

# Quantum bits and superposition of displaced Fock states of the cavity field

L.M. Arévalo Aguilar

*Centro de Investigaciones en Optica, A.C.,  
Prolongación de Constitución No. 607  
Apdo. Post. 507, Aguascalientes, Ags., México.*

H. Moya-Cessa

*Instituto Nacional de Astrofísica, Óptica y Electrónica,  
Apdo. Post. 51 y 216, 72000 Puebla, Pue., México.*

Recibido el 14 de noviembre de 2001; aceptado el 22 de julio de 2002

We study the effects of counter rotating terms in the interaction of quantized light with a two-level atom, by using the method of small rotations. We give an expression for the wave function of the composed system atom plus field and point out one initial wave function that generates a quantum bit of the electromagnetic field with arbitrary amplitudes.

*Keywords:* Qubits; rotating wave approximation; Jaynes-Cummings model.

Estudiamos los efectos de los términos contra-rotantes en la interacción entre un campo electromagnético cuantizado y un átomo de dos niveles, usando el método de pequeñas rotaciones. Obtenemos una expresión para la función de onda del sistema compuesto átomo-campo y elegimos un estado inicial de la función de onda que genera un *bit* cuántico del campo electromagnético.

*Descriptores:* Bits cuánticos; aproximación de onda rotante; modelo de Jaynes y Cummings.

PACS: 42.50.-p; 32.80.-t; 42.50.Ct

## 1. Introduction

The Jaynes-Cummings model (JCM) [1] is a very simplified version of a much more complex problem, the interaction between electromagnetic radiation and atoms. It models this interaction using the rotating wave approximation (RWA) that allows it to be fully solvable. Its simplicity allows physicists to apply the fundamental laws of quantum electrodynamics to it and still be able to solve it analytically. In the early days of its existence, the JCM was regarded as a theoretical curiosity because of the inherent difficulties in its experimental realization. Over the past few years, however, there have been a number of experiments [2] that can be modeled by the Jaynes-Cummings Hamiltonian or generalizations of it [3]. Therefore, the JCM has been a subject of great interest because it enables one to study, in a realistic way, not only the coherent properties of the quantized field, but also its influence on atoms. Collapses and revivals [4], squeezing [5], generation of Schrödinger cat states [6], etc. have been predicted with this model, and recent developments in Cavity QED techniques have made it possible to observe those phenomena [2]. Moreover, there has been a *revival* of the JCM because its Hamiltonian can be used to model some other systems, such as the interaction of a trapped ion with a laser field. On this topic, multiphonon and anti JCM may be produced [7–11], giving rise to a variety of phenomena thought technologically difficult to realize in cavity QED. Recent advances in quantum information processing have given importance to quantum state engineering, as one needs to produce and control quantum bits or qubits (superposition of the ground and first excited level) of the system (see for instance [10]). However, quantum noise,

that destroys quantum coherences very fast, can be very difficult to overcome, and ways of protecting states have been published [12]. Here we would like to treat the problem of a two-level atom interacting with a quantized field but not considering the RWA, because a state produced by the JCM (with RWA) could be thought as if it had some noise (because a small correction with a further evolution can mislead a desired result). Therefore, it is important to give the most exact possible solution to the problem of interaction of a two-level atom with a quantized field. Recently some eigenstates for the complete Hamiltonian have been found [13]. However, as they do not form a complete basis, exact solutions may be found only for those (eigen) states (that of course may be regarded as trapping states).

Former studies on the effects of counter rotating [15, 14] terms have used the path integral technique in coherent state representation (the coherent state propagator) and have shown that even under conditions in which the RWA is considered to be justified, there is significant contribution to the atomic inversion due to counter rotating terms [14]. They obtained results to first order for the atomic inversion [14] and the average number of photons [15]; however, there is no explicit result for the wave function. We believe that it is important to have expressions that are easy to manipulate, specifically of the wave function, because of the possibility of generating non-classical states of light, which in the past has open the field of quantum state engineering. Therefore, we reconsider the problem in this manuscript, and show that non-classical states, namely, qubits and superposition of displaced number states [16] of the quantized electromagnetic field may be produced.

We study the problem from the point of view of the small

rotations method proposed recently by Klimov and Sánchez-Soto [17], to obtain a first order correction to the wave function. The RWA breaks down as the atomic frequency and the field frequency are detuned, and we consider this (detuned) case. With the expression for the evolved wave function, we show what initial state has to be used in order to obtain a qubit of the quantized field with arbitrary amplitudes. In Sec. 2 we transform the Hamiltonian for the atom-field interaction by applying the method of small rotations to have an effective Hamiltonian that can be fully solved. In Sec. 3 we apply it to an initial wave function in order to obtain qubits, and displaced superposition of Fock states and Sec. 4 is devoted to conclusions.

## 2. Method of small rotations

The Hamiltonian for the system of a single two-level atom interacting with a single-mode quantized field in the dipole approximation is given by (we have set  $\hbar = 1$ ) [1]

$$\hat{H} = \omega \hat{n} + \frac{\omega_0}{2} \hat{\sigma}_z + \lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger), \quad (1)$$

where  $\hat{a}^\dagger$  and  $\hat{a}$  are the creation and annihilation operators for the field mode, respectively, obeying  $[\hat{a}, \hat{a}^\dagger] = 1$ ,  $\hat{n} = \hat{a}^\dagger \hat{a}$  and  $\hat{\sigma}_+ = |e\rangle\langle g|$  and  $\hat{\sigma}_- = |g\rangle\langle e|$  are the raising and lowering atomic operators, respectively,  $|e\rangle$  being the excited state and  $|g\rangle$  the ground state of the two-level atom. The atomic operators obey the commutation relation  $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$ .  $\omega$  is the field frequency,  $\omega_0$  the atomic frequency and  $\lambda$  is the interaction constant.

We apply the transformation

$$\hat{T} = e^{-\delta(\hat{a}-\hat{a}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-)}, \quad (2)$$

to Eq. (1) to obtain

$$\begin{aligned} \hat{H} &= \omega(\hat{n} - \delta(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+) + \delta^2) \\ &\quad + \frac{\omega_0}{2}(\sigma_z \cosh[2\delta(\hat{a} - \hat{a}^\dagger)] \\ &\quad - (\sigma_- - \sigma_+) \sinh[2\delta(\hat{a} - \hat{a}^\dagger)]) \\ &\quad + \lambda(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+) - 2\lambda\delta. \end{aligned} \quad (3)$$

By considering the quantity  $\delta$  much smaller than one, we can approximate (3) to first order

$$\begin{aligned} \hat{H} &\approx \omega(\hat{n} - \delta(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+)) \\ &\quad + \frac{\omega_0}{2}(\sigma_z + 2\delta(\sigma_+ - \sigma_-)(\hat{a} - \hat{a}^\dagger)) \\ &\quad + \lambda(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+), \end{aligned} \quad (4)$$

where we dropped constant terms that contribute to a shift of the overall energy. By setting

$$\delta = \frac{\lambda}{\omega + \omega_0}, \quad (5)$$

we finally obtain a Hamiltonian similar to the one obtained when the RWA is applied:

$$\hat{\mathcal{H}} = \omega \hat{n} + \frac{\omega_0}{2} \hat{\sigma}_z + \epsilon(\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-). \quad (6)$$

However it should be noticed that the *new* interaction constant  $\epsilon$  has changed from the initial one ( $\lambda$ ), something that does not occur when the RWA is applied. The new interaction constant is now

$$\epsilon = \lambda \frac{2\omega_0}{\omega_0 + \omega}. \quad (7)$$

The expression for  $\delta$  is exactly equal to the expression for the first order approximation used in Refs. 14 and 15 using path integral approach to the problem in the  $\omega = \omega_0$ . Note that both methods give first order approximations and the expansion parameter here agrees with reference Refs. 14 and 15. We would like however to stress that the present method allows visualization of the form for the evolved wave function that allows the generation of some non-classical states.

The evolved wave function may be found now by applying the transformed unitary evolution operator to an initial wave function

$$|\Psi(t)\rangle = \hat{T}^\dagger \hat{U} \hat{T} |\Psi(0)\rangle, \quad (8)$$

where  $\hat{U}$  is given by

$$\hat{U} = e^{-it(\omega \hat{n} + \frac{1}{2}\omega_0 \hat{\sigma}_z)} e^{-it[\frac{\Delta}{2} \hat{\sigma}_z + \epsilon(\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-)]}, \quad (9)$$

where  $\Delta = \omega_0 - \omega$ . Equation (9) may be re-written as

$$\begin{aligned} \hat{U} &= e^{-it(\omega \hat{n} + \frac{1}{2}\omega_0 \hat{\sigma}_z)} \left( \frac{1}{2} [\hat{U}_{11} + \hat{U}_{22}] \hat{I} \right. \\ &\quad \left. + \frac{1}{2} [\hat{U}_{11} - \hat{U}_{22}] \hat{\sigma}_z + \hat{U}_{21} \sigma_- + \hat{U}_{12} \sigma_+ \right), \end{aligned} \quad (10)$$

where

$$\hat{U}_{11}(t; \hat{n}) = \cos \hat{\Omega}_{\hat{n}+1} t - i \frac{\Delta}{2} \frac{\sin \hat{\Omega}_{\hat{n}+1} t}{\hat{\Omega}_{\hat{n}+1}}, \quad (11)$$

$$\hat{U}_{12}(t; \hat{n}) = -i\epsilon \hat{a} \frac{\sin \hat{\Omega}_{\hat{n}} t}{\hat{\Omega}_{\hat{n}}}, \quad (12)$$

$$\hat{U}_{21}(t; \hat{n}) = -i\epsilon \hat{a}^\dagger \frac{\sin \hat{\Omega}_{\hat{n}+1} t}{\hat{\Omega}_{\hat{n}+1}}, \quad (13)$$

and

$$\hat{U}_{22}(t; \hat{n}) = \cos \hat{\Omega}_{\hat{n}} t + i \frac{\Delta}{2} \frac{\sin \hat{\Omega}_{\hat{n}} t}{\hat{\Omega}_{\hat{n}}}, \quad (14)$$

with

$$\hat{\Omega}_{\hat{n}} = \sqrt{\frac{\Delta^2}{4} + \epsilon^2 \hat{n}}. \quad (15)$$

### 3. Qubits of the quantized field

If the cavity field is initially prepared in the coherent state  $|\delta\rangle$

$$|\delta\rangle = \hat{D}(\delta)|0\rangle = e^{-\frac{\delta^2}{2}} \sum_{n=0}^{\infty} \frac{\delta^n}{\sqrt{n!}} |n\rangle, \quad (16)$$

where  $\hat{D}(\delta) = \exp[-\delta(\hat{a}^\dagger - \hat{a})]$  and  $|n\rangle$  is the Fock number state of the cavity field, and the atom is prepared in a superposition of the excited and ground states, the initial (total) wave function may be written as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)|\delta\rangle. \quad (17)$$

By inserting (17) in (8) we obtain the entangled state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|\psi_e\rangle|e\rangle + |\psi_g\rangle|g\rangle), \quad (18)$$

with

$$\begin{aligned} |\psi_e\rangle = & (U_{11}(t;0)e^{-i\frac{\omega_0 t}{2}} \cosh[\delta(\hat{a} - \hat{a}^\dagger)] \\ & + U_{22}(t;0)e^{i\frac{\omega_0 t}{2}} \sinh[\delta(\hat{a} - \hat{a}^\dagger)]|0\rangle \\ & + \tilde{U}_{21}(t;0)e^{-i(\omega - \frac{\omega_0}{2})t} \sinh[\delta(\hat{a} - \hat{a}^\dagger)]|1\rangle, \end{aligned} \quad (19)$$

and

$$\begin{aligned} |\psi_g\rangle = & (U_{11}(t;0)e^{-i\frac{\omega_0 t}{2}} \sinh[\delta(\hat{a} - \hat{a}^\dagger)] \\ & + U_{22}(t;0)e^{i\frac{\omega_0 t}{2}} \cosh[\delta(\hat{a} - \hat{a}^\dagger)]|0\rangle \\ & + \tilde{U}_{21}(t;0)e^{-i(\omega - \frac{\omega_0}{2})t} \cosh[\delta(\hat{a} - \hat{a}^\dagger)]|1\rangle, \end{aligned} \quad (20)$$

and where the  $U$ 's are defined as the (vacuum) expectation values

$$U_{ij}(t;0) = \langle 0|\hat{U}_{ij}(t;\hat{n})|0\rangle, \quad i, j = 1, 2, \quad (21)$$

and

$$\tilde{U}_{21}(t;0) = -i\epsilon\langle 0|\frac{\sin \hat{\Omega}_{\hat{n}+1} t}{\hat{\Omega}_{\hat{n}+1}}|0\rangle. \quad (22)$$

By measuring the atom when it leaves the cavity in the state

$$|\Psi_{atom}\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle), \quad (23)$$

we end up with a wave function describing the cavity field that reads

$$\begin{aligned} |\Psi_{field}\rangle = & [U_{11}(t;0)e^{-i\frac{\omega_0 t}{2}} + U_{22}(t;0)e^{i\frac{\omega_0 t}{2}}]|\delta\rangle \\ & + \tilde{U}_{21}(t;0)e^{-i(\omega - \frac{\omega_0}{2})t}|\delta, 1\rangle, \end{aligned} \quad (24)$$

where  $|\delta, k\rangle = \hat{D}(\delta)|k\rangle$  is a displaced number state [18]. Therefore, we have constructed a superposition of displaced number states [16] and by displacing the cavity field by  $-\delta$ , *i.e.* by injecting a field that displaces the cavity field by that effective amplitude, we can generate a qubit (in reconstruction processes is a common technique the displacement of a given wave function [19]).

### 4. Conclusions

We have studied the first order contributions of the counter rotating terms present in the interaction between a two-level atom and a cavity field by using a technique recently introduced in Ref. 17. We have been able to write down the wave function in this case, and to point out an initial state of the atom and the field that would lead to the generation of a quantum bit and superposition of displaced number states of the electromagnetic field.

Besides the solution given here, that allows manipulation of parameters to engineer a given state, we have looked for the initial states to construct superposition of displaced number states and qubits of the electromagnetic field which are considered highly non-classical.

It is worth to note that the qubit generated (after displacement of the cavity field) has arbitrary amplitudes, as the coefficients for the ground and first excited states can be varied arbitrarily. The final qubit state reads

$$\begin{aligned} |\Psi_{dis}\rangle = & [U_{11}(t;0)e^{-i\frac{\omega_0 t}{2}} + U_{22}(t;0)e^{i\frac{\omega_0 t}{2}}]|0\rangle \\ & + \tilde{U}_{21}(t;0)e^{-i(\omega - \frac{\omega_0}{2})t}|1\rangle. \end{aligned} \quad (25)$$

In Ref. 15 it was considered a ratio  $\lambda/\omega \approx 0.1$ . There have been efforts to obtain experimentally such ratios (that would allow the interaction with environments to be negligible [2]), considering the same ratio here, and not considering the correction we have found would indeed mislead the final result.

### Acknowledgments

We would like to thank CONACYT for support.

1. E.T. Jaynes and F.W. Cummings, *Proc. IEEE* **51** (1963) 89; see also B. W. Shore, and P. L. Knight, *J. Mod. Opt.* **40** (1993) 1195.
2. G. Rempe, W. Schleich, M.O. Scully and H. Walther, in *Proc. 3rd Intern. Symposium on Foundations of Quantum Mechanics* (Physical Society of Japan, Tokyo, 1989) pp. 294; M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J.M. Raimond, and S. Haroche, *Phys. Rev. Lett.* **76** (1996) 1800; S. Osnaghi, P. Bertet, A. Auffeves, P. Maioli, M. Brune, J.M. Raimond, and S. Haroche, quant-ph/0105063.
3. L. Davidovich, J.M. Raimond, M. Brune, and S. Haroche, *Phys. Rev. A* **36** (1987) 3771.
4. J.H. Eberly, N.B. Narozhny, and J.J. Sánchez/Mondragón, *Phys. Rev. Lett.* **44** (1980) 1323.
5. J.R. Kuklinski and J.L. Madajczyk, *Phys. Rev. A* **37** (1988) 3175.
6. J. Gea-Banacloche, *Phys. Rev. Lett.* **65** (1990) 3385; *Phys. Rev. A* **44** (1991) 6023; V. Buzek, H. Moya-Cessa, P.L. Knight and S.J.D. Phoenix, *Phys. Rev. A* **45** (1992) 8190.
7. D. M. Meekhof, C. Monroe, B. E. King, W. N. Itano, and D. J. Wineland, *Phys. Rev. Lett.* **76** (1996) 1796.
8. Z. Kis, W. Vogel, and L. Davidovich, *Phys. Rev. A* **64** (2001) 033401.
9. R.L. de Matos Filho and W. Vogel, *Phys. Rev. A* **54** (1996) 4560.
10. H. Moya-Cessa and P. Tombesi, *Phys. Rev. A* **61** (2000) 025401.
11. D.J. Wineland, C. Monroe, W.M. Itano, D. Leibfried, B.E. King, and D.M. Meekhof, *J. Res. Nat. Inst. Stan.* **103** (1998) 259.
12. A. Ekert and C. Macchiavello, *Acta Phys. Polonica A* **93** (1998) 63.
13. H. Moya-Cessa, D. Jonathan, and P.L. Knight, e-print quant-ph/0110167.
14. K. Zaheer and M.S. Zubairy, *Phys. Rev. A* **37** (1988) 1628.
15. K. Zaheer and M.S. Zubairy, *Opt. Commun.* **73** (1989) 325.
16. H. Moya-Cessa, *J. of Mod. Optics* **42** (1995) 1741.
17. A. Klimov and L.L. Sánchez-Soto, *Phys. Rev. A* **61** (2000) 063802.
18. F.A.M. de Oliveira, M.S. Kim, P.L. Knight, and V. Buzek, *Phys. Rev. A* **41** (1990) 2645.
19. L.G. Lutterbach and L. Davidovich, *Phys. Rev. Lett.* **78** (1997) 2547; H.Moya-Cessa, J.A. Roversi, S.M. Dutra, and A.Vidiella-Barranco, *Phys. Rev. A* **60** (1999) 4029.