An estimation method of fractal dimension of self-avoiding roughened interfaces

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Two kinds of methods (graphical and statistical) commonly used for the estimation of fractal dimension of self-avoiding interfaces were investigated. It was determined that the current methods of both kinds have significant errors for this type of profiles. In the present work a novel efficient method for the estimation of fractal dimension of self-avoiding curves embedded in the space $\mathbb{R}^2$ based on the Box-count and Hall-Wood estimators is developed. Some physical implications are discussed.

Keywords: Fractal dimension; Roughened interface; Hall-Wood; Box-count; Image processing.

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1. Introduction

The term “fractal” defines irregular, rough or complicated shapes that cannot be described by the Euclidean geometry [1]. The fractal dimension, D, can be defined as a measure of the complexity or roughness of this kind of shapes and can be treated as the degree to which a set “fills” the Euclidean space in which it is embedded [2]. Due to the above mentioned properties the value of D is fractional. The fractal dimension has a large number of applications in different areas of science and engineering, such as: modeling of forest fires [3], flow in porous media [4], imbibition in disorder media [5], percolation [6], electroosmotic in porous media [7], which can have scaling behavior of self-avoiding walks [8]. In many of these cases, it is necessary to estimate the fractal dimension of bitmap images of rough self-avoiding curves embedded in $\mathbb{R}^2$ captured by different devices, depending on research area. Currently, there are a lot of estimators of D obeying different power laws [9–11], which are divided into graphical (image) and statistical (data) estimators.

In practice, the estimation of fractal dimension of a rough interface from an image is challenging [12] and there are two types of procedures for the estimation. The first type is implemented with the use of estimators (graphical) that operate binary images of interfaces and estimate D directly. For example, software Benoit 1.3 [13] and FracLac [14] perform such routines based on the Box-count [1] estimations, among others. In the second kind of procedures the binary image of a roughened interface, which can be represented as a self-avoiding walk [8], is transformed into a single-valued series, and the estimation of D is performed using the numerical information from this transformed profile [4] choosing an adequate estimator for time series. This kind of methods results in a modification of profile information due to the cutting or filling of the ridges and valleys formed, which leads to an error in the estimation of the fractal dimension of the original interface profile (an example of this procedure is presented below in the Sec. 3).

In this work we demonstrate that both kinds of actual methods do not provide reliable estimates for self-avoiding profiles. The aim of the work is the development of a novel method of estimation of fractal dimension D based on the generalization of the estimators of Box-count and Hall-Wood for self-avoiding rough interfaces embedded in $\mathbb{R}^2$ without transforming them into a series of single-valued profile and demonstration of its efficacy in comparison with the traditional methods.

2. Methods of fractal dimension estimation

Essentially, all the methods designed for the estimation of D are based on a common scheme in which D is induced as a function of scale governed by a power law and the estimate is obtained using a linear regression fit with least squares method. It should be noted that not all the estimators are applicable to the same kind of data [15], therefore in order to obtain a reliable estimate it is necessary to perform a careful choice of estimation method in dependence on the data type and its representation (statistical data series or image).

2.1. Box-count estimator

Box-count estimator is one of the most popular and commonly used estimators, and it is motivated by the following scaling law:

$$D_{BC} = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} \quad (1)$$

where D_{BC} is the Box-counting dimension, $N(\epsilon)$ denotes the smallest number of boxes of width ϵ in the Euclidean space with dimension d, which can cover the point set:

$$X = \{(t, X_t) \in \mathbb{R} \times \mathbb{R}^d \,:\, t \in T,\, X_t \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1} \quad (2)$$

i.e. graph of time series or spatial data X_t observed at finite time set $T \subset \mathbb{R}$.

The main idea of this method is that initially the series graph is covered by a single bounding box, which is divided in four quadrants. This process is repeated subsequently with each of the quadrants until the resulting box width matches the resolution of the data recording the number of boxes required at each step. Thus, if $N(\epsilon)$ is the number of boxes required at scale ϵ, Box-count estimator equals the slope in an ordinary least squares regression fit of $\log N(\epsilon)$ on $\log \epsilon$. This method can be used to quantify the fractal dimension of any set of points in the plane and, particularly, for single-valued interfaces or time series data.

Some authors identified problems with the original Box-count estimator, which includes all scales in the least squares regression fit of $\log N(\epsilon)$ on $\log \epsilon$, and proposed modifications that allow obtaining reliable estimations of D for some kinds of data [15].

2.2. Hall-Wood estimator

Peter Hall y Andrew Wood [16] introduced a Box-count estimator modification for fractal dimension D_{HW}, which operates to the smallest observable scales. Let $A(\epsilon)$ be the total area of the boxes with scale ϵ that intersect with the linearly interpolated graph of X, then if $N(\epsilon)$ is the total number of such boxes, hence

$$A(\epsilon) \propto N(\epsilon)\epsilon^2, \tag{3}$$

which results in a reformulation of definition (1) being:

$$D_{HW} = 2 - \lim_{\epsilon \to 0} \frac{\log A(\epsilon)}{\log(\epsilon)} \tag{4}$$

On scale $\epsilon_\ell = \ell/n$, where n is the total number of data series and $\ell = 1, 2, ..., n$, Hall-Wood estimator works similarly to Box-count estimator, but the main difference is that the scale application operates only over the "columns" and counts $A(\ell/n)$.

This estimator has better accuracy as compared to the original method of Box-count [15], but its disadvantage is that it can only be applied to the case in which the set of points X is a single-valued profile (interface) of form (2), that is, the time series graph or spatial data observable on a finite set $T \subset \mathbb{R}$ of times or equally spaced points [10].

2.3. Madogram and Variogram estimators

In the case when the data is given as a time series data or spatial data observed on a finite set $T \subset \mathbb{R}$ of times or equally spaced points, the best results for the estimation of D are provided by the variational estimators, particularly Madogram and Variogram estimators [15], and, potentially, their use can be extended to anisotropic and non-stationary processes.

If $\{X_t \,:\, t \in T,\ X_t \in \mathbb{R}^d\}$ is a Gaussian process with stationary increments whose variogram or structure function is:

$$\gamma_2(t) = \frac{1}{2}\mathbb{E}(X_u - X_{u+t})^2$$

where $\gamma_2(t)$ is the average value of the squared difference of the values in two points separated by a distance or space t and $\mathbb{E}(.)$ denotes the mathematical expectation, then it holds:

$$\gamma_2(t) = |C_2 t|^\alpha + O\left(|t|^{\alpha+\beta}\right), \quad \text{as} \quad t \to 0,$$

where $\alpha \in (0, 2]$, $\beta \geq 0$, y $C_2 > 0$. Here α is the fractal index, β is roughness exponent, $|\cdot|$ denotes the Euclidean norm, so, a sample's path graph, has the fractal dimension:

$$D = d + 1 - \frac{\alpha}{2}\,. \tag{5}$$

The Variogram estimator is obtained using the classical method of moments estimator for $\gamma_2(t)$. In the publication of Bez and Bertrand [17] it was suggested that the Madogram estimator, that is, variational estimator obtained from the Variogram of order $p > 0$ of the form:

$$\gamma_p(t) = \frac{1}{2}\mathbb{E}(X_u - X_{u+t})^p$$

for $p = 1$, is more efficient and the relation (5) between the fractal dimension D and the fractal index α is more robust as compared to the default case $p = 2$ due to the fact that

$$\gamma_p(t) = |C_p t|^{\alpha p/2} + O\left(|t|^{(\alpha+\beta)p/2}\right), \quad \text{as} \quad t \to 0.$$

Generally, the lower is the value of roughness exponent β, the harder is the estimation of fractal index α, and is more pronounced the finite sample bias of estimators of the fractal index or fractal dimension [15].

3. Results

To compare the efficiency of common methods of fractal dimension estimation of both kinds (graphical and statistical) the degrees of variation of the estimates obtained by the both

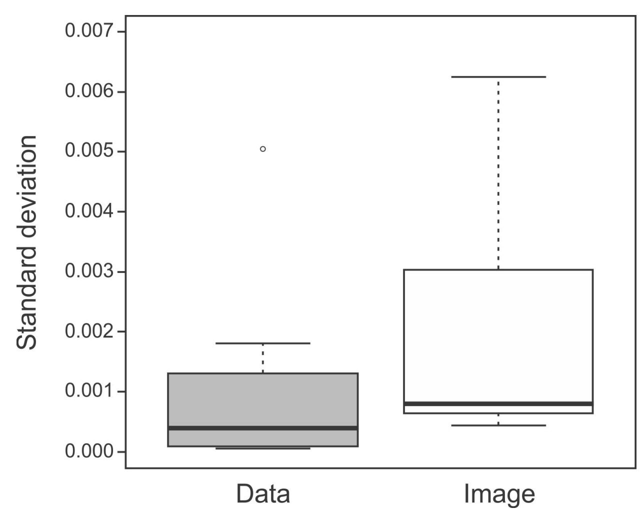

FIGURE 1. Comparison of standard deviations of estimates for same data series obtained by two types of estimators: graphical (image) and statistical (data).

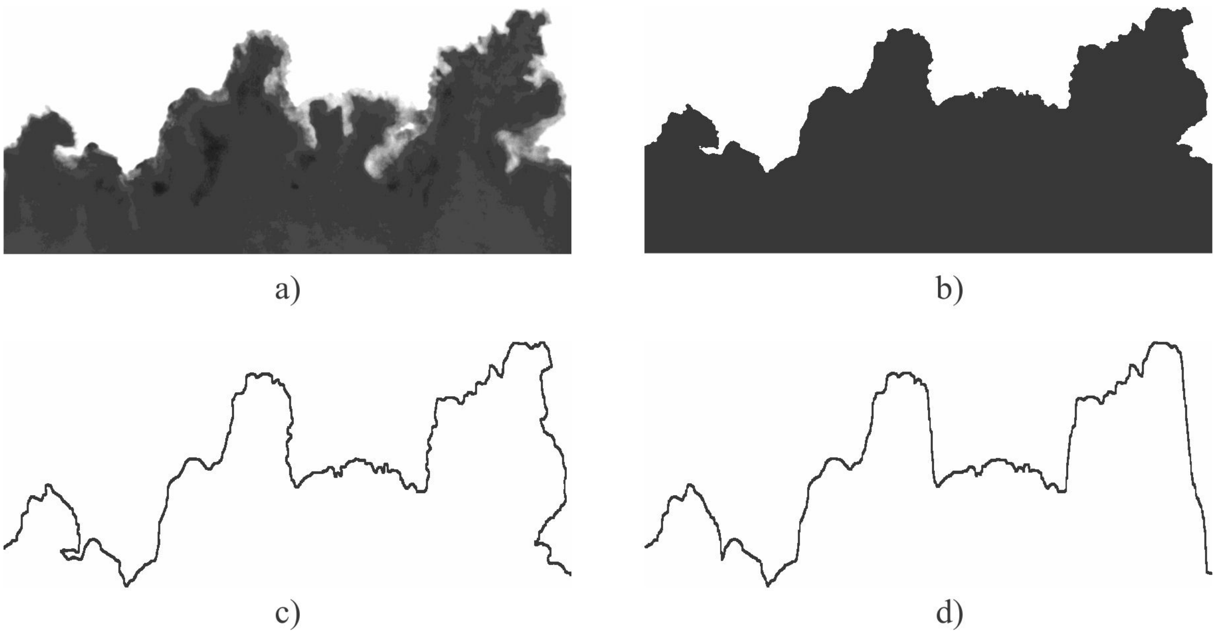

FIGURE 2. Transformation from self-avoiding walk to a single-valued series: a) Original image from a rough imbibition profile, b) Binary image of the rough imbibition profile, c) Self-avoiding profile, d) Self-avoiding profile transformed into a single-valued series.

kinds of estimators were investigated. For this aim, synthetic samples data series with a known D value were generated in MatLab [18]. All the samples had 1500 fractional values and were divided into three groups, with the D values of 1.2, 1.5 and 1.8. Thereafter the fractal dimensions of the data samples were numerically estimated by *fractaldim* package [19], using Box-count, Hall-Wood and Variogram statistical estimators. Furthermore, the data samples were also transformed in graphical bitmaps, binarizing the image, and after that their fractal dimensions were estimated using the *fractaldim* package in the same procedure for the data essays.

The results of the estimates from the two sources of information (data series and graphical bitmaps) of the samples were compared. In Fig. 1 the mean variances obtained for the 4500 samples of both kinds of estimators are shown.

Note that, for the same data series a simple transformation from numerical data to an image results in a notable variance in estimates. Namely, in Fig. 1 it can be seen that the estimates of fractal dimensions for images of data series have a wider range of results (white box), as compared to the estimates obtained by the statistical estimators for the original numerical data samples (grey box) and therefore the statistical estimators provide more reliable estimates for the single-valued profiles (data series).

However, these statistical estimators cannot be used directly for the estimation of fractal dimension of self-avoiding curves embedded in $\mathbb{R}^2$ obtained from an interface image. To apply statistical estimators of D for self-avoiding profiles, at first, it is necessary to transform them into a single-valued profiles. An example of the steps of this transformation is presented in Fig. 2 obtained for an image from imbibition in disordered media experiments.

Nevertheless, below we show that for the Koch curve [20], which has a self-avoiding profile, this transformation leads to a significant error in the estimation of D. Therefore, there is a need to develop an efficient graphical method of fractal dimension estimation for self-avoiding profiles.

In what follows we propose a method of estimation of D for self-avoiding profiles called *Fractionating estimator*, and demonstrate its efficiency applying it to the classic examples of such curves: Koch curve, modified Koch curve [21], G_2 Modified Koch curve [22] and Weierstrass curve [23].

3.1. Fractionating estimator (FE)

Using as reference the Hall-Wood estimator with scale $\epsilon_\ell = \ell/n$ adjusted equidistantly to a data series of form (2), namely, single valued profiles, we developed a procedure, which allows estimating the fractal dimension from a self-avoiding curve embedded in $\mathbb{R}^2$ obtained from an interface image. This method, called *Fractionating estimator* $(\mathbf{FE})$, consists in the following. At the first step the self-avoiding curve is covered by a bounding box and after that at the k-th step, where $k = 2, 3, \ldots, m$, this box is divided into k^2 equal boxes, with this goal in mind each side of the bounding box is divided into k equal segments (Fig. 3a). Then the intersection points between the self-avoiding curve and the obtained grid are determined and a broken line is constructed by joining consecutively the intersection points (Fig. 3b). Thereafter, the segments of the broken line are covered by individual bounding rectangles with consecutive vertices in break points (Fig. 3c) and the number of such rectangles $\tilde{N}(k)$ at step k is counted, and so on (Fig. 3d). The $\mathbf{FE}$ estimation of fractal dimension is obtained as the linear regression fit from the log-log plot, of $\tilde{N}(k)$ *vs.* k^{-1}, for $k = 1, 2, \ldots, m$. Here the number of steps m is an unbounded natural number,

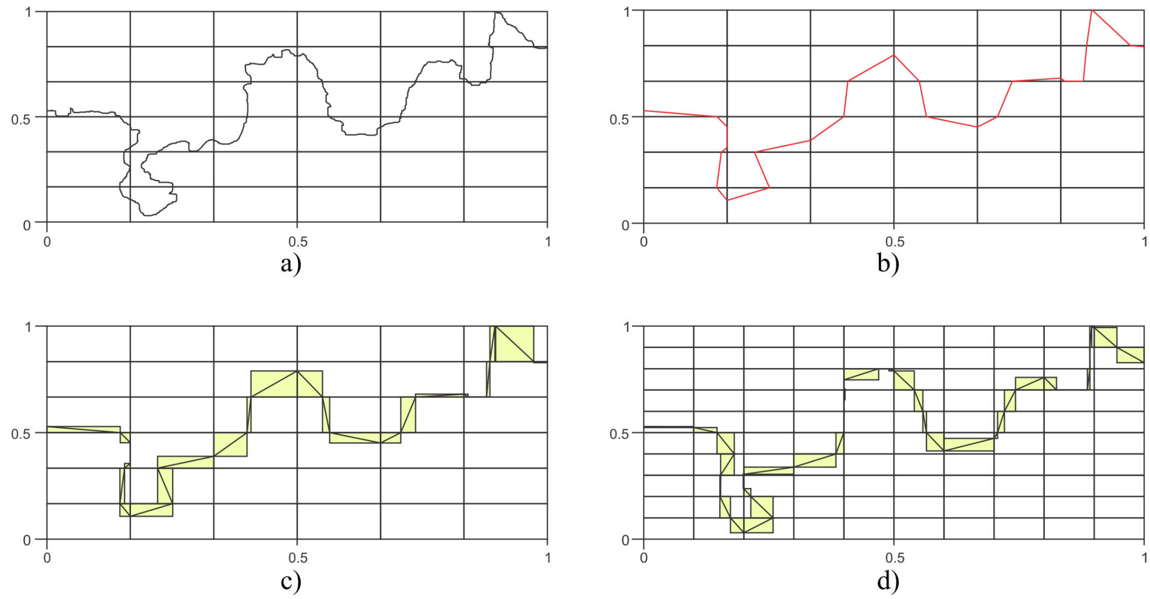

FIGURE 3. Operation of **FE** on a self-avoiding curve: a) grid (k = 6), b) adjusted broken line (k = 6), c) the box cover of the adjusted broken line (k = 6), d) the box cover of the adjusted broken line (k = 10).

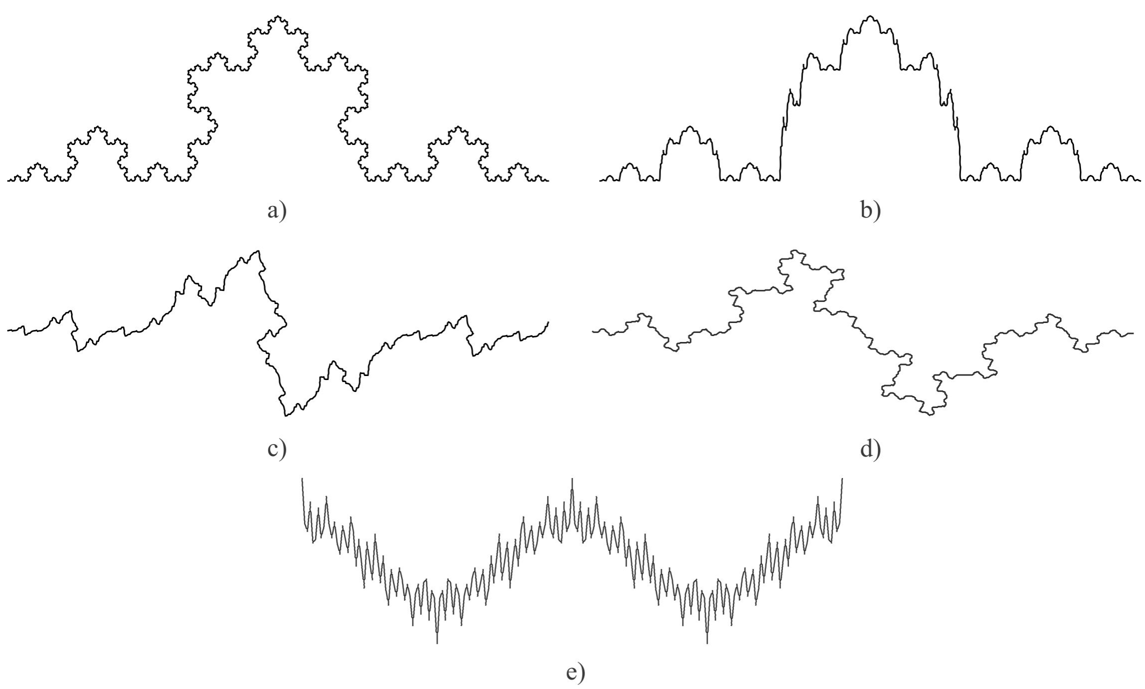

FIGURE 4. a) Koch curve generated with 4200 nodes, b) Single-valued transformation of Koch curve, c) modified Koch curve generated with 3200 nodes, d) G_2 modified Koch curve generated with 6680 nodes, e) Weierstrass curve generated with 1571 nodes.

which depends on the image resolution and computational capacity, the greater the number m the greater the accuracy of the estimate of D.

Note that meanwhile the number of step k increases the broken line of **FE** approaches the interface curve. Furthermore, **FE** splits the rough self-avoiding curve not equidistantly, and in this way, it takes into account as much information from the rough curve both in rows and columns, "breaking it down" into fractional straight segments, preserving the Hall-Wood scale idea but counting boxes as in the Box-count estimator.

3.2. Fractal dimension estimation of Koch curve, modified Koch curves and Weierstrass curve

To verify the efficiency of **FE**, proposed above, a routine in Visual Basic for Applications for Corel Draw X3 was developed. The developed program was used for the estimation of fractal dimension by **FE** of the Koch curve (Fig. 4a) with theoretical $D = 1.2619$, single-value transformation of Koch curve (Fig. 4b), modified Koch curve (Fig. 4c) with theoretical $D = 1.1609$, G_2 modified Koch curve (Fig. 4d) with theoretical $D = 1.2924$ and the Weierstrass curve (Fig. 4e) of the form:

$$w(x) = \sum_{k=0}^{\infty} a^k \cos(\pi b^k x)$$

for $a = 0.5, b = 33$ and theoretical $D = 1.8017$.

In addition the Box-count estimator included in software Benoit 1.31 and the plug-in for *Image J* software of FracLac 1.48, and the estimators Box-count, Hall-Wood, Variogram and Madogram of *fractaldim* package of software *R*, where used for the estimations of D for the considered fractals curves.

The **FE** routine was performed directly to the generated curve without transforming it to a single-valued profile, and the original vector graphic image was exported to *.bmp* format for Benoit 1.31 and FracLac 1.48, where the D was also estimated directly from the image using the Box-count estimator.

As the software *R* works only with numerical data the Koch curve was transformed into a single-valued curve (Fig. 4b) in *.png* format. After that the binarized image was interpreted as a matrix of ones and zeros, each column of ones are accounted to get a series of single-valued profile, from which D was estimated by Box-count, Hall-Wood, Variogram and Madogram estimators.

TABLE I. Average values of fractal dimension estimation for the single-valued Koch curve.

Estimation method	Estimate of D
Box-count	1.1811
Hall-Wood	1.2181
Variogram	1.3195
Madogram	1.1893

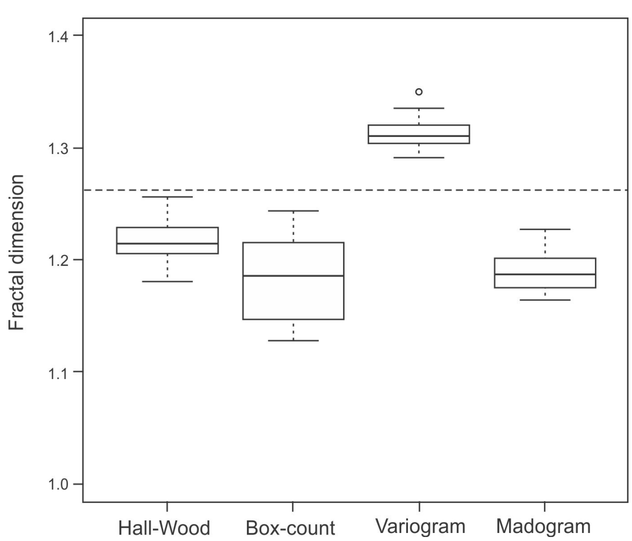

FIGURE 5. Fractal dimension estimations from the transformed single-valued Koch curve.

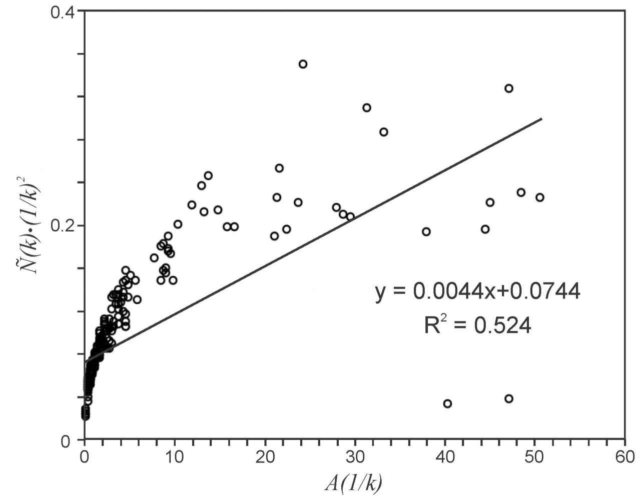

FIGURE 6. Hall-Wood relation (3) of Koch curve.

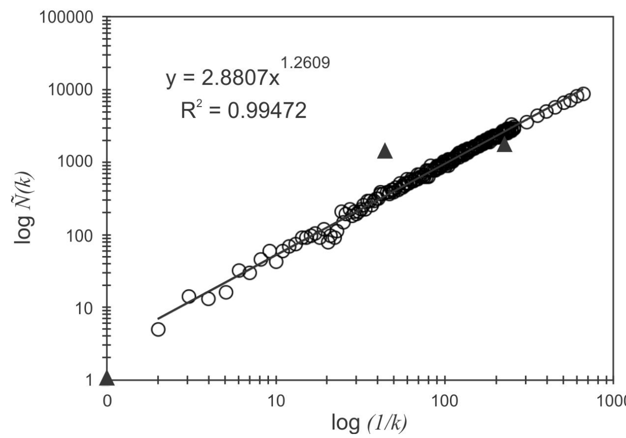

FIGURE 7. Fractal dimension estimation for Koch curve obtained by FE method.

The results of the estimates of the single-valued Koch curve (Fig. 4b), are presented in a synthesized way by Boxplot graph in Fig. 5, where the theoretical and expected dimension $D = 1.2619$ of the Koch curve is marked with dotted horizontal line.

Numerical results of the average estimates for a single-valued profile generated from the Koch curve obtained using the *fractaldim* package are presented in Table I.

From Table I it can be seen that the Hall-Wood estimator has the best performance, followed by Variogram, but none of the considered estimators gives a reliable estimate. Above mentioned results confirm the fact that the transformation of the original Koch curve into a single-valued profile leads to a significant errors in fractal dimension estimation.

Now, performing the estimation of D for the original Koch curve by **FE**, it was observed, as expected, that for self-avoiding interfaces the basic proportionality of Hall-Wood method (3) does not hold (Fig. 6).

TABLE II. Fractal dimension estimates of theoreticall self-avoiding curves: Koch curve, modified Koch curve, G_2 modified Koch curve and Weirstrass curve.

Estimation method	Koch curve	Modified Koch curve	G_2 Modified Koch curve	Weierstrass function
FE	1.2609	1.774	1.2782	1.7847
Box-count (Benoit)	1.9424	1.9715	1.8778	1.9524
Box-count (FracLac)	1.3019	1.2391	1.3401	1.4533

At the same time, from Fig. 7, one can observe a power law behavior with the coefficient of determination R^2 very close to 1 and the estimated value of D obtained by the proposed **FE** method is 1.2609 and approximates the theoretical $D = 1.2619$.

In Fig. 7 the points indicated as triangles correspond to outliers and were not taken into account for the linear regression fit, as the value of the coefficient of determination R^2 in that case reduces to $\mathrm{R}^2 = 0.9873$. The fractal dimension of the original Koch curve was also estimated by the Box-count estimator using the software Benoit and FracLac. The same estimators were used for modified Koch curve (Fig. 4c), G_2 modified Koch curve (Fig. 4d) and Weierstrass curve (Fig. 4e).

The values of estimates of fractal dimensions of the Koch curve, modified Koch curve, G_2 modified Koch curve and Weierstrass curve obtained by **FE**, Benoit and FracLac methods are presented in Table II.

As it can be seen from Tables I and II, the developed **FE** method gives significantly better estimates for both curves studied as compared to the commonly used methods of estimation.

Furthermore, as the scaling behavior of a profile with overhangs (Fig. 2), can lead from self-affinity to multiaffinity by the removal of overhangs in the representation of real interface by single-valued profile [24], the proposed method avoid this problem and there is no need of further study scaling and universality for a varied self-avoiding curves.

4. Conclusions

In this work a study of graphical and statistical estimators such as Box-count, Hall-Wood, Variogram, and Madogram for self-avoiding curves embedded in $\mathbb{R}^2$ was performed. It was determined that both types of actual methods have a significant error and do not give a reliable estimate for this kind of fractals.

A new method for the estimation of fractal dimension for self-avoiding rough interfaces based on the efficiency of Hall-Wood estimator was developed and evaluated. The estimates of fractal dimensions obtained by the proposed method of *Fractionating Estimator* for the considered fractal curves closely approximate the theoretical dimension of the fractals and give more precise estimations as compared with the commonly used estimators. Results obtained validate the application and use of the proposed method for the calculation of D for the rough interfaces as well as time series.

Acknowledgments

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