Multipole expansion of electrostatic and magnetostatic fields

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The multipole moments of an electrostatic or magnetostatic field and the components of the field itself are expressed in terms of two-component spinor and of spin-weighted spherical harmonics, obtaining the correspondence between the Cartesian and spherical components. It is shown that the 2^l-pole moment of a charge or current distribution defines l, not necessarily distinct, directions which determine the angular dependence of the multipole field.

Keywords: Multipole expansion; electrostatic field; magnetostatic field; two-component spinors; spherical harmonics.

Los momentos multipolares de un campo electrostático o magnetostático y las componentes del campo mismo se expresan en términos de espinores de dos componentes y de armónicos esféricos con peso de espín, obteniéndose la correspondencia entre las componentes cartesianas y esféricas. Se muestra que el momento 2^l-polar de una distribución de carga o de corriente define l direcciones, no necesariamente distintas, las cuales determinan la dependencia angular del campo multipolar.

Descriptores: Desarrollo multipolar; campo electrostático; campo magnetostático; espinores de dos componentes; armónicos esféricos.

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1. Introduction

In the standard treatment of the multipole expansion of the electrostatic or the magnetostatic field two alternative approaches are followed, expanding the inverse of the distance between the source point and the field point either in terms of their Cartesian coordinates [1–4] or of the spherical harmonics evaluated at the directions of these points [1,2,5]. The first procedure is more elementary, but the expressions for the terms in the multipole expansion after the dipole or the quadrupole term become very involved and even the counting of the independent components of the multipole moments is somewhat complicated (see, *e.g.*, Refs. 1, 4). On the other hand, the use of the spherical harmonics allows one to write down easily all the terms of the multipole expansion.

The aim of this paper is to show explicitly the equivalence of these two approaches making use of the two-component spinor formalism, obtaining the correspondence between the Cartesian and the spherical components of the multipole moments. In Sec. 2 we give the basic notions about the two-component spinor formalism and the spin-weighted spherical harmonics. In Sec. 3 the multipole expansion of the magnetostatic field is considered, expressing the field and the multipole moments in terms of spin-weighted spherical harmonics. It is shown that the 2^l-pole moment of a bounded current distribution defines l (not necessarily distinct) directions in the three-dimensional space. In Sec. 4 a similar treatment for the electrostatic field is given.

2. The spinor equivalent of a tensor and the spherical harmonics

The two-component spinor formalism in Euclidean three-dimensional space appears in the study of the spin of the electron in non-relativistic quantum mechanics. The components of a (one-index) spinor will be denoted by symbols like ψ_A $(A, B, \ldots = 1, 2)$ and the spinor indices will be lowered or raised according to the rules

$$\psi_A = \varepsilon_{AB}\psi^B \qquad \text{and} \qquad \psi^A = -\varepsilon^{AB}\psi_B, \tag{1}$$

where

$$(\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon^{AB}). \tag{2}$$

Here and in what follows there is implicit sum over repeated indices. The mate of the spinor ψ_A is defined by

$$\widehat{\psi}_A = \overline{\psi^A} \qquad \text{or} \qquad \widehat{\psi}^A = -\overline{\psi_A}, \tag{3}$$

where the bar denotes complex conjugation; thus,

$$\psi^A\widehat{\psi}_A = |\psi_1|^2 + |\psi_2|^2 \geq 0.$$

A symmetric two-index spinor, $v_{AB} = v_{BA}$, corresponds to a (possibly complex) vector with Cartesian components v_i $(i, j, \ldots = 1, 2, 3)$ given by

$$v_i = -\frac{1}{\sqrt{2}}\sigma_i{}^{AB}v_{AB}, \tag{4}$$

where the connection symbols σ_{iAB} are complex constants such that

$$\sigma_{iAB} = \sigma_{iBA} \tag{5}$$

and

$$\sigma_{iAB}\sigma_j{}^{AB} = -2\delta_{ij}. \tag{6}$$

Choosing the connection symbols as

$$(\sigma_{1AB}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma_{2AB}) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$(\sigma_{3AB}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{7}$$

one finds that

$$\overline{\sigma_{iAB}} = -\sigma_i{}^{AB}, \tag{8}$$

therefore, a symmetric two-index spinor v_{AB} corresponds to a real vector if and only if $\overline{v_{AB}} = -v^{AB}$.

From Eq. (6) it follows that

$$\sigma^i{}_{AB}\sigma_{iCD} = -(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}) \tag{9}$$

(the tensor indices are raised or lowered by means of δ^{ij} and δ_{ij}), hence Eq. (4) is equivalent to

$$v_{AB} = \frac{1}{\sqrt{2}}\sigma^i{}_{AB}v_i. \tag{10}$$

Thus, according to Eqs. (7) and (10), the components of the spinor equivalent of a vector are given in terms of its Cartesian components by

$$v_{11} = \frac{1}{\sqrt{2}}(v_x + iv_y), \qquad v_{12} = -\frac{1}{\sqrt{2}}v_z,$$
$$v_{22} = -\frac{1}{\sqrt{2}}(v_x - iv_y). \tag{11}$$

In general, the spinor equivalent of a tensor $t_{ij\cdots k}$ is defined by

$$t_{ABCD\cdots EF} = \frac{1}{\sqrt{2}}\sigma^i{}_{AB}\frac{1}{\sqrt{2}}\sigma^j{}_{CD}\cdots\frac{1}{\sqrt{2}}\sigma^k{}_{EF}t_{ij\cdots k}. \tag{12}$$

Then one finds that $t_{ABCD\cdots EF}$ is totally symmetric if and only if $t_{ij\cdots k}$ is symmetric and tracefree [6,7].

Given a one-index spinor field ψ^A one can construct the real vector field

$$R_i = -\sigma_{iAB}\psi^A\widehat{\psi}^B \tag{13}$$

and the complex vector field

$$M_i = \sigma_{iAB}\psi^A\psi^B, \tag{14}$$

which, according to Eq. (9), satisfy

$$\begin{aligned} R_iR^i &= -(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})\psi^A\widehat{\psi}^B\psi^C\widehat{\psi}^D \\ &= (\psi^A\widehat{\psi}_A)^2, \\ \overline{M_i}M^i &= (\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})\widehat{\psi}^A\widehat{\psi}^B\psi^C\psi^D \\ &= 2(\psi^A\widehat{\psi}_A)^2, \end{aligned}$$

and, similarly, $R_iM^i = 0$, $M_iM^i = 0$; hence, R_i, $\mathrm{Re}\, M_i$, $\mathrm{Im}\, M_i$ are orthogonal to each other and have the magnitude $\psi^A\widehat{\psi}_A$.

In what follows we will make use of the spinor field o^A, with components

$$\begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-i\phi/2}\cos(\theta/2) \\ \mathrm{e}^{i\phi/2}\sin(\theta/2) \end{pmatrix}, \tag{15}$$

where θ and ϕ are the usual polar and azimuth angles associated with the spherical coordinates. The spinor field o^A satisfies $o^A\widehat{o}_A = 1$ and the three mutually orthogonal vectors defined by o^A form the orthonormal basis, $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$, induced by the spherical coordinates r, θ, ϕ, *i.e.*,

$$(\mathbf{e}_r)_j = -\sigma_{jAB}o^A\widehat{o}^B, \qquad (\mathbf{e}_\theta + i\mathbf{e}_\phi)_j = \sigma_{jAB}o^Ao^B. \tag{16}$$

Since the spherical harmonics of order l are of the form $c_{ij\cdots m}(x^i/r)(x^j/r)\cdots(x^m/r)$, where the $c_{ij\cdots m}$ are constants with l indices, symmetric and tracefree, the x^i are Cartesian coordinates and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ (see, *e.g.*, Ref. 8), from Eq. (16) it follows that the spherical harmonics of order l can be written as $c_{ABCD\cdots MN}o^A\widehat{o}^Bo^C\widehat{o}^D\cdots o^M\widehat{o}^N$, where the coefficients $c_{ABCD\cdots MN}$ are constants totally symmetric in their $2l$ spinor indices [6,7]. Explicitly, the spherical harmonics are given by

$$Y_{lm} = (-1)^m\frac{(2l)!}{l!}\left[\frac{2l+1}{4\pi}\frac{1}{(l+m)!(l-m)!}\right]^{1/2} \times \overbrace{\underbrace{o^{(1}o^1\cdots o^1}_{l}\underbrace{\widehat{o}^1\widehat{o}^2\cdots\widehat{o}^{2)}}_{l}}^{(l-m)\text{ 1's, }(l+m)\text{ 2's}}, \tag{17}$$

where the parenthesis denotes symmetrization on the indices enclosed (*e.g.*, $M_{(ABC)} = \frac{1}{6}(M_{ABC} + M_{BCA} + M_{CAB} + M_{ACB} + M_{CBA} + M_{BAC})$). Since the number of o^A's and of $\widehat{o}^A$'s appearing in Eq. (17) coincide, the spherical harmonics (17) are invariant under the transformation

$$o^A \mapsto \mathrm{e}^{i\alpha/2}o^A \tag{18}$$

(which implies that $\widehat{o}^A \mapsto \mathrm{e}^{-i\alpha/2}\widehat{o}^A$). A quantity η has spin weight s if under the transformation in Eq. (18) transforms according to $\eta \mapsto \mathrm{e}^{is\alpha}\eta$. Thus, the ordinary spherical harmonics have spin weight 0.

The spin-weighted spherical harmonics, for an integral spin weight s, can be defined by [9]

$${}_sY_{jm} = \begin{cases} \left[\frac{(j-s)!}{(j+s)!}\right]^{1/2}\eth^s\, Y_{jm}, & \text{if } 0 \le s \le j, \\ \\ (-1)^s\left[\frac{(j+s)!}{(j-s)!}\right]^{1/2}\bar{\eth}^{-s}\, Y_{jm}, & \text{if } -j \le s \le 0, \end{cases} \tag{19}$$

where the operators $\eth$ and $\bar{\eth}$, acting on a function with spin weight s are defined by

$$\begin{aligned} \eth\eta &= -\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - s\cot\theta\right)\eta \\ &= -\sin^s\theta\left(\partial_\theta + \tfrac{i}{\sin\theta}\partial_\phi\right)(\eta\sin^{-s}\theta), \\ \bar{\eth}\eta &= -\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + s\cot\theta\right)\eta \\ &= -\sin^{-s}\theta\left(\partial_\theta - \tfrac{i}{\sin\theta}\partial_\phi\right)(\eta\sin^s\theta), \end{aligned} \tag{20}$$

thus $Y_{jm} = {}_0Y_{jm}$.

The operators $\eth$ and $\bar{\eth}$ appear in the usual vector operators in spherical coordinates

$$\nabla f = (\partial_r f)\,\mathbf{e}_r - \frac{1}{2r}\,\overline{\eth} f\,(\mathbf{e}_\theta + i\mathbf{e}_\phi) - \frac{1}{2r}\,\eth f\,(\mathbf{e}_\theta - i\mathbf{e}_\phi), \tag{21}$$

$$\nabla \cdot \mathbf{F} = -\frac{\sqrt{2}}{r^2}\partial_r(r^2 F_0) + \frac{1}{\sqrt{2}\,r}(\eth F_{-1} - \overline{\eth} F_{+1}), \tag{22}$$

$$\nabla \times \mathbf{F} = \frac{i}{\sqrt{2}\,r}(\eth F_{-1} + \overline{\eth} F_{+1})\,\mathbf{e}_r + \frac{i}{\sqrt{2}\,r}\Big[\partial_r(rF_{-1}) + \overline{\eth} F_0\Big](\mathbf{e}_\theta + i\mathbf{e}_\phi) + \frac{i}{\sqrt{2}\,r}\Big[\partial_r(rF_{+1}) - \eth F_0\Big](\mathbf{e}_\theta - i\mathbf{e}_\phi), \tag{23}$$

where

$$F_0 \equiv -\frac{1}{\sqrt{2}}\,\mathbf{F}\cdot\mathbf{e}_r, \qquad F_{\pm 1} \equiv \pm\frac{1}{\sqrt{2}}\,\mathbf{F}\cdot(\mathbf{e}_\theta \pm i\mathbf{e}_\phi). \tag{24}$$

Owing to Eqs. (9) and (16), these definitions are equivalent to $F_0 = F_{AB} o^A \widehat{o}^B$, $F_1 = F_{AB} o^A o^B$, $F_{-1} = F_{AB}\widehat{o}^A\widehat{o}^B$, where F_{AB} is the spinor equivalent of the vector field $\mathbf{F}$; hence, the component F_s ($s = 0, \pm 1$) has spin weight s. In terms of its spin-weighted components (24), a vector field can be written as

$$\mathbf{F} = -\sqrt{2}\,F_0\,\mathbf{e}_r - \frac{1}{\sqrt{2}}F_{-1}(\mathbf{e}_\theta + i\mathbf{e}_\phi) + \frac{1}{\sqrt{2}}F_{+1}(\mathbf{e}_\theta - i\mathbf{e}_\phi). \tag{25}$$

Equations (19) imply that

$$\begin{aligned} \eth\,{}_sY_{jm} &= [j(j+1) - s(s+1)]^{1/2}\,{}_{s+1}Y_{jm}, \\ \overline{\eth}\,{}_sY_{jm} &= -[j(j+1) - s(s-1)]^{1/2}\,{}_{s-1}Y_{jm}. \end{aligned} \tag{26}$$

Furthermore, any function with spin weight s can be expanded in a series in the spin-weighted spherical harmonics with spin weight s and, for a fixed value of s, the spin-weighted spherical harmonics are orthonormal [9,7]

$$\int_0^{2\pi}\int_0^{\pi} \overline{{}_sY_{jm}}\;{}_sY_{j'm'}\sin\theta d\theta d\phi = \delta_{jj'}\delta_{mm'}. \tag{27}$$

The spin-weighted spherical harmonics, ${}_sY_{jm}$, can also be expressed in terms of o and $\widehat{o}$ as [6,7]

$${}_sY_{jm} = (-1)^m\left[\frac{2j+1}{4\pi}\frac{(2j)!}{(j+m)!(j-m)!} \times \frac{(2j)!}{(j+s)!(j-s)!}\right]^{1/2} \overbrace{\underbrace{o^{(1}o^1\cdots o^1}_{j+s}\underbrace{\widehat{o}^1\widehat{o}^2\cdots\widehat{o}^{2)}}_{j-s}}^{(j-m)\ 1\text{'s},\ (j+m)\ 2\text{'s}}. \tag{28}$$

This expression applies also when s is a half-integer with

$$\begin{aligned} j &= 0, 1/2, 1, \ldots, \\ m &= -j, -j+1, \ldots, j, \\ s &= -j, -j+1, \ldots, j \end{aligned}$$

and shows that ${}_sY_{jm}$ has spin weight s.

3. Multipole expansion of the magnetostatic field

The magnetostatic field in vacuum obeys the equations

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \qquad \nabla\cdot\mathbf{B} = 0, \tag{29}$$

where $\mathbf{J}$ is the current density. By combining Eqs. (29) one finds that

$$\nabla^2(\mathbf{r}\cdot\mathbf{B}) = -\frac{4\pi}{c}\mathbf{r}\cdot\nabla\times\mathbf{J} = \frac{4\pi}{c}\nabla\cdot(\mathbf{r}\times\mathbf{J}) \tag{30}$$

(*cf.* Ref. 1, Eq. (16.86)); hence,

$$\mathbf{r}\cdot\mathbf{B}(\mathbf{r}) = -\frac{1}{c}\int\frac{\nabla'\cdot(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|}\,dv'$$

and assuming that the point $\mathbf{r}$ is outside a sphere enclosing the sources,

$$\begin{aligned} \mathbf{r}\cdot\mathbf{B}(\mathbf{r}) = &-\frac{1}{c}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\ &\times\int r'^l\,\overline{Y_{lm}(\theta',\phi')}\nabla'\cdot(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))\,dv' \\ =&\frac{1}{c}\sum_{l,m}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \\ &\times\int\mathbf{r}'\times\mathbf{J}(\mathbf{r}')\cdot\nabla'(r'^l\,\overline{Y_{lm}(\theta',\phi')})\,dv'. \end{aligned} \tag{31}$$

Thus,

$$\mathbf{e}_r\cdot\mathbf{B} = \sum_{l,m}\frac{4\pi}{2l+1}a_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+2}}, \tag{32}$$

where, making use of Eqs. (21) and (26),

$$\begin{aligned} a_{lm} =&\frac{1}{c}\int\mathbf{r}\times\mathbf{J}(\mathbf{r})\cdot\nabla(r^l\,\overline{Y_{lm}(\theta,\phi)})\,dv \\ =&-\frac{1}{c}\int r^l\mathbf{J}(\mathbf{r})\cdot\mathbf{r}\times\nabla\overline{Y_{lm}(\theta,\phi)}\,dv \\ =&\frac{1}{2c}\int r^l\mathbf{J}(\mathbf{r})\cdot\mathbf{e}_r\times[\overline{\eth}\overline{Y_{lm}}\,(\mathbf{e}_\theta+i\mathbf{e}_\phi) \\ &+\eth\overline{Y_{lm}}\,(\mathbf{e}_\theta - i\mathbf{e}_\phi)]\,dv \\ =&\frac{\sqrt{l(l+1)}}{\sqrt{2}ic}\int r^l(\overline{{}_1Y_{lm}}\,J_{+1} - \overline{{}_{-1}Y_{lm}}\,J_{-1})\,dv \end{aligned} \tag{33}$$

(*cf.* Ref. 5, Eq. (41.e)). Since Y_{00} is a constant (or, equivalently, ${}_{\pm 1}Y_{00} = 0$), we have $a_{00} = 0$. Furthermore, from the relation $\overline{Y_{lm}} = (-1)^m Y_{l,-m}$, it follows that

$$\overline{a_{lm}} = (-1)^m a_{l,-m}.$$

Using Eq. (25) we find that $\mathbf{r} \times \mathbf{J} = (ir/\sqrt{2})[J_{-1}(\mathbf{e}_\theta + i\mathbf{e}_\phi) + J_{+1}(\mathbf{e}_\theta - i\mathbf{e}_\phi)]$, therefore, the spinor equivalent of $\mathbf{r} \times \mathbf{J}$ is given by

$$(\mathbf{r} \times \mathbf{J})^{AB} = ir(J_{+1}\hat{o}^A \hat{o}^B - J_{-1} o^A o^B),$$

and making use of Eq. (28) we see that, for $l \geq 1$, the (complex conjugates of the) multipole coefficients (33) can also be written as

$$\begin{aligned}
\overline{a_{lm}} &= \frac{\sqrt{l(l+1)}}{\sqrt{2}\, ic} \int r^l ({}_1Y_{lm}\, J_{-1} - {}_{-1}Y_{lm}\, J_{+1})\, dv \\
&= \frac{(-1)^m (2l)!}{\sqrt{2}\, ic(l-1)!} \sqrt{\frac{2l+1}{4\pi} \frac{1}{(l+m)!(l-m)!}} \int r^l [J_{-1} \overbrace{\underbrace{o^{(1} o^1 \cdots o^1}_{l+1} \underbrace{\hat{o}^1 \hat{o}^2 \cdots \hat{o}^{2)}}_{l-1}}^{(l-m)\text{ 1's},\, (l+m)\text{ 2's}} - J_{+1} \overbrace{\underbrace{o^{(1} o^1 \cdots o^1}_{l-1} \underbrace{\hat{o}^1 \hat{o}^2 \cdots \hat{o}^{2)}}_{l+1}}^{(l-m)\text{ 1's},\, (l+m)\text{ 2's}}]\, dv \\
&= \frac{(-1)^m (2l)!}{\sqrt{2}\, c(l-1)!} \sqrt{\frac{2l+1}{4\pi} \frac{1}{(l+m)!(l-m)!}} \int r^{l-1} (\mathbf{r} \times \mathbf{J})^{\overbrace{(11\, \underbrace{o^1 \cdots o^2}_{l-1} \underbrace{\hat{o}^2 \cdots \hat{o}^{2)}}_{l-1}}^{(l-m)\text{ 1's},\, (l+m)\text{ 2's}}}\, dv.
\end{aligned} \tag{34}$$

Since $\sqrt{2}\, r o_{(A} \hat{o}_{B)}$ is the spinor equivalent of $\mathbf{r}$, this last expression shows that the coefficients a_{lm} are proportional to the components of the spinor equivalent of the tracefree symmetric part of

$$\int (\mathbf{r} \times \mathbf{J})^i x^j x^k \cdots x^m\, dv$$

(*cf.* Ref. 4, Eq. (78)).

By virtue of the completeness of the spin-weighted spherical harmonics, the remaining components of the magnetic induction can be expressed in the form

$$B_{\pm 1}(\mathbf{r}) = \sum_{l,m} f_{lm}^{(\pm)}(r)\, {}_{\pm 1}Y_{lm}(\theta, \phi), \tag{35}$$

for some functions $f_{lm}^{(\pm)}$. Since $\nabla \cdot \mathbf{B} = 0$, making use of Eqs. (22), (26), (32), and (35) one finds that

$$f_{lm}^{(-)}(r) + f_{lm}^{(+)}(r) = \sqrt{\frac{2l}{l+1}} \frac{4\pi}{2l+1} \frac{a_{lm}}{r^{l+2}}.$$

Similarly, since $\nabla \times \mathbf{B} = 0$ outside the sources, from Eqs. (23) and (35) one obtains $f_{lm}^{(-)}(r) - f_{lm}^{(+)}(r) = 0$. Hence,

$$f_{lm}^{(\pm)}(r) = \sqrt{\frac{l}{2(l+1)}} \frac{4\pi}{2l+1} \frac{a_{lm}}{r^{l+2}}. \tag{36}$$

Thus,

$$\begin{aligned}
\mathbf{B} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{a_{lm}}{r^{l+2}} \Bigg[& Y_{lm}\mathbf{e}_r - \frac{1}{2}\sqrt{\frac{l}{l+1}} \\
\times\, {}_{-1}Y_{lm}(\mathbf{e}_\theta + i\mathbf{e}_\phi) & + \frac{1}{2}\sqrt{\frac{l}{l+1}}\, {}_1Y_{lm}(\mathbf{e}_\theta - i\mathbf{e}_\phi) \Bigg].
\end{aligned} \tag{37}$$

Making use of Eqs. (23) and (26) one finds that the field in Eq. (37) can be written in the form $\mathbf{B} = \nabla \times \mathbf{A}$, with

$$\begin{aligned}
\mathbf{A} = \sum_{l,m} & \frac{4\pi}{2l+1} \frac{a_{lm}}{r^{l+1}} \frac{i}{2\sqrt{l(l+1)}} \\
& \times [{}_{-1}Y_{lm}(\mathbf{e}_\theta + i\mathbf{e}_\phi) + {}_1Y_{lm}(\mathbf{e}_\theta - i\mathbf{e}_\phi)],
\end{aligned} \tag{38}$$

which, in turn, can be expressed as $\mathbf{A} = \mathbf{r} \times \nabla\psi$, with

$$\psi = -\sum_{l,m} \frac{4\pi}{2l+1} \frac{a_{lm}}{l(l+1)} \frac{Y_{lm}}{r^{l+1}},$$

which is a solution of the Laplace equation. Alternatively, the field (37) can be written in terms of a scalar potential, φ_{M}, in the form $\mathbf{B} = -\nabla\varphi_{\mathrm{M}}$, with

$$\varphi_{\mathrm{M}} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{a_{lm}}{l+1} \frac{Y_{lm}}{r^{l+1}}, \tag{39}$$

which also satisfies the Laplace equation.

The components of the spinor equivalent of the vector field (37) are [see Eqs. (16)]

$$\begin{aligned}
B^{CD} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{a_{lm}}{\sqrt{2}\, r^{l+2}} & \Bigg[2Y_{lm} o^{(C} \hat{o}^{D)} \\
& + \sqrt{\frac{l}{l+1}}\, {}_{-1}Y_{lm} o^C o^D + \sqrt{\frac{l}{l+1}}\, {}_1Y_{lm} \hat{o}^C \hat{o}^D \Bigg]
\end{aligned}$$

or, equivalently, making use of Eq. (28)

$$B^{CD} = \sum_{l,m} \left[\frac{4\pi}{2l+1} \frac{1}{(l+m)!(l-m)!} \right]^{1/2} \frac{(-1)^m \sqrt{2}\,(2l+1)!}{(l+1)!} \frac{a_{lm}}{r^{l+2}} \underbrace{o^{(C} o^D \overbrace{o^1 \cdots o^2 \widehat{o}^2 \cdots \widehat{o}^{2)}}^{(l-m)\text{ 1's, } (l+m)\text{ 2's}}}_{(l+1)\ o\text{'s, } (l+1)\ \widehat{o}\text{'s}} \quad , \tag{40}$$

(*cf.* Ref. 4, Eq. (72)). Hence, owing to Eqs. (11) and (17), the Cartesian components of $\mathbf{B}$ are given by

$$\begin{aligned} B_x + iB_y &= \sqrt{2}\, B^{22} = -\sum_{l,m} \frac{4\pi}{2l+1} \left[\frac{2l+1}{2l+3} (l+m+2)(l+m+1) \right]^{1/2} \frac{a_{lm}}{l+1} \frac{Y_{l+1,m+1}}{r^{l+2}}, \\ B_z &= \sqrt{2}\, B^{12} = \sum_{l,m} \frac{4\pi}{2l+1} \left[\frac{2l+1}{2l+3} (l+m+1)(l-m+1) \right]^{1/2} \frac{a_{lm}}{l+1} \frac{Y_{l+1,m}}{r^{l+2}}, \\ B_x - iB_y &= -\sqrt{2}\, B^{11} = \sum_{l,m} \frac{4\pi}{2l+1} \left[\frac{2l+1}{2l+3} (l-m+2)(l-m+1) \right]^{1/2} \frac{a_{lm}}{l+1} \frac{Y_{l+1,m-1}}{r^{l+2}}. \end{aligned} \tag{41}$$

These expressions show explicitly that the Cartesian components of $\mathbf{B}$ satisfy the Laplace equation, which was to be expected since Eqs. (29) imply that, outside the sources, the magnetic induction (and, hence, each Cartesian component of $\mathbf{B}$) satisfies the Laplace equation.

Under a rotation about the origin, each point of the space, $\mathbf{r}$, is transformed into a point $\mathbf{r}' = R(\mathbf{r})$, where R is an orthogonal linear transformation, and any scalar function, f, is transformed into the function $\mathcal{R}f$, with $(\mathcal{R}f)(\mathbf{r}) \equiv f(R^{-1}(\mathbf{r}))$; thus, the scalar potential $\varphi_{\rm M}$ is transformed into

$$\mathcal{R}\varphi_{\rm M} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{a_{lm}}{l+1} \frac{\mathcal{R}Y_{lm}}{r^{l+1}},$$

since r is invariant under the rotations about the origin. On the other hand,

$$\mathcal{R}Y_{lm} = \sum_{m'=-l}^{l} D^l_{m'm}(R)\, Y_{lm'},$$

where the $D^l_{m'm}$ are the Wigner D functions (see, *e.g.*, Refs. 10, 11); hence, under a rotation R, the multipole moment a_{lm} is replaced by

$$\sum_{m'=-l}^{l} D^l_{mm'}(R)\, a_{lm'}. \tag{42}$$

The integrals

$$m^{AB\cdots L} \equiv \int r^{l-1} (\mathbf{r} \times \mathbf{J})^{(AB} \underbrace{o^C \cdots o^D}_{l-1} \underbrace{\widehat{o}^E \cdots \widehat{o}^{L)}}_{l-1} dv. \tag{43}$$

appearing in Eq. (34) are the components of a totally symmetric $2l$-index spinor; therefore there exist $2l$ one-index spinors α^A, β^A, …, λ^A (the principal spinors of $m^{AB\cdots L}$) such that $m^{AB\cdots L} = \alpha^{(A}\beta^B \cdots \lambda^{L)}$ and since $m^{AB\cdots L}$ is the spinor equivalent of a real tensor (which amounts to the conditions $\overline{a_{lm}} = (-1)^m a_{l,-m}$), $m^{AB\cdots L}$ must be of the form [12,7]

$$m^{AB\cdots L} = \alpha^{(A} \widehat{\alpha}^B \beta^C \widehat{\beta}^D \cdots \eta^K \widehat{\eta}^{L)}, \tag{44}$$

which means that the 2^l-pole moment of a given current distribution is determined by l real vectors, $\mathbf{a}$, $\mathbf{b}$, …, $\mathbf{h}$ (the tensor equivalents of $\sqrt{2}\,\alpha^{(A}\widehat{\alpha}^{B)}$, $\sqrt{2}\,\beta^{(A}\widehat{\beta}^{B)}$, …, $\sqrt{2}\,\eta^{(A}\widehat{\eta}^{B)}$) in such a way that the symmetric tracefree part of $\int (\mathbf{r} \times \mathbf{J})^i x^j x^k \cdots x^m\, dv$ is the symmetric tracefree part of $a^i b^j \cdots h^m$.

The directions of $\mathbf{a}$, $\mathbf{b}$, …, $\mathbf{h}$ need not be different; if the product $\alpha^A \widehat{\alpha}^B$ appears p times in the right-hand side of Eq. (44) then by means of a rotation one can align the new z-axis with the vector equivalent of $\alpha^{(A}\widehat{\alpha}^{B)}$ (*i.e.*, α^A is proportional to δ^A_1 after the rotation), then, with respect to the new axes,

$$a_{ll} = a_{l,l-1} = \cdots = a_{l,l-p+1} = 0 \tag{45}$$

(hence $a_{l,-l}$, $a_{l,-l+1}$, …, $a_{l,-l+p-1}$ also vanish). In the extreme case where $p = l$, only a_{l0} is different from zero and the corresponding multipole field is axially symmetric. Thus, by means of a rotation (42) one can eliminate at least two, and at most $2l$, 2^l-pole moments a_{lm}.

For instance, for any given bounded current distribution there exist two vectors, $\mathbf{a}$, $\mathbf{b}$, such that

$$\int (\mathbf{r} \times \mathbf{J})^{(i} x^{j)}\, dv = a^{(i} b^{j)} - \tfrac{1}{3} a^k b_k \delta^{ij}. \tag{46}$$

One can write down analogous expressions for $l \geq 3$, but they become highly involved, by contrast with the spinor expression (44).

The orthogonality of the spin-weighted spherical harmonics is useful if the expansion of $J_{\pm 1}(\mathbf{r})$ in terms of the spin-weighted spherical harmonics of the corresponding spin weight is known. For example, the current density of a rotating uniformly charged sphere is of the form

$$\mathbf{J} = g(r) \sin\theta\, \mathbf{e}_\phi,$$

for some function $g(r)$; therefore,

$$J_{\pm 1} = (\pm i/\sqrt{2}) g(r) \sin\theta.$$

Writing $\sin\theta$ in terms of $\sin(\theta/2)$ and $\cos(\theta/2)$ we obtain

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = 2o^1 o^2 = -2\widehat{o}^1 \widehat{o}^2$$

(see Eqs. (15) and (3)) which, according to Eq. (28), is proportional to ${}_1Y_{10}$ or to ${}_{-1}Y_{10}$. Thus, from Eqs. (33) and (27) one finds that the *only* nonvanishing multipole moment is a_{10}, hence the exterior field is exactly that of an ideal point dipole. Similarly, the field produced by a current distribution whose angular dependence is given by spin-weighted spherical harmonics with a single value of l is a 2^l-pole field.

4. Multipole expansion of the electrostatic field

The basic equations for the electrostatic field in vacuum are

$$\nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{E} = 4\pi\rho, \tag{47}$$

where ρ is the electric charge density. The first of these equations implies the existence of a scalar potential, φ, such that $\mathbf{E} = -\nabla\varphi$, which, therefore, satisfies the Poisson equation

$$\nabla^2\varphi = -4\pi\rho. \tag{48}$$

Hence,

$$\varphi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'$$

and, if $\mathbf{r}$ is outside a sphere enclosing the charges,

$$\begin{aligned}\varphi(\mathbf{r}) &= \sum_{l,m}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\int r'^l\rho(\mathbf{r}')\overline{Y_{lm}(\theta',\phi')}\,dv' \\ &= \sum_{l,m}\frac{4\pi}{2l+1}b_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}},\end{aligned} \tag{49}$$

with

$$b_{lm} \equiv \int r^l\rho(\mathbf{r})\overline{Y_{lm}(\theta,\phi)}\,dv. \tag{50}$$

Note that $\overline{b_{lm}} = (-1)^m b_{l,-m}$ but in the present case the monopole moment, b_{00}, need not be equal to zero.

Substituting Eq. (17) into Eq. (50) we get the equivalent expression for the electric moments

$$\begin{aligned}\overline{b_{lm}} = (-1)^m\frac{(2l)!}{l!}\left[\frac{2l+1}{4\pi}\frac{1}{(l+m)!(l-m)!}\right]^{1/2} \\ \times\int\rho(\mathbf{r})r^l\,\overbrace{\underbrace{o^{(1}o^1\cdots o^1}_{l}\,\underbrace{\widehat{o}^1\widehat{o}^2\cdots\widehat{o}^{2)}}_{l}}^{(l-m)\ 1\text{'s},\ (l+m)\ 2\text{'s}}\,dv.\end{aligned}$$

Recalling that $\sqrt{2}\,ro^{(A}\widehat{o}^{B)}$ is the spinor equivalent of $\mathbf{r}$, one finds that the moments b_{lm} are proportional to the components of the spinor equivalent of the tracefree part of

$$\int \rho x^i x^j\cdots x^m\,dv.$$

By comparing Eqs. (39) and (49) it follows that the analogs of Eqs. (37), (40), and (41) are obtained by replacing a_{lm} by $(l+1)b_{lm}$ in those equations; in this manner we find that the spherical components of the electrostatic field are given by

$$\mathbf{E} = \sum_{l,m}\frac{4\pi}{2l+1}\frac{(l+1)b_{lm}}{r^{l+2}}\left[Y_{lm}\mathbf{e}_r - \frac{1}{2}\sqrt{\frac{l}{l+1}}\,{}_{-1}Y_{lm}(\mathbf{e}_\theta + i\mathbf{e}_\phi) + \frac{1}{2}\sqrt{\frac{l}{l+1}}\,{}_1Y_{lm}(\mathbf{e}_\theta - i\mathbf{e}_\phi)\right] \tag{51}$$

and the components of its spinor equivalent are

$$E^{AB} = \sum_{l,m}\left[\frac{4\pi}{2l+1}\frac{1}{(l+m)!(l-m)!}\right]^{1/2}\frac{(-1)^m\sqrt{2}\,(2l+1)!}{l!}\frac{b_{lm}}{r^{l+2}}\,\underbrace{o^{(A}o^B\overbrace{o^1\cdots o^2\widehat{o}^2\cdots\widehat{o}^{2)}}^{(l-m)\ 1\text{'s},\ (l+m)\ 2\text{'s}}}_{(l+1)\ o\text{'s},\ (l+1)\ \widehat{o}\text{'s}}. \tag{52}$$

Therefore,

$$\begin{aligned}E_x + iE_y &= -\sum_{l,m}\frac{4\pi}{2l+1}\left[\frac{2l+1}{2l+3}(l+m+2)(l+m+1)\right]^{1/2}b_{lm}\frac{Y_{l+1,m+1}}{r^{l+2}}, \\ E_z &= \sum_{l,m}\frac{4\pi}{2l+1}\left[\frac{2l+1}{2l+3}(l+m+1)(l-m+1)\right]^{1/2}b_{lm}\frac{Y_{l+1,m}}{r^{l+2}}, \\ E_x - iE_y &= \sum_{l,m}\frac{4\pi}{2l+1}\left[\frac{2l+1}{2l+3}(l-m+2)(l-m+1)\right]^{1/2}b_{lm}\frac{Y_{l+1,m-1}}{r^{l+2}}.\end{aligned} \tag{53}$$

Since the electrostatic potential is a scalar function, from Eq. (49) it follows that under rotations the multipole moments b_{lm} transform in the same manner as the multipole moments a_{lm} (see Eq. (42)). The coefficients b_{lm} are essentially the components of the totally symmetric $2l$-index spinor

$$\int \rho(\mathbf{r}) r^l \underbrace{o^{(A} \cdots o^D}_{l} \underbrace{\widehat{o}^E \cdots \widehat{o}^{L)}}_{l} \, dv$$

and therefore there exist l one-index spinors, α^A, β^A, $\ldots$, η^A, such that

$$\int \rho(\mathbf{r}) r^l \underbrace{o^{(A} \cdots o^D}_{l} \underbrace{\widehat{o}^E \cdots \widehat{o}^{L)}}_{l} \, dv,$$

which means that the tracefree part of $\int \rho x^i x^j \cdots x^m \, dv$ is equal to the symmetric tracefree part of $a^i b^j \cdots h^m$, where $\mathbf{a}$, $\mathbf{b}$, $\ldots$, $\mathbf{h}$ are the vector equivalents of $\sqrt{2}\,\alpha^{(A}\widehat{\alpha}^{B)}$, $\sqrt{2}\,\beta^{(A}\widehat{\beta}^{B)}$, $\ldots$, $\sqrt{2}\,\eta^{(A}\widehat{\eta}^{B)}$. Aligning the z-axis with $\mathbf{a}$ one obtains

$$b_{ll} = b_{l,l-1} = \cdots = b_{l,l-p+1} = 0,$$

where p is the number of times that the direction of $\mathbf{a}$ is repeated among the directions of $\mathbf{a}$, $\mathbf{b}$, $\ldots$, $\mathbf{h}$. Hence, by means of a suitable rotation, one can eliminate $2p$ multipole moments.

It may be remarked that the expressions for the static fields derived above (*e.g.*, Eqs. (32), (37), (41), (51), and (52)) can be employed directly in the solution of boundary value problems (*e.g.*, if the field is known on the points of the z-axis one can find the multipole moments making use of Eqs. (37) or (51)).

1. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
2. L. Eyges, *The Classical Electromagnetic Field* (Addison-Wesley, Reading, Mass. 1972, reprinted by Dover 1980).
3. J. R. Reitz, F. J. Milford and R.W. Christy, *Foundations of Electromagnetic Theory*, 4th ed. (Addison-Wesley, Reading, Mass., 1993).
4. P. Kielanowski and M. Loewe, *Rev. Mex. Fís.* **44** (1998) 24.
5. E. Ley-Koo and A. Góngora-T., *Rev. Mex. Fís.* **34** (1988) 645.
6. G. F. Torres del Castillo, *Rev. Mex. Fís.* **36** (1990) 446.
7. G. F. Torres del Castillo, *Spinors in Three Dimensions and Spin-weighted Functions*, (unpublished).
8. H. Hochstadt, *The Functions of Mathematical Physics*, (Wiley, New York, 1971, reprinted by Dover, New York, 1986), Chap. 6.
9. E. T. Newman and R. Penrose, *J. Math. Phys.* **7** (1966) 863.
10. A. Messiah, *Quantum Mechanics*, Vol. II, (North Holland, Amsterdam, 1962).
11. D. A. Varshalovich, A.N. Moskalev, and V.K. Khersonskii, *Quantum Theory of Angular Momentum*, (World Scientific, Singapore, 1988).
12. G. F. Torres del Castillo, *Rev. Mex. Fís.* **40** (1994) 713.