One-parameter isospectral special functions

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Using a combination of the ladder operators of Piña [1] and the parametric operators of Mielnik [2] we introduce second order linear differential equations whose eigenfunctions are isospectral to the special functions of the mathematical physics and illustrate the method with several key examples.

Keywords: Supersymmetry; Riccati equation; Liouville operators.

Usando una combinación de los operadores de escalera de Piña [1] y de los operadores parametricos de Mielnik [2] introducimos operadores lineales de segundo orden con eigenfunciones que son formas isoespectrales de las funciones especiales de la física matemática y presentamos algunos ejemplos básicos.

Descriptores: Supersimetría; ecuación de Riccati; operadores de Liouville.

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1. Introduction

The use of the factorization method [3] proved to be a powerful tool for extending the class of exactly solvable Sturm-Liouville problems especially in quantum mechanics, where in the form of *supersymmetric quantum mechanics* led to new potentials, which are isospectral to a given problem [4]. In a paper by Mielnik [2], the usual factorization operators

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \qquad a^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right), \tag{1}$$

of the one dimensional harmonic oscillator Hamiltonian

$$H + \frac{1}{2} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{1}{2} \tag{2}$$

have been replaced by new operators

$$b=\frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta(x)\right), \quad b^*=\frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta(x)\right). \tag{3}$$

In order that these new ladder operators factorize the same Hamiltonian (2)

$$bb^* = aa^* = H + \frac{1}{2}, \tag{4}$$

the function $\beta(x)$ should satisfy a Riccati equation of the form

$$\beta' + \beta^2 = 1 + x^2 \ . \tag{5}$$

Using the solutions of this equation,

$$\beta(x) = x + \frac{e^{-x^2}}{\gamma + \int\limits_0^x e^{-y^2}dy}, \tag{6}$$

one can introduce a new Hamiltonian $\widetilde{H}$, which is defined by the inverse factorization of (4):

$$b^*b = \widetilde{H} - \frac{1}{2} = H + \frac{1}{2} - 1 - \frac{d}{dx}\left[\frac{e^{-x^2}}{\gamma + \int\limits_0^x e^{-y^2}dy}\right] \tag{7}$$

with new potential functions

$$\widetilde{V}(x) = \frac{x^2}{2} - \frac{d}{dx}\left[\frac{e^{-x^2}}{\gamma + \int\limits_0^x e^{-y^2}dy}\right] \tag{8}$$

and whose eigenfunctions

$$\widetilde{\psi}_n = b^*\psi_{n-1} \qquad (n = 1, 2, ...) \tag{9}$$

are isospectral to the harmonic oscillator eigenfunctions ψ_n.

2. Factorization of special functions

The great majority of differential equations appearing in physics can be factorized by means of ladder operators. Therefore, we should be able to apply the procedure described in the previous section to the raising and lowering operators of the important class of Sturm-Liouville problems and get in this way isospectral second order differential equations. To attain this objective we proceed as follows.

One of the possible forms of factorizing a subclass of second order differential operators associated to the special functions of mathematical physics was introduced by Piña [1]. Consider the Sturm-Liouville problem

$$\mathcal{L}_n\psi_n(x) \equiv \left[P(x)\frac{d^2}{dx^2} + Q(x)\frac{d}{dx} + R_n(x)\right]\psi_n(x) = 0, \tag{10}$$

where P, Q and R_n are functions of the variable x, and R_n depend on the index n. Then, it is possible to construct raising and lowering operators [1]

$$A_n^+ = \sqrt{P}\frac{d}{dx} + a_n^+, \qquad A_n^- = \sqrt{P}\frac{d}{dx} + a_n^- \tag{11}$$

that can factorize Eq.(10) in two ways:

$$A_{n+1}^- A_n^+ = \mathcal{L}_n + K_n, \tag{12}$$

$$A_n^+ A_{n+1}^- = \mathcal{L}_{n+1} + K_n\ , \tag{13}$$

where the constant K_n is the same in both factorizations. Furthermore, the functions a_n^+, a_{n+1}^-, turn out to be

$$a_{n+1}^- = \frac{1}{2}\left[\frac{Q}{\sqrt{P}} - \frac{d}{dx}\sqrt{P} + c_n + \int\frac{1}{\sqrt{P}}(R_{n+1} - R_n)dx\right], \tag{14}$$

$$a_n^+ = \frac{1}{2}\left[\frac{Q}{\sqrt{P}} - \frac{d}{dx}\sqrt{P} - c_n - \int\frac{1}{\sqrt{P}}(R_{n+1} - R_n)dx\right], \tag{15}$$

where c_n is an integration constant. From Eq.(12), one may consider the constant K_n as the eigenvalue corresponding to the eigenfunction ψ_n for the operator $A_{n+1}^- A_n^+$.

Let us now define new operators B_n^+, B_{n+1}^- by

$$B_n^+ = A_n^+ + b_n^+,$$
$$B_n^- = A_n^- + b_n^-, \tag{16}$$

and demand that they can also factorize the Sturm-Liouville operator $\mathcal{L}_n(x)$ as

$$B_{n+1}^- B_n^+ = A_{n+1}^- A_n^+ = \mathcal{L}_n(x) + K_n\ . \tag{17}$$

Then, the following relationship should be fulfilled:

$$\sqrt{P}\left(b_n^+\frac{d}{dx} + b_{n+1}^-\frac{d}{dx}\right) + \sqrt{P}\frac{db_n^+}{dx} + b_n^+ a_{n+1}^- + a_n^+ b_{n+1}^- + b_n^+ b_{n+1}^- = 0\ . \tag{18}$$

Therefore, the functions b_n^+, b_{n+1}^- must satisfy

$$b_{n+1}^- = -b_n^+, \tag{19}$$

$$\sqrt{P}\frac{db_n^+}{dx} - {b_n^+}^2 + b_n^+(a_{n+1}^- - a_n^+) = 0\ . \tag{20}$$

Equation (20) is a Riccati type equation, which can be easily solved to get

$$b_n^+(x) = \frac{e^{\delta(x)}}{\gamma - \int\limits_{x_0}^{x}\frac{e^{\delta(y)}}{\sqrt{P(y)}}dy}\ . \tag{21}$$

Here, $\delta(x)$ is defined by the indefinite integral

$$\delta(x) \equiv \int^x \frac{(a_n^+(y) - a_{n+1}^-(y))}{\sqrt{P(y)}}dy, \tag{22}$$

and γ is an integration constant. x_0 may be chosen as the point where the integrand vanishes.

Similarly to Mielnik's new Hamiltonian $\widetilde{H}$, we now introduce the second order differential operator $\widetilde{\mathcal{L}}_n(x)$ given by

$$\widetilde{\mathcal{L}}_{n+1} = B_n^+ B_{n+1}^- - K_n = \mathcal{L}_{n+1} - 2\sqrt{P}\ \frac{db_n^+}{dx}\ . \tag{23}$$

That is, the new operator $\widetilde{\mathcal{L}}_n$ differs from $\mathcal{L}_n$ by the derivative of the solution of the Riccati equation (20), in the same way as Mielnik's Hamiltonian $\widetilde{H}$ differs from the harmonic oscillator Hamiltonian H, as can be seen in Eq.(7).

If we now define the functions

$$\widetilde{\psi}_{n+1} \equiv B_n^+\psi_n, \qquad n = 0, 1, 2, \dots\ , \tag{24}$$

where ψ_n are the eigenfunctions of $\mathcal{L}_n$, we can see that

$$\widetilde{\mathcal{L}}_{n+1}\widetilde{\psi}_{n+1} = \left(B_n^+ B_{n+1}^- - K_n\right) B_n^+\psi_n = B_n^+\mathcal{L}_n\psi_n = 0. \tag{25}$$

Therefore, these $\widetilde{\psi}_n$ are the eigenfunctions for the new operator $\widetilde{\mathcal{L}}_n$, and we can write the eigenvalue equation

$$B_n^+ B_{n+1}^-\widetilde{\psi}_{n+1} = \left(\widetilde{\mathcal{L}}_{n+1} + K_n\right)\psi_{n+1} = K_n\widetilde{\psi}_{n+1}\ . \tag{26}$$

As can be seen from Eq.(21), we have constructed a one paramenter family of operators $\widetilde{\mathcal{L}}_n(x;\gamma)$ which are "isospectral" to the original Sturm-Liouville operator $\mathcal{L}_n(x)$, for the allowed values of the parameter γ that produce a non-divergent function $b_n^+(x)$.

As we can see, $B_n^+,\ B_{n+1}^-$ are not ladder operators as are $A_n^+,\ A_{n+1}^-$. However, one can easily verify that the third order operators,

$$C_n^+ \equiv B_n^+ A_{n-1}^+ B_n^-, \tag{27}$$

$$C_{n+1}^- \equiv B_{n-1}^+ A_n^- B_{n+1}^-, \tag{28}$$

play the role of raising and lowering operators, respectively, for the functions $\widetilde{\psi}_n(x)$. The use of ladder operators of order higher than two is not easy to find in the literature, but some work in this direction has already been reported [5].

Since B_{n+1}^-, and therefore C_{n+1}^-, is not defined for $n=0$, the space defined by the eigenfunctions $\widetilde{\psi}_n$ lack the element with $n = 0$, similarly to what happens in SUSY factorization [4].

3. Examples

Here, we proceed to find the new operators $\widetilde{\mathcal{L}}_n(x;\gamma)$ and their eigenfunctions, from the factorizations of the special functions of mathematical physics.

3.1. Hermite polynomials

The Hermite differential equation

$$\frac{d^2 H_n(x)}{dx^2} - 2x\frac{dH_n(x)}{dx} + 2nH_n(x) = 0 \tag{29}$$

has raising and lowering differential operators given by

$$\left(\frac{d}{dx} - 2x\right) H_n(x) = -H_{n+1}, \tag{30}$$

$$\frac{d}{dx} H_{n+1}(x) = 2(n+1)H_n(x) \; . \tag{31}$$

In this case, the δ-integral in Eq.(22) is

$$\delta = \int^x (-2y)dy = -x^2,$$

and therefore

$$b_n^+ = \frac{e^{-x^2}}{\gamma - \int\limits_0^x e^{-y^2} dy} \; . \tag{32}$$

The integrand in the denominator being positive definite, one should impose the condition $|\gamma| > \sqrt{\pi}/2$ [2] in order to have a well-defined operator.

Now, with the use of Eq.(20) one gets

$$\frac{db_n^+}{dx} = \left(\frac{e^{-x^2}}{\gamma - \int\limits_0^x e^{-y^2} dy}\right)^2 - \frac{2xe^{-x^2}}{\gamma - \int\limits_0^x e^{-y^2} dy} \; , \tag{33}$$

and therefore, the second order differential operator

$$\widetilde{\mathcal{L}}_{n+1}(x;\gamma) = \frac{d^2}{dx^2} - 2x\frac{d}{dx} + 2n + \frac{4xe^{-x^2}}{\gamma - \int\limits_0^x e^{-y^2} dy} - \frac{2e^{-2x^2}}{\left(\gamma - \int\limits_0^x e^{-y^2} dy\right)^2} \tag{34}$$

has parametric eigenfunctions given by

$$\widetilde{H}_{n+1}(x;\gamma) = -H_{n+1}(x) + \frac{e^{-x^2}}{\gamma - \int\limits_0^x e^{-y^2} dy} H_n(x) \; . \tag{35}$$

3.2. Laguerre polynomials

For the Laguerre differential equation

$$x^2\frac{d^2 L_n^\alpha(x)}{dx^2} + [(\alpha+1)x - x^2]\frac{dL_n^\alpha(x)}{dx} + nxL_n^\alpha(x) = 0 \; , \tag{36}$$

the raising and lowering operators are

$$\left(x\frac{d}{dx} + \alpha + n + 1 - x\right) L_n^\alpha(x) = (n+1)L_{n+1}^\alpha(x), \tag{37}$$

$$\left(x\frac{d}{dx} - n - 1\right) L_{n+1}^\alpha(x) = -(\alpha+n+1)L_n^\alpha(x) \; . \tag{38}$$

The δ-integral is

$$\delta = \int^x \frac{\alpha + 2(n+1) - y}{y} dy = \ln x^{[\alpha+2(n+1)]} - x, \tag{39}$$

and hence

$$b_n^+ = \frac{x^{\alpha+2n+2}e^{-x}}{\gamma - \int\limits_0^x y^{(\alpha+2n+1)}e^{-y}dy} \; . \tag{40}$$

For $x > 0$, the integral in the denominator is positive definite, with maximum value

$$\int\limits_0^\infty y^{(\alpha+2n+1)}e^{-y}dy = \Gamma(\alpha+2n+2),$$

and, since there is no upper limit for increasing n, we must have $\gamma < 0$.

The second order differential operator

$$\widetilde{\mathcal{L}}_{n+1}(x;\gamma) = x^2\frac{d^2}{dx^2} + \left[(\alpha+1)x - x^2\right]\frac{d}{dx} + nx + \frac{2[x - (\alpha+2n+2)]x^{\alpha+2n+1}e^{-x}}{\gamma - \int\limits_0^x y^{\alpha+2n+1}e^{-y}dy} - \frac{2x^{2\alpha+4n+2}e^{-2x}}{\left(\gamma - \int\limits_0^x y^{\alpha+2n+1}e^{-y}dy\right)^2} \tag{41}$$

has as parametric eigenfunctions

$$\widetilde{L}_{n+1}^\alpha(x;\gamma) = (n+1)L_{n+1}^\alpha(x) + \frac{x^{\alpha+2n+2}e^{-x}}{\gamma - \int\limits_0^x y^{\alpha+2n+1}e^{-y}dy} L_n^\alpha \; . \tag{42}$$

3.3. Legendre polynomials

The Legendre differential equation

$$(x^2-1)^2\frac{d^2P_n(x)}{dx^2}+2x(x^2-1)\frac{dP_n(x)}{dx} -n(n+1)(x^2-1)P_n(x)=0 \quad (43)$$

has the raising and lowering operators

$$\left[(x^2-1)\frac{d}{dx}+(n+1)x\right]P_n(x)=(n+1)P_{n+1}(x), \quad (44)$$

$$\left[(x^2-1)\frac{d}{dx}-(n+1)x\right]P_{n+1}(x) =-(n+1)P_n(x). \quad (45)$$

The δ-integral is in this case

$$\delta=-2(n+1)\int^x \frac{y}{1-y^2}dy=\ln(1-x^2)^{n+1}, \quad (46)$$

hence

$$b_n^+=\frac{(1-x^2)^{n+1}}{\gamma-\int_{-1}^{x}(1-y^2)^n dy}. \quad (47)$$

The integral in the denominator is positive definite, which, for a given n, has maximum value

$$\int_{-1}^{+1}(1-x^2)^n dx=2\int_0^{\pi/2}\sin^{2n+1}\theta\, d\theta=\frac{2(2n)!!}{(2n+1)!!}$$

and, therefore, $|\gamma|>2$.

Hence, the second order differential operator

$$\widetilde{\mathcal{L}}_{n+1}(x;\gamma)=(1-x^2)\frac{d^2}{dx^2}-2x\left(1-x^2\right)\frac{d}{dx} +n(n+1)(1-x^2)+\frac{4(n+1)(1-x^2)^{n+1}}{\gamma-\int_{-1}^{x}(1-y^2)^n dy} -\frac{2(1-x^2)^{2n+2}}{\left(\gamma-\int_{-1}^{x}(1-y^2)^n dy\right)^2} \quad (48)$$

has as parametric eigenfunctions

$$\widetilde{P}_{n+1}(x;\gamma)=-(n+1)P_{n+1}(x) +\frac{(1-x^2)^{n+1}}{\gamma-\int_{-1}^{x}(1-y^2)^n dy}P_n(x). \quad (49)$$

3.4. Chebyshev polynomials

For the Chebyshev differential equation

$$(1-x^2)^2\frac{d^2T_n(x)}{dx^2}-x(1-x^2)\frac{dT_n(x)}{dx} +n^2(1-x^2)T_n(x)=0, \quad (50)$$

the corresponding raising and lowering operators are

$$\left[(1-x^2)\frac{d}{dx}-nx\right]T_n(x)=-nT_{n+1}(x), \quad (51)$$

$$\left[(1-x^2)\frac{d}{dx}+(n+1)x\right]T_{n+1}(x) =(n+1)T_n(x). \quad (52)$$

In this case, the δ-integral is

$$\delta=-(2n+1)\int^x\frac{y}{1-y^2}dy=\left(n+\frac{1}{2}\right)\ln(1-x^2) \quad (53)$$

and, hence

$$b_n^+=\frac{(1-x^2)^{n+\frac{1}{2}}}{\gamma-\int_1^x(1-y^2)^{n-\frac{1}{2}}dy}. \quad (54)$$

The integral in the denominator has maximum value

$$\int_{-1}^{1}(1-x^2)^{n-\frac{1}{2}}dx=2\int_0^{\pi/2}\sin^{2n}\theta d\theta=\frac{\pi(2n-1)!!}{(2n)!!},$$

which imposes the condition $\gamma>\pi$.

We can thus construct the second order differential operator

$$\widetilde{\mathcal{L}}_{n+1}(x;\gamma)=(1-x^2)\frac{d^2}{dx^2}-x\left(1-x^2\right)\frac{d}{dx}+n^2(1-x^2) +\frac{2(2n+1)x(1-x^2)^{n+\frac{1}{2}}}{\gamma-\int_{-1}^{x}(1-y^2)^{n-\frac{1}{2}}dy}-\frac{2(1-x^2)^{2n+1}}{\left(\gamma-\int_{-1}^{x}(1-y^2)^{n-\frac{1}{2}}dy\right)^2}, \quad (55)$$

with the parametric eigenfunctions

$$\widetilde{T}_{n+1}(x;\gamma)=-nT_{n+1}(x) +\frac{(1-x^2)^{n+\frac{1}{2}}}{\gamma-\int_{-1}^{x}(1-y^2)^{n-\frac{1}{2}}dy}T_n(x). \quad (56)$$

3.5. Jacobi polynomials

The differential equation defining the Jacobi polynomials

$$(1-x^2)^2\,\frac{d^2P_n^{\alpha\beta}(x)}{dx^2}+(1-x^2)[\beta-\alpha-(\alpha+\beta+2)x] \times\frac{dP_n^{\alpha\beta}(x)}{dx}++(1-x^2)n(n+\alpha+\beta+1)P_n^{\alpha\beta}(x) =0, \quad (57)$$

has raising and lowering operators given by

$$\left[(1-x^2)\frac{d}{dx}+(n+1+\alpha+\beta)\left(-x+\frac{\beta-\alpha}{2n+2+\alpha+\beta}\right)\right]P_n^{\alpha\beta}(x)=-\frac{2(n+1)(n+1+\alpha+\beta)}{2n+2+\alpha+\beta}P_{n+1}^{\alpha\beta}(x), \qquad (58)$$

$$\left[(1-x^2)\frac{d}{dx}+(n+1)(x+\frac{\beta-\alpha}{2n+2+\alpha+\beta})\right]P_{n+1}^{\alpha\beta}(x)=\frac{2(n+1+\alpha)(n+1+\beta)}{2n+2+\alpha+\beta}P_n^{\alpha\beta}(x)\,. \qquad (59)$$

In this case, we have

$$\delta=\int^x\frac{p-qy}{1-y^2}dy=\frac{1}{2}(q+p)\ln(1+x)+\frac{1}{2}(q-p)\ln(1-x)\,,$$

where

$$p=\frac{\beta^2-\alpha^2}{2n+2+\alpha+\beta}\,,\qquad q=2n+2+\alpha+\beta\,.$$

Hence

$$b_n^+=\frac{(1+x)^{\frac{1}{2}(q+p)}(1-x)^{\frac{1}{2}(q-p)}}{\gamma-\int\limits_{-1}^{x}(1+y)^{\frac{1}{2}(q+p)-1}(1-y)^{\frac{1}{2}(q-p)-1}dy}\,. \qquad (60)$$

For the parameter γ, since $q>p$, we demand that

$$\gamma>\int\limits_{-1}^{1}(1+x)^{\frac{1}{2}(q+p)-1}(1-x)^{\frac{1}{2}(q-p)-1}dx =2^{q-1}\frac{\Gamma(\frac{q+p}{2})\Gamma(\frac{q-p}{2})}{\Gamma(q)}.$$

From here, we construct the second order differential operator

$$\widetilde{\mathcal{L}}_{n+1}(x;\gamma)=(1-x^2)^2\frac{d^2}{dx^2}+(1-x^2)\left[\beta-\alpha-(\alpha+\beta+2)x\right]\frac{d}{dx}+(1-x^2)n(n+\alpha+\beta+1)+ \frac{2(qx-p)(1+x)^{\frac{1}{2}(q+p)}(1-x)^{\frac{1}{2}(q-p)}}{\gamma-\int\limits_{-1}^{x}(1+y)^{\frac{1}{2}(q+p)-1}(1-y)^{\frac{1}{2}(q-p)-1}dy}-\frac{2(1+x)^{q+p}(1-x)^{q-p}}{\left(\gamma-\int\limits_{-1}^{x}(1+y)^{\frac{1}{2}(q+p)-1}(1-y)^{\frac{1}{2}(q-p)-1}dy\right)^2}\,, \qquad (61)$$

whose parametric eigenfunctions are

$$\widetilde{P}_{n+1}^{\alpha\beta}(x;\gamma)=-\frac{2(n+1)(n+1+\alpha+\beta)}{2n+2+\alpha+\beta}P_{n+1}^{\alpha\beta}(x) +\frac{(1+x)^{\frac{1}{2}(q+p)}(1-x)^{\frac{1}{2}(q-p)}}{\gamma-\int\limits_{-1}^{x}(1+y)^{\frac{1}{2}(q+p)-1}(1-y)^{\frac{1}{2}(q-p)-1}dy}P_n^{\alpha\beta}(x). \qquad (62)$$

3.6. Bessel functions

For the Bessel differential equation

$$\frac{d^2J_n(x)}{dx^2}+\frac{1}{x}\frac{dJ_n(x)}{dx}+(1-\frac{n^2}{x^2})J_n(x)=0\,, \qquad (63)$$

the raising and lowering operators are

$$\left(\frac{d}{dx}-\frac{n}{x}\right)J_n(x)=-J_{n+1}(x) \qquad (64)$$

$$\left(\frac{d}{dx}+\frac{n+1}{x}\right)J_{n+1}(x)=J_n(x)\,. \qquad (65)$$

The δ-integral is found to be

$$\delta=\int^x\left(-\frac{n}{y}-\frac{n+1}{y}\right)dy=\ln x^{-(2n+1)}\,,$$

and therefore

$$b_n^+=\frac{x^{-(2n+1)}}{\gamma'-\int\limits_{\infty}^{x}y^{-(2n+1)}dy}=\frac{2n}{\gamma x^{2n+1}+x} \qquad (66)$$

with $\gamma=2n\gamma'\geq 0$.

From here, we can construct a second order differential operator defined by

$$\widetilde{\mathcal{L}}_{n+1}(x;\gamma)=\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+\left(1-\frac{(n+1)^2}{x^2}\right) +\frac{4n+4n\gamma(2n+1)x^{2n-1}}{(\gamma x^{2n+1}+x)^2}\,, \qquad (67)$$

whose parametric eigenfunctions are given by

$$\widetilde{J}_{n+1}(x;\gamma)=-J_{n+1}(x)+\frac{2n}{\gamma x^{2n+1}+x}J_n(x)\,. \qquad (68)$$

Hence, in the case of Bessel functions the new eigenfunctions $\widetilde{J}_{n+1}(x;\gamma)$ are not regular at $x=0$, except in the case $\gamma=0$, for which $\widetilde{\mathcal{L}}_{n+1}(x;\gamma)=\mathcal{L}_{n-1}$ and $\widetilde{J}_{n+1}(x;\gamma)=J_{n-1}(x)$.

4. Conclusion

We elaborated here on a combination of a class of Sturm-Liouville ladder operators and one-parameter operators of Mielnik type that allowed us to construct isospectral Sturm-Liouville second-order linear differential operators with parametric eigenfunctions. Calculations are worked out for a few important cases.

1. E. Piña, *Rev. Mex. Fís*. **41** (1995) 913. See also, H.C. Rosu, M. Reyes, and O. Obregón, *Rev. Mex. Fís*. **43** (1997) 224.
2. B. Mielnik, *J. Math. Phys.* **25** (1984) 3387.
3. P.A.M. Dirac, *Principles of Quantum Mechanics*, (Clarendon Press, Oxford, Second Ed.,1935); E.Schrödinger, *Proc. Roy. Irish Acad. A* **46** (1940) 183; *A* **47** (1941) 53; L. Infeld and T.E. Hull, *Rev. Mod. Phys.* **23** (1951) 21; A. Stahlhofen and K. Bleuler, *Nuovo Cim. B* **104** (1989) 447.
4. F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry in Quantum Mechanics,* (World Scientific, Singapore, 2001).
5. A. Pérez-Lorenzana, *Rev. Mex. Fís*. **42** (1996) 1060.