The spectra of a Hamiltonian with a linear radial potential derived by a variational calculation based on a set of harmonic oscillator states

M. Moshinsky* and A. Suárez Moreno
Instituto de Física, Universidad Nacional Autónoma de México
Apartado postal 20-364, 01000 México, D.F., Mexico

Recibido el 23 de octubre de 2001; aceptado el 8 de noviembre de 2001

In relation with a quark interaction confinement potential a linear radial one is frequently used. As the spectrum of the resulting Hamiltonian is not available in an analytic fashion, we determine it in this paper using a variational procedure based on a set of appropriate harmonic oscillator states.

Keywords: Spectra; linear radial potential

Para el potencial de confinamiento de los quarks se usa frecuentemente uno radial que es lineal. Como el espectro del hamiltoniano resultante no está disponible en forma analítica, lo determinamos en este artículo por un procedimiento variacional basado en un conjunto apropiado de funciones de oscilador armónico.

Descriptores: Espectros; potencial radial lineal

PACS: 13.39.Pn; 31.15.Pf; 02.70.Hm

1. Introduction

One of the more usual potential between quarks is a linear combination of the one gluon exchange that gives the effect at short distances plus another that takes into account the confinement. The first one, in analogy of what happens in the one photon exchange for two electrons, gives rise to an attractive potential of the Coulomb type, while the second one is frequently assumed as linear radial potential. In cgs units our potential can then be written as [1]

$$V(r') = q'r' - \frac{b'^2}{r'}, \tag{1}$$

where we used primed letters for the radial coordinate r' and the parameters q', b' as we wish to reserve the non-primed ones for a more convenient set of units. As V is an energy potential in a cgs system of units it is given by ergs, while r' is in centimeters so the units of q', b' are, respectively,

$$\begin{aligned} q' &\to (\mathrm{ergs/cm}) \\ b'^2 &\to (\mathrm{ergs\ cm}). \end{aligned} \tag{2}$$

A two body problem of a quark-antiquark of masses m_1, m_2 respectively, when considered non relativistically, has a Hamiltonian

$$H' = \frac{1}{2\mu}p'^2 + q'r' - \frac{b'^2}{r'}, \tag{3}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{4}$$

which is a good approximation to the physical situation when m_1, m_2 are large as is the case of charm and bottom quarks.

Our final objective will be to give the matrix representation of the Hamiltonian of Eq. (3) in a basis of harmonic oscillator states of frequency ω. For this purpose we make the change to dimensionless variables

$$\begin{aligned} H &= \frac{H'}{\mu c^2}, & \mathbf{p}' &= \sqrt{\hbar\mu\omega}\,\mathbf{p}, \\ \mathbf{r}' &= \sqrt{\frac{\hbar}{\mu\omega}}\,\mathbf{r}, & \epsilon &\equiv \sqrt{\frac{\hbar\omega}{\mu c^2}} \end{aligned} \tag{5}$$

and we get

$$H = \frac{1}{2}\epsilon^2 p^2 + \frac{q}{\epsilon}r - b^2\epsilon\frac{1}{r}, \tag{6}$$

where

$$\begin{aligned} q &= \frac{q'\hbar}{\mu^2 c^3}, \\ b^2 &= \frac{b'^2}{\hbar c}. \end{aligned} \tag{7}$$

We now can get the matrix element of H of Eq. (6) with respect to harmonic oscillator states of unit frequency and mass as they are already incorporated in the parameter ϵ of Eq. (5). It turns out though that variational calculation of the part of the Coulomb potential *i.e.* (b^2/r), is very slowly convergent as the corresponding discrete levels bunch up at energy 0 while those of the harmonic oscillator continue to grow indefinitely with the same spacing.

Thus in this article we restrict ourselves to the confining potential qr assuming $b^2 = 0$. Before discussing the variational procedure in detail we first indicate some well known properties of the spectra of a linear radial potential.

2. Properties of the Hamiltonian for a linear radial potential

If in Eq. (6) we take $b^2 = 0$ and $\epsilon = 1$ and write the wave function $\psi = (\phi/r)$ we have the radial equation

$$\left\{\frac{1}{2}\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right]+qr\right\}\phi = E\phi. \tag{8}$$

There is no analytical solution for Eq. (8) if $l \neq 0$, and in this case the energy eigenvalues have to be determined numerically.

For $l = 0$ we divide Eq. (8) by $(2q)^{2/3}$ and it can then be written as

$$\left(\frac{1}{(2q)^{2/3}}\frac{d^2}{dr^2}-\frac{2qr-2E}{(2q)^{2/3}}\right)\phi = 0. \tag{9}$$

Defining now

$$z \equiv \frac{2qr-2E}{(2q)^{2/3}},$$

$$dz = (2q)^{1/3}\,dr. \tag{10}$$

we get the equation

$$\frac{d^2\phi}{dz^2} - z\phi = 0, \tag{11}$$

which is the one discussed by Airy [2] *i.e.*,

$$\phi(r) = \mathrm{Ai}(z) = \mathrm{Ai}\left[\frac{2qr-2E}{(2q)^{2/3}}\right]. \tag{12}$$

This solution applies to the linear potential in the whole plain z, but we want it only for $r > 0$ and it should vanish when $r = 0$. In that case we have that the values of the energy E_n, $n = 1, 2, \ldots$, are given by the zeros of the Airy function

$$\mathrm{Ai}\left[-\frac{2E_n}{(2q)^{2/3}}\right] = 0. \tag{13}$$

In Ref. 2 (Table 10.13 p. 478), the first ten zeros of the Airy function are denoted by the negative numbers a_n, so the corresponding energies become

$$E_n = -\frac{(2q)^{2/3}}{2}a_n, \tag{14}$$

or

$$\frac{2^{1/3}E_n}{q^{2/3}} = -a_n, \tag{15}$$

and their values up to a factor $(q^{2/3}/2^{1/3})$ are given in Table I, where we also have the corresponding numbers obtained variationally by a procedure outlined in the next section.

TABLE I. On the left hand side we give the negative of the zeros of the Airy function *i.e.* $(-a_n)$ given in p. 478, Table 10.13 of Ref. 2. On the right hand side we give the results for the energies obtained from the diagonalization of the matrix whose elements are given by Eq. (23) with $l = 0, n, n' = 0, 1, 2, \ldots 35$. As all matrix elements contain the factor $(q^{2/3}/2^{1/3})$ we multiplied the energies by the inverse of this factor as indicated in Eq. (14) to be able to compare the left and right hand side of Table I. Note that the values on the left and right hand side of Table I coincide except for the last three digits in each row which can no longer be trusted in both computations, and is the reason for separating them a bit from the rest.

n	$-a_n$		$2^{1/3}q^{-2/3}E_n$	
1	2.33810	741	2.33810	753
2	4.08794	944	4.08794	957
3	5.52055	983	5.52055	997
4	6.78670	809	6.78670	824
5	7.94413	359	7.94413	375
6	9.02265	085	9.02265	103
7	10.04017	434	10.04017	453
8	11.00852	430	11.00852	451
9	11.93601	556	11.93601	591
10	12.82877	675	12.82878	491

3. Energy spectra of our problem based on a set of harmonic oscillator states

The eigenstates of an harmonic oscillator in which $\hbar = \mu = \omega = 1$, are given by the ket

$$|nlm\rangle = R_{nl}(r)Y_{lm}(\theta,\varphi), \tag{16}$$

where Y_{lm} is a spherical harmonic and the radial wave function is given by

$$R_{nl}(r) = \left[\frac{2(n!)}{\Gamma(n+l+3/2)}\right]^{1/2} r^l \mathrm{e}^{-r^2/2} L_n^{l+1/2}(r^2), \tag{17}$$

where $L_n^{l+1/2}$ in a Laguerre polynomial [3].

Our Hamiltonian is the one given by Eq. (6) when $b^2 = 0$ and, as it is invariant under rotations it commutes with the orbital angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{18}$$

so the kets of Eq. (16) are characterized by the eigenvalues $l(l+1)$ of L^2 and m of L_z. The matrix elements of our Hamiltonian with respect to the states $|nlm\rangle$ will then be diagonal in l, m and, from the analysis of p. 3–7 of Ref. 4, are given by

$$\left\langle n'lm\left|\frac{1}{2}\epsilon^2p^2+\frac{q}{\epsilon}r\right|nlm\right\rangle = \frac{\epsilon^2}{2}\left\{\left[n\left(n+l+\frac{1}{2}\right)\right]^{1/2}\delta_{n'n-1}+\left(2n+l+\frac{3}{2}\right)\delta_{n'n}+\left[(n+1)\left(n+l+\frac{3}{2}\right)\right]^{\frac{1}{2}}\delta_{n'n+1}\right\}$$
$$+\frac{q}{\epsilon}\sum_{p=l}^{n+n'+l} B(n'l,n,l,p)\frac{(p+1)!}{\Gamma\left(p+\frac{3}{2}\right)}, \tag{19}$$

where the coefficient $B(n'l, nl, p)$ is defined in Eq. (2.6) of Ref. 4.

To be able to diagonalize the matrix whose elements are given by Eq. (19) we have to specify the values of $\epsilon = (\hbar\omega/\mu c^2)^{1/2}$ or, in other words, the frequency of the oscillator that we shall use for the particular value of l we wish to investigate. Following the analysis of p. 7 of Ref. 4 we can start with the matrix element Eq. (19) for the lowest possible values for n, n' *i.e.* $n = n' = 0$ and thus have

$$\left\langle 0\, lm \left| \frac{1}{2}\epsilon^2 p^2 + \frac{qr}{\epsilon} \right| 0\, lm \right\rangle = \frac{1}{2}\left(l+\frac{3}{2}\right)\epsilon^2 + \frac{q}{\epsilon}\frac{(l+1)!}{\Gamma\left(l+\frac{3}{2}\right)}. \quad (20)$$

We can find the minimum value of the right hand side of Eq. (20) if we take its derivative with respect to ϵ and equate it to 0.

In this way we get the equation

$$\left(l+\frac{3}{2}\right)\epsilon - \frac{q}{\epsilon^2}\frac{(l+1)!}{\Gamma\left(l+\frac{3}{2}\right)} = 0, \quad (21)$$

from which we obtain

$$\epsilon_l = \left[\frac{q(l+1)!}{\Gamma\left(l+\frac{5}{2}\right)}\right]^{\frac{1}{3}}, \quad (22)$$

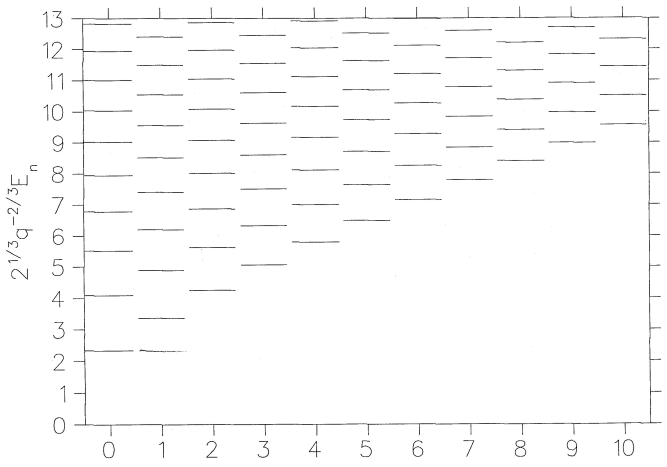

FIGURE 1. We give the energies in Eq. (8) multiplied by the factor $(2^{1/3}q^{-2/3})$ indicated in Eq. (14), as function of the orbital angular momentum l shown in abscissa, and enumerated by the number n corresponding to increasing order for a fixed l. For $l = 0$ the levels are given numerically in Table I.

where we added the index l to the ϵ on the left hand side, as the minimum value of this parameter depends on l.

We can now replace the value ϵ_l of Eq. (22) on the right hand side of Eq. (19) and the matrix we need to diagonalize has elements

$$\frac{q^{2/3}}{2^{1/3}}\left\{\left[\frac{1}{2}\frac{(l+1)!}{\Gamma\left(l+\frac{5}{2}\right)}\right]^{\frac{2}{3}}\left\{\left[n\left(n+l+\frac{1}{2}\right)^{\frac{1}{2}}\right]\delta_{n'n-1} + \left(2n+l+\frac{3}{2}\right)\delta_{n'n} + \left[(n+1)\left(n+l+\frac{3}{2}\right)\right]^{\frac{1}{2}}\delta_{n'n+1}\right\}\right.$$

$$\left. +2^{1/3}\left[\frac{\Gamma\left(l+\frac{5}{2}\right)}{(l+1)!}\right]^{\frac{1}{3}}\sum_{p=l}^{n+n'+l}\left[B(n'l,nl,p)\frac{(p+1)!}{\Gamma\left(p+\frac{3}{2}\right)}\right]\right\}. \quad (23)$$

Immediately we see that all eigenvalues obtained by diagonalizing this matrix with $n, n' = 0, 1, 2, \ldots N$, but keeping l fixed, will have a factor $q^{2/3}$, the same that appears in the energy E_n in Eq. (14) when $l = 0$ where we can solve the problem in terms of Airy functions. Thus our variational results are universal in the sense that they depend on the strength q of the interaction only through the factor $q^{2/3}$.

The numerical analysis was done by taking the maximum value N of n, n' up to $N = 35$ and the first ten energies for $l = 0$ are given in Table I and they coincide, to the order indicated, with those derived from Eq. (15).

In Fig. 1 we give the spectra of the Hamiltonian with a linear radial potential for different values of l supressing the constant factor $q^{2/3}$. We note that it resembles the spectra of an harmonic oscillator but the spacings are contracted as we go to higher energies and the degeneracies are partially removed.

4. Conclusions

We see that a variational procedure based on harmonic oscillator states gives a good approximation to the spectrum of a Hamiltonian with a linear radial potential.

Acknowledgments

The authors thank project 32421-E of CONACyT for support of this work.

*. Member of El Colegio Nacional and the Sistema Nacional de Investigadores.

1. W. Lucha, F. Schöberl, and D. Gromes, *Phys. Rep.* **200** (1991) 137, Eq. (4.2).
2. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, (Dover Publications Inc., New York, 1965), p. 478, Table 10.13.
3. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, (Academic Press, New York, 1965), p. 1037.
4. M. Moshinsky and Yu.F. Smirnov, *The harmonic oscillator in modern physics*, (Hardwood Academic Publishers, Netherlands, 1996).