Universal relations for three-dimensional thermal, electric and magnetic properties

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A periodic fibre-reinforced two phase material is considered here. The material properties of the constituents are anisotropic and may be specified by either the thermal, electrical conductivity, dielectric or magnetic permeability tensor. The orientation of each of the characteristic directions of the material is not parallel to the axis of geometric symmetry. The asymptotic homogenization method is used to study this doubly periodic composite. The interface between fibres and matrix must conform with the periodicity, but otherwise is arbitrary and may be disjoint in the periodic cell. A connection is found among the solutions of the three local problems that appear in the implementation of the method. This relation is basic for the elementary derivation of three universal relations for this three dimensional scalar problem. Several examples show how to relate the structure of the material tensor considered in this paper to that provided by the material data.

Keywords: Universal relations; asymptotic homogenization method; thermal and electrical conductivity; dielectric properties; magnetic permeability.

Se considera un material periódico de dos fases reforzado por fibras. Las propiedades materiales de las componentes son anisótropas y pueden especificarse por el tensor, ya sea térmico, de conductividad eléctrica, de permeabilidad dieléctrica o magnética. La orientación de cada una de las direcciones características del material no es paralela al eje de simetría geométrica. Se usa el método de homogeneización asintótica para estudiar este compuesto doblemente periódico. La intercara entre fibras y matriz debe cumplir con la periodicidad, pero por lo demás es arbitraria y puede ser disjunta en la celda periódica. Se encuentra una conexión entre las soluciones de los tres problemas locales que aparecen en la instrumentación del método. Esta relación es básica para la derivación elemental de tres relaciones universales para este problema tridimensional escalar. Varios ejemplos muestran cómo relacionar la estructura del tensor material considerado en este trabajo con los datos del material.

Descriptores: Relaciones universales; método de homogeneización asintótica; conductividad térmica y eléctrica; propiedades dieléctricas; permeabilidad magnética.

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1. Introduction

The asymptotic homogenization method is a useful mathematical tool for dealing with partial differential equations that have rapidly oscillating coefficients [1–2]. The presence of two characteristic lengths l (fast) and L (slow) in the ratio $\epsilon = l/L$ allows the use of the two-scale asymptotic method, when ϵ is a small parameter. The original boundary value problem is transformed into one with constant coefficients, so-called effective or homogenized. An explicit formula is obtained for the coefficients, which depends on the periodic solution of certain cell problems (local). There are a few exact solutions of these local problems, some obtained very recently [3–9]. Otherwise the main thrust of the implementation of the method is numerical. (See references in Refs. 6 and 7]). New theoretical results within the frame of this method were obtained in [5–9]. One result relates three solutions of the local problems. The other one is a new derivation of a series of universal relations which reproduce well known results: those of Hill [10] and Dvorak [11] in elasticity and Benveniste and Dvorak [12] and Schulgasser [13] in piezoelectricity. The equation that connects the solution of several local problems is the key to derive the universal relations for the effective coefficients, besides the symmetry considerations of the latter. It was pointed out in the four papers already mentioned that this new derivation of universal relations may be useful to derive other new ones, which is indeed the case here, where the problem considered is the simplest that can be thought of. A related question is that of exact relations. Nowadays the field of exact relations is plentiful. The search for necessary and sufficient conditions for exact relations is currently under way [14-17] using complicated algebraic methods. The methods used here to obtain universal relations, however, are elementary. More references to recent developments in composites can be found in Ref. 18.

2. Universal relations

A two-phase uniaxial reinforced material is considered here, with parallel cylindrical fibres embedded in a matrix material. Both media may have anisotropic properties. The cross-section of the fibres lies in the x_1x_2-plane, none of the axes of the characteristic material properties directions coincides with the fibre direction, which is taken as the Ox_3 axis. The fibres are periodically distributed without overlapping in two non-parallel directions in the x_1x_2-plane. Either the square or hexagonal symmetry may be considered. Moreover, within the periodic cell, the fibre cross-section shape has square or hexagonal symmetry and is otherwise arbitrary and it may be disjoint.

The overall properties of the above periodic medium are sought by means of the well-known asymptotic homogenization method. Appendix A has a brief outline of the technique leading to (A.4) and (A.5) which are the equations to be considered here at the outset. Then, it follows that, in terms of the fast variable **y**, the appropiate periodic cell Y (per unit length) is taken in the y_1y_2-plane so that $Y = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = \emptyset$, where the domain Y_1 is occupied by the matrix and its complement Y_2 is filled up with the fibre, which may be disjoint. The common interface betwen the matrix and the fibre is denoted by Γ, and it is arbitrary as long as it conforms with the symmetry. The thermal conductivity tensors $k_{ij}^{(1)}$ and $k_{ij}^{(2)}$ refer to the properties of the matrix and the fibre, respectively.

A connection among the solutions of the local problems ${}_pL$ in (A.5) can now easily be found. Among all possible tensors, let $k_{11}^{(\Upsilon)} = k_{22}^{(\Upsilon)}$ and $k_{13}^{(\Upsilon)} = k_{23}^{(\Upsilon)}$, $\Upsilon = 1, 2$ (the properties relative to directions Ox_1 and Ox_2 are the same). Then it is easy to get

$$\parallel k_{13}^{(\Upsilon)} \parallel ({}_1\Theta^{(\Upsilon)} + {}_2\Theta^{(\Upsilon)}) = \parallel k_{11}^{(\Upsilon)} + k_{12}^{(\Upsilon)} \parallel {}_3\Theta^{(\Upsilon)}. \quad (1)$$

Relations among the overall material properties can next be obtained. Take the expression (A.5), set $i = 1, p = 1, 2, 3$, then

$$\bar{k}_{11} = \quad \langle k_{11} + k_{11}\ {}_1\Theta_{,1} + k_{12}\ {}_1\Theta_{,2} + k_{13}\ {}_1\Theta_{,3} \rangle, \quad (2a)$$

$$\bar{k}_{12} = \quad \langle k_{12} + k_{11}\ {}_2\Theta_{,1} + k_{12}\ {}_2\Theta_{,2} + k_{13}\ {}_2\Theta_{,3} \rangle, \quad (2b)$$

$$\bar{k}_{13} = \quad \langle k_{13} + k_{11}\ {}_3\Theta_{,1} + k_{12}\ {}_3\Theta_{,2} + k_{13}\ {}_3\Theta_{,3} \rangle. \quad (2c)$$

The addition of (2a) and (2b), the substitution of (1) and the elimination of the common term with (2c) leads to

$$\frac{\bar{k}_{11} + \bar{k}_{12} - \langle k_{11} + k_{12} \rangle}{\bar{k}_{13} - \langle k_{13} \rangle} = \frac{\parallel k_{11}^{(\Upsilon)} + k_{12}^{(\Upsilon)} \parallel}{\parallel k_{13}^{(\Upsilon)} \parallel}, \quad (3)$$

a universal relation among $\bar{k}_{11}, \bar{k}_{12}$ and $\bar{k}_{13}$ in terms of material properties contrast $\parallel k_{11}^{(\Upsilon)} + k_{12}^{(\Upsilon)} \parallel$ and $\parallel k_{13}^{(\Upsilon)} \parallel$ and matrix and fibre area fractions V_1 and V_2, respectively since, say,

$$\langle k_{13} \rangle = V_1 k_{13}^{(1)} + V_2 k_{13}^{(2)} \quad (4)$$

is the arithmetic or Voigt mean of $k_{13}^{(\Upsilon)}$. Note that Eq. (3) was derived without solving any local problem.

Two similar expressions to Eq. (3) can similarly be derived, which yields

$$\frac{\bar{k}_{21} + \bar{k}_{22} - \langle k_{21} + k_{22} \rangle}{\bar{k}_{23} - \langle k_{23} \rangle} = \frac{\bar{k}_{31} + \bar{k}_{32} - \langle k_{31} + k_{32} \rangle}{\bar{k}_{33} - \langle k_{33} \rangle} = \frac{\parallel k_{11}^{(\Upsilon)} + k_{12}^{(\Upsilon)} \parallel}{\parallel k_{13}^{(\Upsilon)} \parallel}, \quad (5)$$

another two universal relations with similar characteristics as Eq. (3). The knowledge of three of the properties can fix the other three. Note that the right hand side of Eqs. (3) and (5) is the same. These relations were derived for a three-dimensional case.

3. Relation to data

Anisotropic data are usually given as the characteristic (two or three) values of the material (see, for instance, Refs. 19 and 20). These values, which are denoted K_{11}, K_{22} and K_{33}, relate to the components of the tensor k_{ij} of Sec. 2 by means of simple expressions as will be shown below. The corresponding characteristic directions are given by the set of orthogonal vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, respectively. None of these particular directions is parallel to the axis of geometric symmetry, the Ox_3 axis.

4. Three- and two-parameter families

(a) Let the thermal conductivity tensor k_{ij} of the matrix be isotropic and that of the fibre have the form, consistent with the structure of Sec. 2,

$$\begin{pmatrix} k & m & l \\ m & k & l \\ l & l & k \end{pmatrix}, \quad (6)$$

where k, l, m are three nonzero parameters. This is a three-parameter family. The characteristic values and directions of (6) can easily be found. They are given by the expressions

$$K_{11} = k + 2lS_-, \quad (4.7a)$$

$$K_{22} = k + 2lS_+, \quad (4.7b)$$

$$K_{33} = k - m, \quad (4.7c)$$

$$\mathbf{e}_1 = (S_-, S_-, 1), \quad (4.7d)$$

$$\mathbf{e}_2 = (S_+, S_+, 1), \quad (4.7e)$$

$$\mathbf{e}_3 = (-1, 1, 0), \quad (4.7f)$$

where

$$4lS_\pm = m \pm (m^2 + 8l^2)^{1/2}. \quad (8)$$

Moreover, the following simple expressions follow from

Eqs. ($7a, b, c$):

$$
\begin{aligned}
k &= (K_{11} + K_{22} + K_{33})/3, \\
m &= (K_{11} + K_{22} - 2K_{33}/3, \\
l &= \big[9(K_{11} - K_{22})^2 \\
&\quad - (K_{11} + K_{22} - 2K_{33})^2\big]^{1/2}/(6 \cdot 2^{1/2}). \quad (9)
\end{aligned}
$$

Given the data K_{11}, K_{22} and K_{33}, two real solutions k, l, m are possible, whenever the radicand is positive. This means that, for this composite, there exist two possible geometric directions (that of the fibre) for which the universal relations (3, 5) hold.

(b) A two-parameter family is obtained from (6) setting $m = l$, thus

$$
\begin{aligned}
K_{11} &= K - l, \\
K_{22} &= K - 2l, \\
K_{33} &= K - l, \\
\mathbf{e}_1 &= (-1/2, -1/2, 1), \\
\mathbf{e}_2 &= (1, 1, 0), \\
\mathbf{e}_3 &= (-1, 1, 0). \quad (10)
\end{aligned}
$$

Also, it is easily obtained that

$$
\begin{aligned}
k &= (2K_{11} + K_{33})/3, \\
l &= (K_{22} - K_{33})/3. \quad (11)
\end{aligned}
$$

Many more examples of this type could be shown by conveniently choosing the elements of k_{ij} in accordance to the limitations of Sec. 2. For the sake of brevity, the above examples suffice to show that there exists particular directions for which universal relations in three dimensions hold in the case of a fibre-reinforced composite.

5. Concluding remarks

The method of asymptotic homogenization is used to give a simple derivation of three universal relations. The application of the method, in general, leads to an equation for the effective or homogenized parameters of the concerned periodic medium, which depends on the solution of certain cell problems, the so-called local problems. Here there are three such problems. A relation among their solutions is found, which is instrumental in deducing the above mentioned connections. Although the geometry of the periodic fibre-reinforced medium is two dimensional, the fact that none of the characteristic directions of the material properties is parallel to the axis of geometric symmetry makes the studied problem three-dimensional. It is also shown in two examples, that two and three parameter families of the thermal conductivity tensor can be related to the two or three measured values of certain materials. The elementary method used in this paper to derive universal relations may be applicable to other problems where new results of this nature may be found. The universal relations may be useful for checking numerical codes, experimental data and the range of validity of certain approximations.

Appendix A

A periodic medium is considered which occupies a bounded region $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ and diameter L. A rapid variation of material properties over the small scale l is taking place within Ω, so that $\epsilon = l/L$ is a small parameter and $\epsilon << 1$. For the sake of argument, a temperature $\theta^{(\epsilon)}$ is sought such that

$$
\begin{aligned}
-\frac{\partial}{\partial x_i}\left[k_{ij}\,(x/\epsilon)\,\frac{\partial\theta^{(\epsilon)}}{\partial x_j}\right] &= f \quad \text{in} \quad \Omega, \\
\theta^{(\epsilon)} &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned} \quad (12)
$$

where f is the heat source, k_{ij} is the thermal conductivity tensor, the summation convention is understood, where the indices run from 1 to 3. The material properties show the dependence on ϵ and they obey the symmetry $k_{ij} = k_{ji}$.

The method of asymptotic homogenization can be applied to find an asymptotic solution of Eq. (12). Due to the rapidly oscillating properties, the temperature is assumed to depend on two independent spatial variables $\mathbf{x}$ and $\mathbf{y} = \mathbf{x}/\epsilon$, the so-called slow and fast variables, respectively, and the method of two scales is used; the following ansatz is posed:

$$
\theta^{(\epsilon)} = \theta^{(0)}(\mathbf{x}, \mathbf{y}) + \epsilon\theta^{(1)}(\mathbf{x}, \mathbf{y}) + O(\epsilon^2) \quad (13)
$$

The substitution of Eq. (13) into the boundary-value problem in Eq. (12), the application of the chain rule of differentiation and the comparison of similar powers of ϵ yields a sequence of problems to seek $\theta^{(0)}(\mathbf{x})$, which is found to depend only on $\mathbf{x}$, and $\theta^{(1)}$. (For details, see, [21], 59–62). A product solution is found for $\theta^{(1)}$ as follows:

$$
\theta^{(1)}(\mathbf{x}, \mathbf{y}) = {}_p\Theta(\mathbf{y})\frac{\partial\theta^{(0)}(\mathbf{x})}{\partial x_p}, \quad (14)
$$

where the set of p-functions ${}_p\Theta(\mathbf{y})$, which depend only on $\mathbf{y}$, are the unique solution of the so-called p-local problems, denoted by ${}_pL$, over the periodic cell Y. For the case where two materials of different properties are contained in Y, for which the temperature and heat flux across the common interface $\Gamma \subset Y$ are continuous, the p-local problem is

$$
\begin{aligned}
k_{ij}\;{}_p\Theta_{,ij} &= 0 \quad \text{in} \quad Y, \\
\|\;{}_p\Theta\;\| &= 0 \quad \text{on} \quad \Gamma, \\
\|\;k_{ij}\;{}_p\Theta_{,j} n_i\;\| &= -\,\|\;k_{ip}\;\|\;n_i \quad \text{on} \quad \Gamma, \\
\langle{}_p\Theta\rangle &= 0, \quad (15)
\end{aligned}
$$

where the comma notation is used to denote differentiation with respect to the y_i coordinate, *i.e.*, ${}_p\Theta_{,i} \equiv \partial_p\Theta/\partial y_i$, the outward unit normal vector to the interface Γ is $\mathbf{n}$, the double bar notation defines the jump of the function across the interface Γ and the angular brackets represent the volume average of the function over the cell.

The homogenized thermal conductivites $\bar{k}_{ip} = \bar{k}_{pi}$ are then given by the expression

$$\bar{k}_{ip} = \langle k_{ip} + k_{ij\ p}\Theta_{,j} \rangle. \tag{16}$$

It should be noted that the boundary value problem (12) also describes phenomena related to dielectrics, electrical conductivity and magnetic permeability [19].

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