we have to take into account that $Q_m^\dagger$ acts on the radial part of a wavefunction with angular dependence $\mathrm{e}^{im\theta}$, yielding the radial part of a wavefunction with angular dependence $\mathrm{e}^{i(m+2)\theta}$

and that the radial part of the wavefunction $\psi(r,\theta)$ is equal to $r^{-1/2}u(r)$; hence, $Q_m^\dagger$ corresponds to the operator

$$Q^\dagger = \frac{\hbar^2}{2M} e^{2i\theta} r^{-1/2} \left[\frac{\partial^2}{\partial r^2} - \frac{2}{r}\left(1 - i\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{2} - i\frac{\partial}{\partial\theta}\right)\left(\frac{5}{2} - i\frac{\partial}{\partial\theta}\right) - \frac{M^2\omega^2 r^2}{\hbar^2}\right] r^{1/2}$$

$$= \frac{\hbar^2}{2M} e^{2i\theta} \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{2i}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r} - \frac{2i}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{M^2\omega^2 r^2}{\hbar^2}\right)$$

$$= \frac{\hbar^2}{2M} \left[e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) - \frac{\partial^2}{\partial\theta^2} - \frac{M^2\omega^2}{\hbar^2}(re^{i\theta})^2\right],$$

which does not contain the quantum number m, or, in terms of Cartesian coordinates,

$$Q^\dagger = \frac{\hbar^2}{2M}\left[\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^2 - \frac{M^2\omega^2}{\hbar^2}(x+iy)^2\right]$$
$$= -\frac{1}{2M}\left[(p_x + ip_y)^2 + M^2\omega^2(x+iy)^2\right]. \qquad (14)$$

A straightforward computation shows that $Q^\dagger$ indeed commutes with the Hamiltonian of the TIHO. Then, writing $Q^\dagger = -\omega(Q_1 + iQ_2)$, with Q_1 and Q_2 Hermitean, it follows that Q_1 and Q_2 are two Hermitean constants of motion, which are components of the second-rank symmetric traceless tensor

$$q_{ij} = \frac{1}{M\omega}(p_i p_j + M^2\omega^2 x_i x_j) - \frac{1}{2M\omega}\left[p_x^2 + p_y^2 + M^2\omega^2(x^2+y^2)\right]\delta_{ij},$$

for $i, j = 1, 2$ ($Q_1 = q_{11}, Q_2 = q_{12}$).

3. Concluding remarks

The results presented here, as well as those of Ref. 1, show that in some cases the hidden symmetries of a Hamiltonian can be deduced making use of the supersymmetry quantum mechanics.

If one considers the time-independent Schrödinger equation in three dimensions with a central potential $V(r)$, the radial equation can be reduced to a one-dimensional Schrödinger equation of the form (7), with $m^2 - 1/4$ replaced by $l(l+1)$, and in case of the isotropic harmonic oscillator the steps between Eqs. (10) and (13) also apply, but while in the two-dimensional case it is easy to replace the quantum number m by a differential operator, owing to the simplicity of the angular dependence of the wavefunction, in the three-dimensional case, where the angular dependence of the wavefunction is given by the spherical harmonics, it is not so easy to obtain an analog of the operator Q (see also Ref. 1).

1. J.M. Lyman and P.K. Aravind, *J. Phys. A: Math. Gen.* **26** (1993) 3307.
2. V.A. Kostelecky, M.M. Nieto, and D.R. Traux, *Phys. Rev. D* **32** (1985) 2627.
3. C.V. Sukumar, *J. Phys. A: Math. Gen.* **18** (1985) L57.
4. C.V. Sukumar, *J. Phys. A: Math. Gen.* **18** (1985) 2917.
5. R.W. Haymaker and A.R.P. Rau, *Am. J. Phys.* **54** (1986) 928.
6. R. Dutt, A. Khare, and U.P. Sukhatme, *Am. J. Phys.* **56** (1988) 163.
7. O.L. de Lange and R.E. Raab, *Operator Methods in Quantum Mechanics*, (Oxford University Press, Oxford, 1991).
8. G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*, (Springer-Verlag, Berlin, 1996).