Hamiltonian structures for the acoustic field

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It is shown that the Hamiltonian structures for the acoustic field obtained by means of the canonical formalism using as field variables the components of the molecular displacements or the variation of the gas density are different and lead to different Poisson brackets. It is shown that by requiring that the values of the Hamiltonians coincide, the Poisson brackets also coincide.

Keywords: Acoustic field; Hamiltonian structures.

Se muestra que las estructuras hamiltonianas para el campo acústico que se obtienen a través del formalismo canónico usando como variables de campo las componentes de los desplazamientos moleculares o la variación de la densidad del gas son diferentes y llevan a paréntesis de Poisson diferentes. Se muestra que si se propone que los valores de las hamiltonianas coincidan, entonces los paréntesis de Poisson también coinciden.

Descriptores: Campo acústico; estructuras hamiltonianas.

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1. Introduction

As is well known, the Lagrangian and Hamiltonian formalisms employed in the study of mechanical systems with a finite number of degrees of freedom can be applied in the case of continuous media and fields. The Hamiltonian formulation is usually obtained from the Lagrangian formulation by means of the Legendre transformation, but in the case of fields this canonical procedure presents difficulties since not always the momentum densities are independent of the field variables, which is usually mended by the introduction of constraints.

However, it is possible to avoid these complications and give a Hamiltonian formulation for a given continuous system, without making reference to the Lagrangian formulation, if its evolution equations can be written in the form

$$\dot{\phi}_\alpha = D_{\alpha\beta} \frac{\delta H}{\delta \phi_\beta}, \qquad (1)$$

where the field variables ϕ_α represent the state of the system, H is some functional of the ϕ_α, $\delta H/\delta\phi_\alpha$ is the functional derivative of H with respect to ϕ_α, and the $D_{\alpha\beta}$ are operators that must satisfy certain conditions that allow the definition of a Poisson bracket between functionals of the ϕ_α (see, *e.g.*, Refs. 1–3). Here and in what follows a dot denotes partial differentiation with respect to the time and there is summation over repeated indices.

In this paper we consider Hamiltonian structures for the acoustic field in a perfect gas. An interesting feature of this simple example is the fact that one can use as field variables either the components of the vector field representing the displacement of the particles of the gas (three variables per space point) or the variations of the density of the gas (one variable per space point) [4,5]. In Sec. 2 we compare the Hamiltonian structures for the acoustic field obtained from the corresponding Lagrangians using the displacement vector field or the variation of the gas density as field variables and we find that the Poisson brackets and the Hamiltonians obtained in these two cases do not coincide. In Sec. 3 we show that if the value of the Hamiltonian is the same, regardless of which variables are employed, then the Poisson brackets also coincide.

2. Canonical formalism

In the study of the acoustic field in a perfect gas one can make use of the Cartesian components of the vector field, $\boldsymbol{\eta}$, that represents the small displacements of the gas particles with respect to their positions in the absence of sound waves. A suitable Lagrangian density is [4]

$$\mathcal{L}_v = \frac{1}{2}\mu_0 \dot{\boldsymbol{\eta}}^2 + P_0 \nabla\cdot\boldsymbol{\eta} - \frac{1}{2}\gamma P_0 (\nabla\cdot\boldsymbol{\eta})^2, \qquad (2)$$

where μ_0 is the mass density of the gas in equilibrium, P_0 is the pressure in equilibrium and $\gamma = C_p/C_v$, with C_p and C_v being the heat capacities at constant pressure and at constant volume, respectively (the subscript v is introduced to remark the vector character of the field variables). The Lagrangian density (2) has the usual structure found in the case of mechanical systems with a finite number of degrees of freedom, being the difference between a term corresponding to the kinetic energy density and another term identifiable as the potential energy density.

The Euler–Lagrange equations applied to (2) yield

$$\mu_0 \ddot{\boldsymbol{\eta}} - \gamma P_0 \nabla(\nabla\cdot\boldsymbol{\eta}) = 0. \qquad (3)$$

Following the standard procedure one finds that the conjugate momenta to the field variables η_i are $\pi_i = \partial\mathcal{L}_v/\partial\dot{\eta}_i = \mu_0\dot{\eta}_i$ $(i, j, \ldots = 1, 2, 3)$ and the Hamiltonian density is given by

$$\mathcal{H}_v = \pi_i\dot{\eta}_i - \mathcal{L}_v = \frac{\boldsymbol{\pi}^2}{2\mu_0} - P_0\nabla\cdot\boldsymbol{\eta} + \frac{\gamma P_0}{2}(\nabla\cdot\boldsymbol{\eta})^2. \quad (4)$$

Then, the Hamilton equations [4,2]

$$\dot{\eta}_i = \frac{\delta H_v}{\delta\pi_i}, \qquad \dot{\pi}_i = -\frac{\delta H_v}{\delta\eta_i}, \quad (5)$$

where $H_v = \int \mathcal{H}_v dv$, reproduce Eq. (3).

If F and G are two functionals of η_i and π_i, their Poisson bracket is defined as

$$\{F, G\}_v = \int \left(\frac{\delta F}{\delta\eta_i}\frac{\delta G}{\delta\pi_i} - \frac{\delta G}{\delta\eta_i}\frac{\delta F}{\delta\pi_i}\right) dv. \quad (6)$$

Letting $(\phi_1, \ldots, \phi_6) \equiv (\eta_1, \eta_2, \eta_3, \pi_1, \pi_2, \pi_3)$ we have

$$\phi_\alpha(\mathbf{r}', t) = \int \delta_{\alpha\beta}\delta(\mathbf{r}' - \mathbf{r})\phi_\beta(\mathbf{r}, t)dv \quad (7)$$

$(\alpha, \beta, \ldots = 1, 2, \ldots, 6)$, and

$$\frac{\delta\phi_\alpha(\mathbf{r}', t)}{\delta\phi_\beta(\mathbf{r}, t)} = \delta_{\alpha\beta}\delta(\mathbf{r}' - \mathbf{r}).$$

Hence,

$$\{\phi_\alpha(\mathbf{r}', t), \phi_\beta(\mathbf{r}'', t)\}_v = D_{\alpha\beta}\delta(\mathbf{r}' - \mathbf{r}''), \quad (8)$$

where

$$(D_{\alpha\beta}) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad (9)$$

and I is the 3×3 identity matrix. Explicitly,

$$\{\pi_i(\mathbf{r}', t), \pi_j(\mathbf{r}'', t)\}_v = 0 = \{\eta_i(\mathbf{r}', t), \eta_j(\mathbf{r}'', t)\}_v \quad (10)$$

and

$$\{\eta_i(\mathbf{r}', t), \pi_j(\mathbf{r}'', t)\}_v = \delta_{ij}\delta(\mathbf{r}' - \mathbf{r}''). \quad (11)$$

Making use of the matrix $(D_{\alpha\beta})$, Eqs. (5) and (6) can be rewritten as

$$\dot{\phi}_\alpha = D_{\alpha\beta}\frac{\delta H_v}{\delta\phi_\beta} \quad (12)$$

and

$$\{F, G\}_v = \int \frac{\delta F}{\delta\phi_\alpha}D_{\alpha\beta}\frac{\delta G}{\delta\phi_\beta}dv, \quad (13)$$

respectively. It may be pointed out that, owing to the linearity of the evolution equations considered here, there exists a set of functions $G_{\alpha\beta}(\mathbf{r}', t'; \mathbf{r}, t)$ such that

$$\phi_\alpha(\mathbf{r}', t') = \int G_{\alpha\beta}(\mathbf{r}', t'; \mathbf{r}, t)\phi_\beta(\mathbf{r}, t)dv,$$

hence

$$\{\phi_\alpha(\mathbf{r}', t'), \phi_\beta(\mathbf{r}'', t'')\}_v = D_{\alpha\gamma}G_{\beta\gamma}(\mathbf{r}'', t'', \mathbf{r}', t').$$

The relative variation of the gas density, denoted by σ, is related with $\boldsymbol{\eta}$ according to [4,5]

$$\sigma = -\nabla\cdot\boldsymbol{\eta} \quad (14)$$

and the divergence of Eq. (3) yields the wave equation

$$\mu_0\ddot{\sigma} - \gamma P_0\nabla^2\sigma = 0. \quad (15)$$

As can be readily seen, Eq. (15) follows from the Euler–Lagrange equations with

$$\mathcal{L}_s = \frac{1}{2}\left[\mu_0\dot{\sigma}^2 - \gamma P_0(\nabla\sigma)^2\right], \quad (16)$$

considering σ as the field variable.

The conjugate momentum to σ is therefore

$$\rho = \partial\mathcal{L}_s/\partial\dot{\sigma} = \mu_0\dot{\sigma}$$

and the corresponding Hamiltonian density is given by

$$\mathcal{H}_s = \rho\dot{\sigma} - \mathcal{L}_s = \frac{\rho^2}{2\mu_0} + \frac{\gamma P_0}{2}(\nabla\sigma)^2. \quad (17)$$

With the variables σ, ρ there is associated a Poisson bracket given by

$$\{F, G\}_s = \int \left(\frac{\delta F}{\delta\sigma}\frac{\delta G}{\delta\rho} - \frac{\delta G}{\delta\sigma}\frac{\delta F}{\delta\rho}\right) dv, \quad (18)$$

for any pair of functionals of σ and ρ. Thus, by analogy with Eqs. (10) and (11),

$$\{\sigma(\mathbf{r}', t), \sigma(\mathbf{r}'', t)\}_s = 0 = \{\rho(\mathbf{r}', t), \rho(\mathbf{r}'', t)\}_s$$

and

$$\{\sigma(\mathbf{r}', t), \rho(\mathbf{r}'', t)\}_s = \delta(\mathbf{r}' - \mathbf{r}''). \quad (19)$$

On the other hand, since $\sigma = -\nabla\cdot\boldsymbol{\eta}$ and

$$\rho = \mu_0\dot{\sigma} = -\mu_0\nabla\cdot\dot{\boldsymbol{\eta}} = -\nabla\cdot\boldsymbol{\pi},$$

the variables σ and ρ can be regarded as functionals of η_i and π_i. Then, making use of the expression (6) we obtain

$$\{\sigma(\mathbf{r}', t), \rho(\mathbf{r}'', t)\}_v = -\nabla'^2\delta(\mathbf{r}' - \mathbf{r}''), \quad (20)$$

which differs from (19); hence, the Poisson brackets (6) and (18) are different and define different Hamiltonian structures for the acoustic field.

The linear momentum of the acoustic field (defined as the generator of translations) is given by the general expression [4,2]

$$G_i = -\int \pi_k\frac{\partial\eta_k}{\partial x_i}dv. \quad (21)$$

But, when σ is the field variable, the components of the linear momentum are

$$G_i = -\int \rho \frac{\partial \sigma}{\partial x_i} dv \tag{22}$$

and, apart from the difference in the dimensions of the functionals (21) and (22), it seems clear that their values are not related (though both are conserved, *e.g.*, if there are no boundaries), which adds another difference between the two Hamiltonian structures.

We note, in passing, that if the Lagrangian density $\mathcal{L}_s$ is expressed in terms of $\boldsymbol{\eta}$, then the Euler–Lagrange equations give

$$\mu_0 \nabla(\nabla \cdot \ddot{\boldsymbol{\eta}}) - \gamma P_0 \nabla[\nabla^2(\nabla \cdot \boldsymbol{\eta})] = 0,$$

which is the gradient of the divergence of Eq. (3).

3. Alternative definition of the Hamiltonian structure

We shall consider again the components η_i as the field variables without making use of a Lagrangian, but demanding that the value of the Hamiltonian density coincides with that obtained in the case of the scalar variables, $\mathcal{H}_s$.

The idea is to write the equations of motion in the form (12), looking for the appropriate $D_{\alpha\beta}$, which may be constants, functions or operators, with the condition that $D_{\alpha\beta} = -\overline{D^{\dagger}_{\beta\alpha}}$, where $D^{\dagger}_{\alpha\beta}$ is the adjoint of $D_{\alpha\beta}$ and the bar denotes complex conjugation, in order for the Poisson bracket (13) to be antisymmetric (*i.e.*, $\{F,G\} = -\{G,F\}$) [2,3]. When the $D_{\alpha\beta}$ are constants, the Poisson bracket automatically satisfies the Jacobi identity, but in other cases one has to verify that this identity is satisfied [1].

Thus, assuming that the η_i are the field variables, we take as the Hamiltonian density

$$\mathcal{H}'_v = \frac{(\nabla \cdot \boldsymbol{\pi})^2}{2\mu_0} + \frac{\gamma P_0}{2} [\nabla(\nabla \cdot \boldsymbol{\eta})]^2, \tag{23}$$

which is obtained from $\mathcal{H}_s$ given in (17), substituting σ and ρ by the corresponding expressions in terms of η_i and π_i (it may be noticed that $\mathcal{H}'_v$ cannot be obtained from a Lagrangian density in the standard way). The functional derivatives of $H'_v = \int \mathcal{H}'_v dv$ are then

$$\frac{\delta H'_v}{\delta \eta_i} = \gamma P_0 \frac{\partial}{\partial x_i} \nabla^2 (\nabla \cdot \boldsymbol{\eta}) = \mu_0 \frac{\partial}{\partial x_i} \nabla \cdot \ddot{\boldsymbol{\eta}},$$

where we have made use of Eq. (3), and

$$\frac{\delta H'_v}{\delta \pi_i} = -\frac{1}{\mu_0} \frac{\partial}{\partial x_i} \nabla \cdot \boldsymbol{\pi}.$$

Substituting these derivatives into Eq. (12) with $(D_{\alpha\beta})$ written as the block matrix

$$(D_{\alpha\beta}) = \begin{pmatrix} (C_{ij}) & (E_{ij}) \\ (F_{ij}) & (G_{ij}) \end{pmatrix}, \tag{24}$$

and making use of the fact that $\pi_i = \mu_0 \dot{\eta}_i$ we have

$$\begin{aligned} \dot{\eta}_i &= C_{ij} \frac{\delta H'_v}{\delta \eta_j} + E_{ij} \frac{\delta H'_v}{\delta \pi_j} \\ &= C_{ij} \left(\mu_0 \frac{\partial}{\partial x_j} \nabla \cdot \ddot{\boldsymbol{\eta}} \right) + E_{ij} \left(-\frac{1}{\mu_0} \frac{\partial}{\partial x_j} \nabla \cdot \boldsymbol{\pi} \right) \\ &= \frac{\pi_i}{\mu_0} \end{aligned} \tag{25}$$

and, similarly,

$$\begin{aligned} \dot{\pi}_i &= F_{ij} \frac{\delta H'_v}{\delta \eta_j} + G_{ij} \frac{\delta H'_v}{\delta \pi_j} \\ &= F_{ij} \left(\mu_0 \frac{\partial}{\partial x_j} \nabla \cdot \ddot{\boldsymbol{\eta}} \right) + G_{ij} \left(-\frac{1}{\mu_0} \frac{\partial}{\partial x_j} \nabla \cdot \boldsymbol{\pi} \right) \\ &= \mu_0 \ddot{\eta}_i. \end{aligned} \tag{26}$$

These equations are satisfied if we take $C_{ij} = 0 = G_{ij}$,

$$E_{ij} \left(\frac{\partial}{\partial x_j} \nabla \cdot \boldsymbol{\pi} \right) = -\pi_i \tag{27}$$

and $F_{ij} = -E_{ij}$.

Taking the curl on both sides of Eq. (3) we obtain $\nabla \times \ddot{\boldsymbol{\eta}} = 0$, and therefore we can assume that $\nabla \times \boldsymbol{\pi}$ also vanishes (otherwise $\nabla \times \boldsymbol{\eta}$ would not be bounded). Thus Eq. (27) can be rewritten as

$$\begin{aligned} -\pi_i &= E_{ij} \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \pi_k \right) = E_{ij} \left(\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \pi_k \right) \\ &= E_{ij} \left(\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} \pi_j \right) = E_{ij} (\nabla^2 \pi_j), \end{aligned}$$

hence, if there are no boundaries present, the E_{ij} are the integral operators

$$E_{ij}(A_j) = \frac{1}{4\pi} \int \frac{A_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'. \tag{28}$$

Indeed, making use of $\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}')$, from Eq. (28) we have

$$E_{ij} \left(\frac{\partial}{\partial x_j} \nabla \cdot \pi \right) = E_{ij}(\nabla^2 \pi_j) = \frac{1}{4\pi} \int \frac{(\nabla'^2 \pi_i)(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'$$

and integrating by parts twice

$$\begin{aligned} E_{ij} \left(\frac{\partial}{\partial x_j} \nabla \cdot \pi \right) &= \frac{1}{4\pi} \int \pi_i(\mathbf{r}') \nabla'^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dv' \\ &= -\int \pi_i(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dv' = -\pi_i(\mathbf{r}), \end{aligned}$$

as required by Eq. (27).

From Eq. (28) it follows that for any pair of vector fields, **A** and **B**,

$$\int B_i E_{ij}(A_j)dv = \frac{1}{4\pi}\int B_i(\mathbf{r})\left[\int \frac{A_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'\right]dv$$
$$= \frac{1}{4\pi}\int A_i(\mathbf{r}')\left[\int \frac{B_i(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}dv\right]dv'$$
$$= \frac{1}{4\pi}\int A_i(\mathbf{r})\left[\int \frac{B_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'\right]dv$$
$$= \int A_i E_{ij}(B_j)dv = \int A_j E_{ji}(B_i)dv$$

therefore, $E^\dagger_{ij} = E_{ji}$ and, since $E_{ij} = -F_{ij}$, we have $E^\dagger_{ij} = -F_{ji}$, which means that

$$D^\dagger_{\alpha\beta} = -D_{\beta\alpha} = -\overline{D_{\beta\alpha}}$$

as required in order for the Poisson bracket to be antisymmetric.

We now compute the Poisson bracket between σ and ρ using the definition (13) with $(D_{\alpha\beta})$ given by (24), with $C_{ij} = 0 = G_{ij}$, $F_{ij} = -E_{ij}$ and E_{ij} given by Eq. (28)

$$\{\sigma(\mathbf{r},t),\rho(\mathbf{r}',t)\}'_v = \int \frac{\delta\sigma(\mathbf{r},t)}{\delta\eta_k(\mathbf{r}'',t)}E_{kl}\frac{\delta\rho(\mathbf{r}',t)}{\delta\pi_l(\mathbf{r}'',t)}dv'' - \int \frac{\delta\sigma(\mathbf{r},t)}{\delta\pi_k(\mathbf{r}',t)}E_{kl}\frac{\delta\rho(\mathbf{r}',t)}{\delta\eta_l(\mathbf{r},t)}dv'' \quad (29)$$

hence

$$\{\sigma(\mathbf{r},t),\rho(\mathbf{r}',t)\}'_v = \int \frac{\partial}{\partial x''_k}\delta(\mathbf{r}''-\mathbf{r}) \times \left[\frac{1}{4\pi}\int \frac{\frac{\partial}{\partial x'''_k}\delta(\mathbf{r}'-\mathbf{r}''')}{|\mathbf{r}''-\mathbf{r}'''|}dv'''\right]dv''$$

and integrating by parts both integrals we find that

$$\{\sigma(\mathbf{r},t),\rho(\mathbf{r}',t)\}'_v = \frac{1}{4\pi} \times \int\int \delta(\mathbf{r}''-\mathbf{r})\delta(\mathbf{r}'-\mathbf{r}''')\frac{\partial}{\partial x''_k}\frac{\partial}{\partial x'''_k}\frac{1}{|\mathbf{r}''-\mathbf{r}'''|}dv''dv'''$$

but

$$\frac{\partial}{\partial x''_k}\frac{\partial}{\partial x'''_k}\frac{1}{|\mathbf{r}''-\mathbf{r}'''|} = -\frac{\partial}{\partial x''_k}\frac{\partial}{\partial x''_k}\frac{1}{|\mathbf{r}''-\mathbf{r}'''|} = -\nabla''^2\frac{1}{|\mathbf{r}''-\mathbf{r}'''|} = 4\pi\delta(\mathbf{r}''-\mathbf{r}''')$$

and therefore

$$\{\sigma(\mathbf{r},t),\rho(\mathbf{r}',t)\}'_v = \delta(\mathbf{r}-\mathbf{r}'), \quad (30)$$

which agrees with Eq. (19).

4. Conclusions

The example considered here shows that despite the difference in the number of field variables employed to deal with the acoustic field, we can obtain equivalent Hamiltonian structures. This example also illustrates the fact that the Hamiltonian functional and the Poisson brackets can be chosen in many ways (*cf.* also Ref. 6). The Hamiltonian is in all these cases a constant of the motion, but to the authors' knowledge, it is not known if there exist additional conditions for a constant of motion to be a Hamiltonian (with the corresponding Poisson bracket satisfying the Jacobi identity). As pointed out at the beginning of Sec. 3, when the $D_{\alpha\beta}$ are constants the Jacobi identity is always satisfied [1,2]. Since the Poisson bracket constructed in Sec. 3 agrees with that defined by Eq. (18), which satisfies the Jacobi identity, it is to be expected that it also satisfies this identity.

Finally, it should be remarked that it would be wrong to talk about *the* Lagrangian or Hamiltonian density of a given system. If these densities are defined by the only condition that they reproduce the equations of motion of the system in question through the Euler–Lagrange equations or the Hamilton equations, there may be many acceptable choices, which may not have a direct physical meaning. Of course, with any appropriate Lagrangian or Hamiltonian density the corresponding formalism must yield valid results (such as conservation laws).

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