

# Supersymmetric quantum solution for FRW cosmological model with matter

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Recibido el 13 de agosto de 2001; aceptado el 22 de enero de 2002

Using the technique of supersymmetric quantum mechanics we present new cosmological quantum solutions, in the regime for FRW cosmological model using a barotropic perfect fluid as matter field.

*Keywords:* Witten's supersymmetric quantum mechanics; factor ordering; Einstein-Hamilton-Jacobi equation; exact solutions.

Usando la técnica de la mecánica cuántica supersimétrica, se presentan nuevas soluciones cosmológicas cuánticas para el modelo cosmológico FRW, para un fluido perfecto barotrópico como campo de materia.

*Descriptores:* Mecánica cuántica supersimétrica a la Witten; ordenamiento de factores; ecuación de Einstein-Hamilton-Jacobi; soluciones exactas.

PACS: 04.20.Jb; 04.40.Nr; 12.60.Jv; 98.80.Hw

## 1. Introduction

We are interested to study some cosmological models in the supersymmetric quantum mechanics scheme. Recently, particular exact solutions to the Wheeler-DeWitt (WDW) equation in Witten's [1] supersymmetric quantum mechanics for all Bianchi Cosmological Class A Models in the Einstein theory were found [2].

Our goal in this work is to try to solve an ambiguity in the factor ordering of the position and momenta operators and give *selection rules* that fix the parameter that measures this ambiguity. Such ambiguities always arise, when there are expressions containing the product of non-commuting quantities that depend on  $q^\mu$  and  $P_\mu$  as in our case. It is then necessary to find some criteria to know which factor ordering should be selected. The factor ordering in the semiclassical approximation is irrelevant, but not so in the exact theory. In a previous work [2] the global factor was dropped by hand and the factor ordering ambiguity was avoided when they factorized the WDW equation.

Thus, the idea of Witten [1] is to find the supersymmetric super-charges operators  $Q$ ,  $\bar{Q}$  that produce a super-hamiltonian  $H_{ss}$ , and that satisfies the closed superalgebra

$$\{Q, \bar{Q}\} = H_{ss}, \quad [H_{ss}, Q] = 0, \quad [H_{ss}, \bar{Q}] = 0, \quad (1)$$

where the super-hamiltonian  $H_{ss}$  has the following form:

$$H_{ss} := H_0 + \frac{\partial^2 \Sigma(x, y)}{\partial q^\nu \partial q^\mu} [\bar{\psi}^\nu, \psi^\mu], \quad (2)$$

here  $H_0$  is the bosonic Hamiltonian and  $\Sigma$  is known as the super-potential term that is related with the potential term that appears in the bosonic hamiltonian. This idea was applied in reference [2] for all Bianchi type cosmological models.

In this approach, the hamiltonian  $H_{ss} = \frac{1}{2}Q^2$  is positive semi-definite and a supersymmetric state with  $Q|\Psi\rangle = 0$  is

automatically a zero energy ground state. This simplifies the problem of finding supersymmetric ground state because the energy is known a priori and also because the factorization of  $H_{ss}|\Psi\rangle = 0$  into  $Q|\Psi\rangle = 0$ ,  $\bar{Q}|\Psi\rangle = 0$  often provides a simple first-order equation for the ground state wave function. The simplicity of this factorization is related to the solubility of certain bosonic hamiltonians. For example, in this work we find for the empty (+) and filled (-) sector of the fermion Fock space zero energy solution

$$|\Psi_\pm\rangle = e^{\pm\Sigma}|\pm\rangle, \quad (3)$$

where  $\Sigma$  denotes a superpotential, and  $\Psi_+$  and  $\Psi_-$  are the corresponding components for the empty and filled sector in the wave function. We also observe a tendency for supersymmetric vacua to remain close to their semi-classical limits, because in this work and others [2], the exact solutions (3) are also the lowest-order WKB approximations.

This paper is organized in the following way: In Sec. 2, the ADM lagrangian of our model is constructed and also, we propose the known classical solution for this model. In Sec. 3, we derive the corresponding Hamiltonian that allows us to obtain the quantum WDW equation for the FRW cosmological model with matter field. In Sec. 4 we derive the WDW solution in the supersymmetric quantum mechanics approach. Sec. 5 is devoted to conclusions.

## 2. ADM Lagrangian formulation

We consider the total lagrangian, where one part is geometry and the other one corresponds to the matter field. We will consider a perfect fluid with barotropic equation of state as our matter field:

$$\mathcal{L}_{total} = \mathcal{L}_{geom} + \mathcal{L}_{matter}, \quad (4)$$

where the lagrangian density for geometry is the usual

$$\mathcal{L}_{geom} = \sqrt{-^{(4)}g} R, \tag{5}$$

where  $R$  is the Ricci scalar.

Using the metric for FRW:

$$ds^2 = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu = -N^2 dt^2 + A^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{6}$$

where  $N$  is the lapse function,  $A$  is the scale factor of the model, and  $\kappa$  is the curvature index of the universe ( $\kappa = 0, +1, -1$  plane, close and open, respectively)

The covariant components for the tensor metric are:

$$g_{tt} = -N(t)^2, \quad g_{rr} = \frac{A(t)^2}{1 - \kappa r^2},$$

$$g_{\theta\theta} = A^2 r^2, \quad g_{\phi\phi} = A^2 r^2 \sin^2 \theta, \tag{7}$$

and the contravariant components

$$g^{tt} = -\frac{1}{N^2}, \quad g^{rr} = \frac{1 - \kappa r^2}{A^2},$$

$$g^{\theta\theta} = \frac{1}{A^2 r^2}, \quad g^{\phi\phi} = \frac{1}{A^2 r^2 \sin^2 \theta}. \tag{8}$$

With these elements, we can calculate the nonzero Christoffel symbols:

$$\Gamma_{tt}^t = \frac{\dot{N}}{N}, \quad \Gamma_{tr}^r = \Gamma_{t\theta}^\theta = \Gamma_{\theta t}^\theta = \Gamma_{t\phi}^\phi = \Gamma_{\phi t}^\phi = \frac{\dot{A}}{A},$$

$$\Gamma_{rr}^r = \frac{-A\dot{A}}{N^2(\kappa r^2 - 1)}, \quad \Gamma_{rr}^r = \frac{-\kappa r}{\kappa r^2 - 1}, \quad \Gamma_{\theta\theta}^t = r(\kappa r^2 - 1),$$

$$\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \frac{1}{r}, \quad \Gamma_{\theta\theta}^t = \frac{Ar\dot{A}}{N^2},$$

$$\Gamma_{\phi\phi}^\theta = -\cos\theta\sin\theta, \quad \Gamma_{\phi\theta}^\phi = \Gamma_{\theta\phi}^\phi = \frac{\cos\theta}{\sin\theta},$$

$$\Gamma_{\phi\phi}^t = \frac{Ar^2\dot{A}\sin^2\theta}{N^2}, \quad \Gamma_{\phi\phi}^r = r\sin^2\theta(\kappa r^2 - 1), \tag{9}$$

where  $\dot{A} = dA/dt$ .

The Ricci scalar becomes

$$R = -\frac{6}{AN^2} \frac{d^2A}{dt^2} - \frac{6}{A^2N^2} \left(\frac{dA}{dt}\right)^2 + \frac{6}{AN^3} \frac{dA}{dt} \frac{dN}{dt} - \frac{6\kappa}{A^2}. \tag{10}$$

We consider a perfect fluid energy-momentum tensor

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho) U_\mu U_\nu, \tag{11}$$

where  $p, \rho, U_\mu$  are the pressure, energy density and the four-velocity of the system, respectively. Using the covariance of this tensor:

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{12}$$

we obtain the following partial differential equation:

$$3\frac{dA}{dt}\rho + 3\frac{dA}{dt}p + \frac{d\rho}{dt}A = 0, \tag{13}$$

and using the barotropic state function between the pressure and the energy density,  $p = \gamma\rho$ , with  $\gamma$  constant, we have the solution for the energy density as a function of the scale factor of the FRW universe as:

$$\rho = \frac{M_\gamma}{A^{3(\gamma+1)}}, \tag{14}$$

where  $M_\gamma$  is an integration constant.

The corresponding Einstein field equations are

$$G_0^0 = -8\pi GM_\gamma A^{-3(\gamma+1)} + \frac{3}{A^2 N^2} \left(\frac{dA}{dt}\right)^2 + 3A^{-2}\kappa = 0, \tag{15}$$

$$G_1^1 = \frac{2}{AN^2} \frac{d^2A}{dt^2} + \frac{1}{A^2 N^2} \left(\frac{dA}{dt}\right)^2 - \frac{2}{AN^2} \frac{dA}{dt} \frac{dN}{dt} + \frac{\kappa}{A^2} + \frac{8\pi GM_\gamma}{A^{3(\gamma+1)}} = 0. \tag{16}$$

We now consider the gauge  $N = 1$  and making some algebra we get the following master equation for the scale factor  $A$ :

$$\frac{d^2A}{dt^2} + \frac{4\pi GM_\gamma(3\gamma+1)}{3} \frac{1}{A^{3\gamma+2}} = 0. \tag{17}$$

Here we use a power law ansatz

$$A = A_0(t - t_0)^q, \tag{18}$$

with  $q$  and  $A_0$  parameters to be determined in terms of the parameters of the models.

Introducing this ansatz in Eq. (17) we obtain that

$$q = \frac{2}{3(\gamma+1)}, \quad A_0 = \left[6\pi GM_\gamma(\gamma+1)^2\right]^{\frac{1}{3(\gamma+1)}}. \tag{19}$$

Thus, the scale factor  $A$  has the following well-known classical behaviour [3]:

$$A = \left[6\pi GM_\gamma(\gamma+1)^2\right]^{\frac{1}{3(\gamma+1)}} (t - t_0)^{\frac{2}{3(\gamma+1)}}. \tag{20}$$

Taking different values for the constant  $\gamma$  we have the following subcases:

$$A = \begin{cases} \left[ \frac{32}{3} \pi G M_{\frac{1}{3}} \right]^{\frac{1}{4}} (t - t_0)^{\frac{1}{2}} & \text{for } \gamma = \frac{1}{3} \text{ radiation} \\ [6\pi G M_0]^{\frac{1}{3}} (t - t_0)^{\frac{2}{3}} & \text{for } \gamma = 0 \text{ dust} \\ [24\pi G M_1]^{\frac{1}{6}} (t - t_0)^{\frac{1}{3}} & \text{for } \gamma = 1 \text{ stiff fluid} \end{cases} \quad (21)$$

However, for the case  $\gamma = -1$ , we will solve the Eq. (17), whose solution is exponential:

$$A = A_0 e^{Ht}, \quad \text{with } H = 2\sqrt{\frac{2}{3}\pi G M_{-1}}, \quad (22)$$

here, we consider the sign (+) in the exponential function, because we consider the inflationary behaviour.

Using the line element for FRW, the density lagrangian for geometry has the following structure:

$$\begin{aligned} \mathcal{L}_{geom} &= \sqrt{-^{(4)}g} R = -\frac{6A^2}{N} \frac{d^2 A}{dt^2} - \frac{6A}{N} \left( \frac{dA}{dt} \right)^2 \\ &\quad + \frac{6A^2}{N^2} \frac{dA}{dt} \frac{dN}{dt} - 6\kappa N A \\ &= \frac{d}{dt} \left( \frac{-6A^2 \dot{A}}{N} \right) + \frac{6A}{N} \left( \frac{dA}{dt} \right)^2 - 6\kappa N A, \end{aligned} \quad (23)$$

and the matter density lagrangian [4, 5] is:

$$\begin{aligned} \mathcal{L}_{matter} &= -16\pi N \rho \left\{ (\gamma + 1) (1 + g^{km} U_k U_m)^{\frac{1}{2}} \right. \\ &\quad \left. - \gamma (1 + g^{km} U_k U_m)^{-\frac{1}{2}} \right\} + 16\pi \rho (\gamma + 1) U_m N^m. \end{aligned} \quad (24)$$

In  $\mathcal{L}_{matter}$  we consider the comovil fluid ( $U_k = 0$ ), and the gauge  $N^k = 0$ , obtaining

$$\mathcal{L}_{matter} = 16\pi N M_{\gamma} A^{-3\gamma}. \quad (25)$$

Finally, the total density lagrangian has the following form

$$\begin{aligned} \mathcal{L}_{tot} &= \frac{d}{dt} \left( \frac{-6A^2 \dot{A}}{N} \right) + \frac{6A}{N} \left( \frac{dA}{dt} \right)^2 \\ &\quad - 6\kappa N A + 16\pi G N M_{\gamma} A^{-3\gamma}. \end{aligned} \quad (26)$$

### 3. Hamiltonian formulation

Following the well-known procedure for obtaining the canonical hamiltonian function, we define the canonical momentum conjugate to the generalized coordinate  $A$  (scale factor) as  $\Pi_A \equiv \partial L / \partial \dot{A}$ :

$$L = 6 \left[ \frac{A}{N} \left( \frac{dA}{dt} \right)^2 - N\kappa A + \frac{8}{3}\pi G N M_{\gamma} A^{-3\gamma} \right], \quad (27)$$

or in its canonical form

$$\begin{aligned} L &= \Pi_A \dot{A} - N H \\ &= \Pi_A \dot{A} - N \left[ \frac{\Pi_A^2}{24A} + 6\kappa A - 16\pi G M_{\gamma} A^{-3\gamma} \right], \end{aligned} \quad (28)$$

where

$$H = \frac{\Pi_A^2}{24A} + 6\kappa A - 16\pi G M_{\gamma} A^{-3\gamma}, \quad (29)$$

when we perform the variation of this lagrangian (28) with respect to  $N, \partial L / \partial N = 0$ , implying  $H = 0$ .

The quantization procedure will be made in the usual way, considering the momentum as operators and taking representation for them, but it is possible to realize other type of quantization for this same model. For example, the supersymmetric quantum mechanics scheme [6-8] is

$$H \rightarrow \hat{H}\Psi = 0, \quad \Pi_A \equiv -i\hbar \frac{\partial}{\partial A}, \quad (30)$$

where  $\Psi(A)$  is the wave function of the FRW universe model. In this work we take  $\hbar = 1$ .

With these assumptions, (29) is transformed into a non linear differential equation:

$$\hat{H} = \frac{1}{24A} \left[ -\frac{d^2}{dA^2} + 144\kappa A^2 - 384\pi G M_{\gamma} A^{-3\gamma+1} \right]. \quad (31)$$

In Ref. 7 was shown that closed, radiation-filled FRW quantum universe for arbitrary factor ordering obey the Whittaker equation.

One important result yields at the level of WKB method, where we perform the transformation  $\Pi_A \rightarrow d\Phi/dA$ , then (29) is transformed in the Einstein-Hamilton-Jacobi equation, where  $\Phi$  is the superpotential function, that is related to the physical potential under consideration.

Introducing this ansatz in (29),

$$H = \frac{1}{24A} \left[ \left( \frac{d\Phi}{dA} \right)^2 + 144\kappa A^2 - 384\pi G M_{\gamma} A^{-3\gamma+1} \right], \quad (32)$$

thus, the superpotential  $\Phi$  has the following form:

$$\Phi = \pm \int \sqrt{384\pi G M_{\gamma} A^{-3\gamma+1} - 144\kappa A^2} dA, \quad (33)$$

where for whatever  $\gamma$  the integral has the solution

$$\int \sqrt{384\pi GM_\gamma A^{-3\gamma+1} - 144\kappa A^2} dA = \sqrt{-144\kappa A^2 + 384\pi GM_\gamma A^{1-3\gamma}} \times \left\{ \frac{1}{2} A + \frac{\gamma + \frac{1}{3}}{(-1 + \gamma)(3A^{1+3\gamma\kappa} - 8GM_\gamma\pi)} \left( AGM_\gamma \sqrt{2\pi} \sqrt{\frac{-3A^{1+3\gamma\kappa} + 8GM_\gamma\pi}{GM_\gamma}} \times {}_2F_1 \left[ -\frac{3(-1 + \gamma)}{2(1 + 3\gamma)}, \frac{1}{2}, 1 - \frac{3(-1 + \gamma)}{2(1 + 3\gamma)}, \frac{3A^{1+3\gamma\kappa}}{8GM_\gamma\pi} \right] \right) \right\}, \quad (34)$$

where  ${}_2F_1$  is the hypergeometric function.

However, we can solve the integral in (33) for particular cases of the  $\gamma$  parameter, that is:

- radiation case:  $\gamma = \frac{1}{3}$

$$\int \sqrt{384\pi GM_{1/3} - 144\kappa A^2} dA = \frac{A}{2} \sqrt{384\pi GM_{1/3} - 144\kappa A^2} + \frac{16iG\pi M_{1/3}}{\sqrt{\kappa}} \ln \left\{ -24iA\sqrt{\kappa} + 2\sqrt{384\pi GM_{1/3} - 144\kappa A^2} \right\}, \quad (35)$$

- dust fluid:  $\gamma = 0$

$$\int \sqrt{384\pi GM_0 A - 144\kappa A^2} dA = \left( \frac{A}{2} - \frac{2G\pi M_0}{3\kappa} \right) \sqrt{384\pi GM_0 A - 144\kappa A^2} - \frac{16G^2 M_0^2 \pi^2 \sqrt{384\pi GM_0 A - 144\kappa A^2} \ln \left( 2\sqrt{3\kappa A} + 2\sqrt{3\kappa A - 8G\pi M_0} \right)}{3\sqrt{3\kappa^3 A} \sqrt{3\kappa A - 8G\pi M_0}}, \quad (36)$$

- inflation like case:  $\gamma = -1$

$$\int \sqrt{384\pi GM_{-1} A^4 - 144\kappa A^2} dA = \frac{1}{2A} \left( \frac{2A^2}{3} - \frac{\kappa}{4G\pi M_{-1}} \right) \sqrt{384\pi GM_{-1} A^4 - 144\kappa A^2}, \quad (37)$$

- stiff fluid:  $\gamma = 1$

$$\int \sqrt{384\pi GM_1 A^{-2} - 144\kappa A^2} dA = \sqrt{384\pi GM_1 A^{-2} - 144\kappa A^2} \times \left\{ \frac{1}{2A} + \frac{iA\sqrt{2\pi GM_1}}{\sqrt{3\kappa A^4 - 8\pi GM_1}} \ln \left( -\frac{4i\sqrt{2\pi GM_1}}{A^2} + \frac{2\sqrt{3\kappa A^4 - 8\pi GM_1}}{A^2} \right) \right\}. \quad (38)$$

These results will be used in the next section, to obtain the solution according to the supersymmetric quantum mechanics scheme.

### 4. Supersymmetric quantum solutions

In order to include the factor ordering problem, we substitute the following relation into (31), that corresponds to what Hartle and Hawking [9] called a semi-general factor ordering

$$A^{-1} \frac{d^2\Psi}{dA^2} \rightarrow A^{-1+p} \frac{d}{dA} A^{-p} \frac{d\Psi}{dA} = A^{-1} \left( \frac{d^2\Psi}{dA^2} - pA^{-1} \frac{d\Psi}{dA} \right), \quad (39)$$

where the real parameter  $p$  measures the ambiguity in the factor ordering. So, the Wheeler-DeWitt equation can be written as follows:

$$\mathcal{H}_0 \Psi = -A \frac{d^2\Psi}{dA^2} + p \frac{d\Psi}{dA} - V(A) \Psi = 0, \quad (40)$$

with  $V(A) = 384\pi GM_\gamma A^{-3\gamma+2} - 144\kappa A^3$

In this scheme, we start giving the following super-hamiltonian:

$$H_{super} := \left( \mathcal{H}_0 + F \frac{\partial^2 \Sigma(A)}{\partial q^\nu \partial q^\mu} [\bar{\psi}, \psi] \right), \quad (41)$$

where the bosonic hamiltonian  $\mathcal{H}_0$  corresponds to the one in eq. (40),  $F$  is a complex function and  $\Sigma$  is the superpotential function. We write the super-charges as follows:

$$Q = \psi \left( f(A) \frac{d}{dA} + i \frac{d\Sigma(A)}{dA} \right), \quad (42)$$

$$\bar{Q} = \bar{\psi} \left( f(A) \frac{d}{dA} - i \frac{d\Sigma(A)}{dA} \right), \quad (43)$$

where  $f(A)$  is an auxiliary function to be determined via the analogy with the hamiltonian under study.

We suppose the following algebra for the variables [2]  $\psi$  and  $\bar{\psi}$ :

$$\{\psi, \bar{\psi}\} = -1, \quad \{\psi, \psi\} = 0, \quad \{\bar{\psi}, \bar{\psi}\} = 0. \quad (44)$$

Using the representation  $\psi = -d/d\theta^0$  and  $\bar{\psi} = \theta^0$ , one finds the superspace hamiltonian to be written in the form

$$\begin{aligned} H_{super} \Psi &= \{Q, \bar{Q}\} \Psi = (Q\bar{Q} + \bar{Q}Q) \Psi \\ &= \left( -f^2(A) \frac{d^2}{dA^2} - f(A) \frac{df}{dA} \frac{d}{dA} - \left( \frac{d\Sigma(A)}{dA} \right)^2 \right. \\ &\quad \left. + i f(A) \frac{d^2 \Sigma(A)}{dA^2} [\bar{\psi}, \psi] \right) \Psi. \end{aligned} \quad (45)$$

This last equation is similar to (41).

Comparing Eqs. (40) and (45) we obtain the following relations:

$$f^2(A) = A, \quad p = -\frac{1}{2}, \quad V(A) = \left( \frac{d\Sigma(A)}{dA} \right)^2. \quad (46)$$

We can see that the parameter that gives the measure of the factor ordering is fixed at  $p = -\frac{1}{2}$  in this approach, leading to

$$-f(A) \frac{df}{dA} \equiv p. \quad (47)$$

Then, any hamiltonian equation in one dimension that obeys this relation, will have the factor ordering fixed in the supersymmetric regime. In other cases, it will be necessary to study the particular hamiltonian equation and the supersymmetric scheme.

Moreover, in this scheme, any physical state must obey the following quantum constraints:

$$\bar{Q}\Psi = 0, \quad (48)$$

$$Q\Psi = 0. \quad (49)$$

The wave function has the following decomposition in the Grassmann variables representation

$$\Psi = \Pi_+ + \Pi_- \theta^0, \quad (50)$$

where the component  $\Pi_+$  is the contribution of the bosonic sector, and whereas  $\Pi_-$  is the contribution of the fermionic sector.

The supercharges read as

$$Q = - \left( \sqrt{A} \frac{d}{dA} + i D_A \Sigma \right) \frac{d}{d\theta^0}, \quad (51)$$

$$\bar{Q} = \theta^0 \left( \sqrt{A} \frac{d}{dA} - i D_A \Sigma \right), \quad (52)$$

where  $D_A = d/dA$ .

Using Eq. (48), we get the following differential equation:

$$\left( \sqrt{A} \frac{d\Pi_+}{dA} - i D_A \Sigma \Pi_+ \right) = 0. \quad (53)$$

The solution of the latter equation is

$$\Pi_+ = \Pi_{0+} e^{i \int \frac{1}{\sqrt{A}} D_A \Sigma dA}, \quad (54)$$

where  $\Pi_{0+}$  is an integration constant.

Employing Eq. (49) one gets

$$\left( \sqrt{A} \frac{d\Pi_-}{dA} + i D_A \Sigma \Pi_- \right) = 0, \quad (55)$$

where  $\Pi_-$  has the form:

$$\Pi_- = \Pi_{0-} e^{-i \int \frac{1}{\sqrt{A}} D_A \Sigma dA}. \quad (56)$$

Equations (54) and (56) can be written in the following way:

$$\Pi_{\pm} = \Pi_{0\pm} e^{\pm i \int \frac{1}{\sqrt{A}} D_A \Sigma dA}. \quad (57)$$

The integration in these equations corresponds exactly to equation (33). Thus, the supersymmetric quantum solutions are obtained in closed form.

In the supersymmetric fashion, the calculation by means of the Grassmann variables of  $|\Psi|^2$  given by (50) is well known [10]:

$$\begin{aligned} (\Psi_1 | \Psi_2) &= \\ &= \int (\Psi_1(\theta^*))^* \Psi_2(\theta^*) e^{-\sum_i \theta_i^* \theta_i} \prod_i d\theta_i^* d\theta_i, \end{aligned} \quad (58)$$

where the operation  $*$  is defined as:

$$(C\theta_1 \dots \theta_n)^* = \theta_n^* \dots \theta_1^* C^*,$$

with the usual algebra for the Grassman numbers  $\theta_i \theta_j = -\theta_j \theta_i$ . The rules to integrate over these numbers are the following:

$$\int \theta_1 \theta_1^* \dots \theta_n \theta_n^* d\theta_n^* d\theta_n \dots d\theta_1^* d\theta_1 = 1, \quad (59)$$

$$\int d\theta_i^* = \int d\theta_i = 0. \quad (60)$$

In our case, we have  $\Psi_1 = \Psi_2 = \Psi$ . So, when we integrate to the Grassmann numbers, employing also the relations (59) and (60), we obtain

$$|\Psi|^2 = \bar{\square}_+ \square_+ + \bar{\square}_- \square_-, \quad (61)$$

where the  $\bar{\square}$  symbol means the complex operation.

Using the expressions for the functions  $\square_+$  and  $\square_-$  given in (54) and (56), respectively, we arrive to the following expression for the probability density:

$$|\Psi|^2 = \square_{0+}^2 + \square_{0-}^2. \quad (62)$$

Thus, we are able to express Eq. (58) for our particular problem. As can be seen, we can infer that the contributions

of the bosonic and fermionic sectors of the density probability are equal.

## 5. Conclusions

The main results in this work are to provide the methodology to find the general form for all contributions that occurred in the expansion of the FRW wave function of the Universe with matter, within the approach of Witten's supersymmetric quantum mechanics. In addition, we find one criterion for fixing the parameter that measure the factor ordering of the operators. Besides, we find that the exact solutions for the empty (+) and filled (-) sector of the fermion Fock space are at the same time the lowest-order WKB approximations (Einstein-Hamilton-Jacobi equation). Finally, we find the general form of the probability density, Eq. (58), for the FRW case, including matter fields.

## Acknowledgments

We thank H.C. Rosu and N. Lopez for critical reading of the manuscript. This work was partially supported by CONACYT.

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1. E. Witten, *Nucl. Phys. B* **188** (1981) 513.
  2. J. Socorro and E.R. Medina, *Phys. Rev. D* **61** (2000) 087702-1.
  3. Jorge Cervantes Cota, *Induced Gravity and Cosmology* in Konstanz University dissertation, Hartung-Gorre Verlag, (1996).
  4. Michael Ryan Jr, in *Hamiltonian Cosmology*, (Springer Verlag 1972).
  5. Enrique Pazos (tesis licenciatura), *Aplicación del formalismo lagrangiano ADM a un modelo cosmológico*, Univ. de San Carlos de Guatemala (2000).
  6. H.C. Rosu and J. Socorro, *Phys. Lett. A* **233** (1996) 28.
  7. H.C. Rosu and J. Socorro, *Il Nuovo Cimento B* **112** (1998) 119.
  8. H.C. Rosu, *Mod. Phys. Lett. A* **15** (2000) 979.
  9. J. Hartle and S. W. Hawking, *Phys. Rev. D* **28** (1983) 2960.
  10. L.D. Faddeev and A.A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, (Addison-Wesley, Reading, MA.), sec. 2.5. (1991).