Obtaining the gravitational force corresponding to arbitrary spacetimes. The Schwarzschild's case.

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Making use of the classical Binet's equation a general procedure to obtain the gravitational force corresponding to an arbitrary 4-dimensional spacetime is presented. This method provides, for general relativistic scenarios, classics expressions that may help to visualize certain effects that Newton's theory can not explain. In particular, the force produced by a gravitational field which source is spherically symmetrical (Schwarzschild's spacetime) is obtained. Such expression uses a redefinition of the classical reduced mass, in the limit case it can be reduced to Newton's universal law of gravitation and it produces *two* different orbital velocities for test particles that asimptotically coincide with the Newtonian one. As far as we know this is a new result.

Keywords: Universal gravitational law; perihelionshift; Schwarzschild potential; reduced mass.

Se presenta, haciendo uso de la ecuación de Binet clásica, un procedimiento general para obtener la expresión de la fuerza gravitacional correspondiente a un espacio-tiempo tetradimensional arbitrario. Este método provee expresiones clásicas para escenarios relativistas, lo que podría ayudar a visualizar efectos que no pueden ser explicados por la teoría newtoniana. En particular, se obtiene la fuerza producida por un campo gravitacional, cuya fuente es esféricamente simétrica (espacio-tiempo de Schwarzschild). Tal expresión emplea una redefinición de la masa reducida clásica, en el caso límite se reduce a la ley de gravitación universal de Newton y produce *dos* velocidades orbitales diferentes para partículas de prueba insertas en el campo, que coinciden asintóticamente con la velocidad orbital newtoniana. Hasta donde conocen los autores, éste es un resultado nuevo.

Descriptores: Ley de gravitación universal; corrimiento del perihelio; potencial de Schwarzschild; masa reducida.

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1. A historical introduction

The obtention of an expression for the gravitational force was first published in 1687 in the well known book *Philosophiae Naturalis Principia Mathematica* [1], in which Newton, using his three laws, deduces [2] an expression according to which the intensity of gravitational attraction is proportional to the product of the interacting masses and decreases as the square of the distance between them. In vectorial notation this expression can be written as

$$\overrightarrow{F} = -G\frac{Mm}{r^2}\hat{u}_r, \tag{1}$$

where G is the constant of universal gravitation, M and m the source and the test particle masses respectively, r the distance separating the bodies and $\hat{u}_r$ a versor in radial direction. As everyone knows, with this set of laws Newton satisfactorily explained the movement of planets and the other celestial bodies, giving, in this way, the foundations of modern astronomy.

Even though Newton's gravitation theory gave a good explanation of orbital phenomena happening in the heavens, with time, observations demonstrated that certain discrepancies that could not be adequately explained did exist. After Newton, Laplace and Poisson rewrote the gravitational law, giving it a both mathematical and physical more elegant formalism, but which still did not solve the aforesaid discrepancies. In fact, in 1895 Simon Newcomb, after numerous attempts to find a solution to the problem, suggested that perhaps the Newtonian law of the inverse square "is inexact when applied on short distances" [3]. It is with the release of Einstein's theory of general relativity (TGR) in 1915 when our conception of gravitational phenomena radically changed. This theory is able to account for the discrepancies that we have been talking about, particularly, the one which turned out to be one of its most notorious predictions: Mercury's perihelionshift. Additionally, TGR demonstrated that, in the low velocity limit relative to the speed of light and in presence of weak gravitational fields, it was equivalent to Newton's gravitation theory.

It is important to remark that the new theory, for the sake of a covariant expression (universal principle of covariance) of physical laws, does not use Newton's formulation based on forces, which arbitrarily depends on the selected coordinate system to describe the phenomenon, but instead uses tensor field equations which expressions are independent from the chosen coordinate systems. In this way, this equations, called Einstein's Field Equations (EFE), satisfy the theoretical need to express a covariant law of gravitation in the TGR. As we know, in tensor notation and conventional units, we can write them as

$$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}, \tag{2}$$

where $G_{\mu\nu}$ is Einstein's tensor, c the light's speed and $T_{\mu\nu}$ is the energy-momentum tensor. The way this equation relates geometry (left side) with physics (right side) is the reason why the TGR is often called "geometrodynamics". However, to get the EFE's solution means to solve, in the general case, ten second-order partial differential equations, a tiresome work even with today's constant advances in the computation processors' technology. This is an important reason because, unless physical conditions demands TGR (speeds close to the speed of light and strong gravitational fields), Newton's gravitation law is still commonly used as a perfectly valid theory.

In recent years the exploration of non-relativistics schemes to calculate astrodynamics effects has become more and more active (see for example the references of Wang's quantum-corrected newtonian force [4], the Calura *et al.* post-Newtonian planetary equations [5] or the Jefimenko's Gravito-Cogravitism theory [6]). However, it is important to remind that it is indeed possible [7–9] to generalize the notion of force, originally defined in a 3d-space, in order to be able to use it in TGR's 4d-space (Minkowski' s force). For the case of the obtention of a gravitational force generalized to a 4d-space that includes relativistic effects, Adler, Bazin, and Schiffer [10] and Weinberg [11] propose interesting procedures worth consulting. However, it is also possible to find an expression for a gravitational force of this kind beginning with a given solution of the EFE and the use of an equation which possesses -as Eq. (2) does- a geometro-dynamical character: Binet's equation.

In this paper we want to present a general scheme developed for obtain classical gravitational forces from general relativistic settings. This expressions can describe purely relativistic effects (as perihelionshift, for example) with just the use of Newtonian theory. The application of the procedure is briefly described in the Sec. 2 and 3 for the Schwarzschild case. The general method is summarized and the discussion is presented in Sec. 4. Finally, we are using here the MKS system.

2. Movement of a test particle in a gravitational field

We are interested in the dynamic study of a test particle placed in a gravitational field. In any advanced Mechanic's text [12–14], it is possible to find the procedure to obtain Binet's equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(u^{-1})}{\mu h^2 u^2}, \tag{3}$$

where $u = 1/r$, $\mu = (Mm)/(M+m)$ is the reduced mass, and $h = r^2\dot{\theta}$ (the dot means a classical time derivative) is the angular momentum per mass unit and F is the force involved. Thus, this equation relates the orbit's geometry (left side) to the particle's dynamics (right side). As long as the orbit is given, the force that determines it can be obtained and vice versa.

Now, if F is given by Newton's universal gravitation law, which means, if we study the classical case, we can find with (3) that

$$u(\theta) = \frac{A}{h^2}\left[1 + e\cos(\theta)\right], \tag{4}$$

where $A \equiv G(M+m)$, $e \equiv \frac{Ch^2}{A}$ (eccentricity) and C is an integration constant.

This is the essence of the method we want to introduce. For the relativistic case, it is first necessary to solve the EFE for certain physical conditions. In this paper, to make it simpler, a static and spherically symmetrical mass distribution is going to be taken as the source of the gravitational field. Under these conditions, from (2), Schwarzschild' s solution [15] is obtained, and we can write it as the following line element:

$$ds^2 = \left(1 - \frac{2m_g}{r}\right)dt^2 - \left(1 - \frac{2m_g}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{5}$$

where, $d\Omega^2 \equiv d\theta^2 + sen^2\theta d\phi^2$ is the differential solid angle and

$$m_g = \frac{GM}{c^2} \tag{6}$$

is called geometrical mass or gravitational radius.

From the study of the geodesics obtained from (5) it results that the trajectory followed by the test particle in the discussed gravity field is given, using a first-order perturbation method and after some simplifications, by [8]

$$\widetilde{u}(\theta) \cong \frac{m_g c^2}{\tilde{h}^2}\left\{1 + e\cos\left[\theta\left(1-\epsilon\right)\right]\right\} \tag{7}$$

in non-geometrized units, where

$$\epsilon = 3\frac{m_g^2 c^2}{\tilde{h}^2} \tag{8}$$

is the perturbation parameter, a very small quantity, and $\tilde{h}$ is the relativistic angular momentum per mass unit.

Note the similarity between Eqs. (7) and (4). From this relativistic result Mercury's perihelionshift can be obtained, whereas with (4) this effect does not appear. Note also that (7) must be equal to (4) when $\epsilon = 0$, obtaining this way the relation between the classical and relativistic momentum:

$$\tilde{h} = \sqrt{\frac{M}{M+m}}h. \tag{9}$$

3. Gravitational force from the Schwarzschild's metric

Having already noticed that Eq. (7) contains relativistic information (in fact, it is a solution of the EFE) and that it is precisely an orbital equation, we can, from a mathematical point of view, introduce it into Binet' s equation (3), obtain the corresponding force and study the physical consequences of it all.

We would like to know how should the force's expression be on (3) in order for (7) to become its solution. Thus, let us rewrite (3) in this way

$$\frac{d^2\widetilde{u}}{d\theta^2} + \widetilde{u} = -\frac{\tilde{F}(\widetilde{u}^{-1})}{\tilde{\mu}h^2\widetilde{u}^2}, \tag{10}$$

where $\tilde{u} = 1/r$ and $\tilde{\mu}$ has been written instead of μ (the reduced mass) because we do not know if, in such a procedure, the definition of reduced mass remains unaltered.

Noting that Eq. (7) involve a first-order perturbation method, from (8), (9) and (10), after tedious calculations and neglecting second order terms in ϵ, we obtain

$$\tilde{F} = -\frac{G(M+m)(6m_g + r)}{r^3}\tilde{\mu} + \left[\frac{G(M+m)(3m_g + 2r)}{r^3}\tilde{\mu}\right]\epsilon. \tag{11}$$

We must now require that as long as $\epsilon \to 0$, our

$$\tilde{F} \to -G\frac{Mm}{r^2},$$

doing this we can find that

$$\tilde{\mu} = \frac{Mmr}{(M+m)(6m_g + r)}, \tag{12}$$

which, as we shall see, is some kind of "generalized reduced mass".

If we now substitute (12) in (11), the general first-order gravitational force for the Schwarzschild's solution is obtained:

$$\tilde{F} = -G\frac{Mm}{r^2} + 3\frac{m_g^2 c^2 GMm\left(2c^2 r + 3MG\right)}{\tilde{h}^2 r^2\left(c^2 r + 6MG\right)}. \tag{13}$$

We can see in the second term of (13) the relativistic angular momentum per mass unit $\tilde{h} = r^2\dot{\theta}$ (here the point indicates a derivative with respect of proper-time). Remember that for conservative systems the angular momentum is conserved, so $\tilde{h}$ is constant. But we can still rewrite (13); for example, note that if very low speeds related to the speed of light, c, are considered (as in the Solar System), it is possible to write

$$\tilde{h} \cong Vr, \tag{14}$$

where V is the orbital velocity of the test particle. We then obtain,

$$\tilde{F} = -G\frac{Mm}{r^2} + 3\frac{G^3 M^3 m\left(2c^2 r + 3MG\right)}{c^2 r^4 V^2\left(c^2 r + 6MG\right)}. \tag{15}$$

If we wish to express the force $\tilde{F}$ in (15) as a function of r alone, without the orbital velocity V of the test particle, we proceed as in the classical scheme: given that we are working with a classical force, we assume Newton's laws are valid for it, and we make (15) equal to the centrifuge force

$$F_c = m\frac{V^2}{r}, \tag{16}$$

obtaining

$$V_1 = \pm\frac{c}{6rD} \times\sqrt{3D\left(D-r\right)\left(Dr + \sqrt{Dr\left(3r^2 - Dr - D^2\right)}\right)} \tag{17}$$

and

$$V_2 = \pm\frac{c}{6rD} \times\sqrt{3D\left(D-r\right)\left(Dr - \sqrt{Dr\left(3r^2 - Dr - D^2\right)}\right)}, \tag{18}$$

where $D = r + 6m_g$.

Substituting any of these expressions in (15) we can write $\tilde{F}$ as a function of r alone.

4. Analysis and conclusions

We have presented the procedure directly applied to the Schwarszchild solution. It should be convenient to summarize the general method as follows:

1. Select an arbitrary spacetime. The line element must be completely determined.

2. Obtain the radial equation of motion for a test particle via the corresponding geodesic equation. Note the order of the perturbation method(s) involved, if any.

3. Introduce the equation of motion in the Binet's equation, (10). The corresponding force should be writen in terms of the perturbation parameter(s). Check out the concordance with the order of the perturbation(s).

4. Require that as the perturbation parameter(s) tends to zero, the preliminar force tends to Newtonian force. This condition provides the $\tilde{\mu}$ factor.

5. Write the resulting expression only in terms of classical variables.

With the application of this method, we have derived the expression (15), the Schwarzschild's force from now on, including relativistic effects and which, in strong fields, differs from Newton's law of universal gravitation. Figure 1 shows this difference for a Sun-Mercury-like system at very small distances from the source. Note that this is in accord with Newcomb's idea [3]. Certainly, when we come near the source, the field's intensity increases, and it is precisely at this moment when the relativistic effects contained in the second term of the Schwarzschild's force are notorious. The asymptotic behaviour of the expression (15) is in perfect agreement with the predictions of Newton's gravitational law (1) as it might have been expected.

The Schwarzschild's force, is also in concordance with the predicted general relativity correction to Newtonian gravitational motion (see Refs. 10 and 16; the r^{-3} corrective potential can be derived from Eq. (25.42) in Ref. 17). In fact, for weak fields, but still non Newtonian fields (for example, in the Sun case $m_g = 1.4766\ km$), we have

$$\frac{G^3M^3m\left(2c^2r+3MG\right)}{c^2r^4V^2\left(c^2r+6MG\right)} \simeq 2\frac{G^3M^3m}{c^2r^4V^2}, \qquad (19)$$

and the Schwarzschild Force has an explicit r^{-4} term.

Naturally, from (15) is possible to find the perihelionshift advance, so it can help to show, just via classical mechanics, this important result of the TGR. Bearing this in mind, it is also illustrative to compare the Schwarzschild's force with the results of the Gravito-Cogravitism approach to find the periastron advance [18]: the total gravito-cogravitational force has a roughly similar r^{-4} dependent term.

Figure 2 shows the comparisson between the classical velocity and the obtained expressions (17) and (18). It is important to remark that for short distances velocities V_1 and V_2 are slower than the Newtonian one, but asimptotically agrees with the former. This is due to the fact that the second term in Eq. (15) is important when the field is strong, making the Schwarzschild's force $\tilde{F}$ less intense than the classical force (1), as seen in Fig. 1; therefore, the equilibrium condition between gravitational and centrifugal force demands Eqs. (17) and (18) to be lesser than the classical orbital velocity. As the field becomes weaker, the Schwarzschild's force matchs the Newton's force, and there is not *a priori* way to distinguish between the velocities. Note that the velocities matching occurs for diferents distances. Indeed, for a Sun-Mercury-like system, as the one presented in Fig. 2, V_1 coincides with Newtonian orbital velocity for $r_1 \gtrsim 4x10^2 Km$ from the source, while V_2 coincides for $r_2 \gtrsim 5x10^5 Km$.

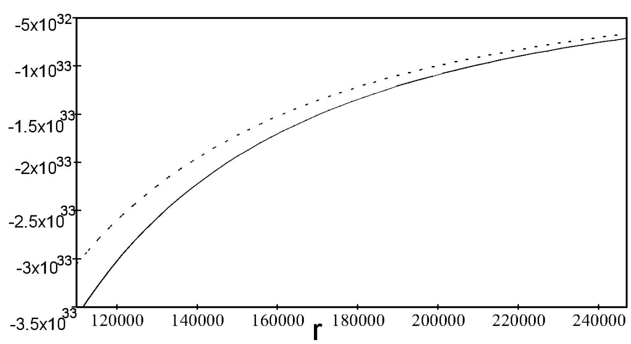

FIGURE 1. Newtonian (solid) and obtained (dots) forces.

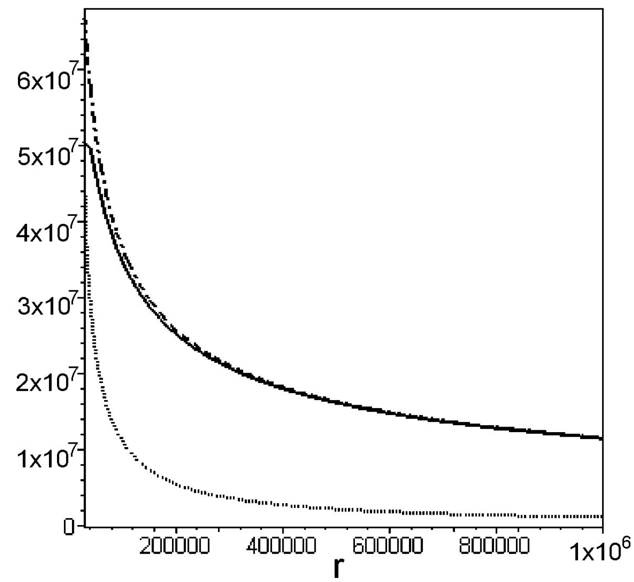

FIGURE 2. Classical velocity (dotdashed) and expressions (17), with solid line, and (18) with dots.

It is an interesting issue that there are *two* velocities, [(17) and (18)], that satisfies the equilibrium condition between centrifugal and gravitational forces instead of one. We report that as a curiosity of the ansatz we have used since we had not find any physical reason to discard Eq. (18). As far as we know this is a new result, and, since this bifurcation occurs for very near distances from the source it could be quite difficult to prove the existence (or not) of this effect.

On the other hand, we also have obtained the interesting Eq. (12) that is a mass depending on the separation between the interacting bodies and which, in the weak fields limit (small m_g), turns into the known expression for the reduced mass of a two bodies system. Also observe that the same asymptotic behaviour occurs when the bodies' separation is quite large (Fig. 3). The origin of (12) can be attributed to the fact that the effects spawned from space-time curvature in TGR have been transfered, into the present scheme, in the variation of this generalized mass with respect to the interacting bodies' distances.

As we have mentioned at the introduction, Eq. (10) possesses a geometro-dynamical character in the sense that it relates the test particle's trajectory (geometry) to the force (dynamics) acting on it. Note too that $\tilde{\mu}$ is present in the denominator of the dynamical member of (10): from this we can deduce that as long as we approach the source, the effects on the geometrical member become increasingly notorious, meaning that the trajectory will progressively differ from the one predicted by Newton's law of gravitation. Examples of this are Mercury and Icarus because of their nearness to the Sun.

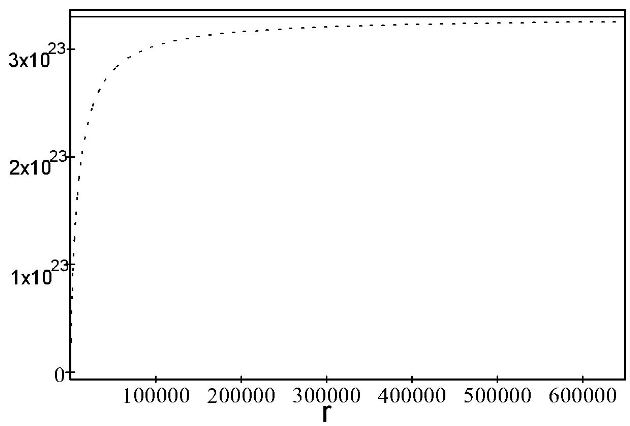

FIGURE 3.Reduced mass (solid line) and $\tilde{\mu}$ parameter (dots line).

Finally, we want to point out that more detailed future investigations are necessary to explore all the wealth of this ansatz; this method possesses a great deal of conceptual value, because it shows the way to describe purely relativistic effects through the use of a completely classical scheme. Moreover, it is general: given a trajectory predicted in a relativistic setting, it can be obtained a corresponding gravitational force "only" expressed in classical notions.

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