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Symplectic structures and Hamiltonians of a mechanical system

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It is shown that in the case of a mechanical system with a finite number of degrees of freedom in classical mechanics, any constant of motion can be used as Hamiltonian by defining appropriately the symplectic structure of the phase space (or, equivalently, the Poisson bracket) and that for a given constant of motion, there are infinitely many symplectic structures that reproduce the equations of motion of the system.

Keywords: Symplectic structure; Hamilton equations.

Se muestra que en el caso de un sistema mecánico con un número finito de grados de libertad en la mecánica clásica, cualquier constante de movimiento puede usarse como hamiltoniana definiendo apropiadamente la estructura simpléctica del espacio fase (o, equivalentemente, el paréntesis de Poisson) y que para una constante de movimiento dada, existe una infinidad de estructuras simplécticas que reproducen las ecuaciones de movimiento del sistema.

Descriptores: Estructura simpléctica; ecuaciones de Hamilton.

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1. Introduction

The equations of motion of a mechanical system with a finite number of degrees of freedom in classical mechanics are usually the Euler–Lagrange equations for the Lagrangian $L = T - U$, where T denotes the kinetic energy and U is the potential energy (assuming that the forces are derivable from a potential). The equations of motion can also be expressed in the form of Hamilton's equations, which are equivalent to $df/dt = \{f, H\}$, for any function f that does not depend explicitly on the time, defined on the phase space, where H is the Hamiltonian and $\{\ ,\ \}$ is the Poisson bracket. The Hamiltonian is usually obtained from the Lagrangian by means of the Legendre transformation and, frequently, but not always, H corresponds to the total energy (see, *e.g.*, Refs. 1, 2). As shown below, for a given mechanical system with a finite number of degrees of freedom there are infinitely many Hamiltonians, which need not be derived from a Lagrangian, and for each choice of the Hamiltonian, there are infinitely many Poisson brackets that allow us to express the equations of motion in Hamiltonian form (see also Refs. 3–5).

In Sec. 2 the basic theory is reviewed. In Sec. 3 some concrete examples are given, considering two simple systems with two degrees of freedom, and in Sec. 4 the general results are established. Throughout this paper the summation convention is employed.

2. Symplectic structures

Hamilton's equations, expressed in terms of canonical coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$, are given by

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \tag{1}$$

where H is the Hamiltonian function of the mechanical system. In what follows we shall consider only the case where the right-hand sides of Eqs. (1) do not depend on the time, then it can be assumed that H does not depend explicitly on the time and making use of Eqs. (1) and the chain rule it follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i} = 0,$$

which means that H must be a constant of motion.

Similarly, one finds that for an arbitrary (differentiable) function $f = f(q^i, p_i)$,

$$\frac{df}{dt}=\frac{\partial f}{\partial q^i}\frac{dq^i}{dt}+\frac{\partial f}{\partial p_i}\frac{dp_i}{dt}=\frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i} \equiv \{f, H\}, \tag{2}$$

where $\{\ ,\ \}$ is the Poisson bracket. If

$$x^\mu \ (\mu, \nu, \ldots = 1, 2, \ldots, 2n)$$

is an arbitrary system of coordinates in the phase space, the Poisson bracket of two arbitrary functions, f, g, is expressed as

$$\{f, g\}=\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}=\left(\frac{\partial x^\mu}{\partial q^i}\frac{\partial x^\nu}{\partial p_i}-\frac{\partial x^\mu}{\partial p_i}\frac{\partial x^\nu}{\partial q^i}\right) \times\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} = \sigma^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu}, \tag{3}$$

with

$$\sigma^{\mu\nu} \equiv \{x^\mu, x^\nu\}. \tag{4}$$

Since the Poisson bracket satisfies the Jacobi identity, $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, we have

$\{\{x^\mu, x^\nu\}, x^\lambda\} + \{\{x^\nu, x^\lambda\}, x^\mu\} + \{\{x^\lambda, x^\mu\}, x^\nu\} = 0$ which, according to Eqs. (3) and (4) amounts to

$$\sigma^{\rho\lambda}\frac{\partial\sigma^{\mu\nu}}{\partial x^\rho} + \sigma^{\rho\mu}\frac{\partial\sigma^{\nu\lambda}}{\partial x^\rho} + \sigma^{\rho\nu}\frac{\partial\sigma^{\lambda\mu}}{\partial x^\rho} = 0. \tag{5}$$

By combining Eqs. (2) and (3) it follows that,

$$\frac{dx^\mu}{dt} = \sigma^{\mu\nu}\frac{\partial H}{\partial x^\nu}, \tag{6}$$

where the functions $\sigma^{\mu\nu}$ form an antisymmetric matrix, $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ [see Eq. (4)] which must obey the nonlinear differential equations (5). Equations (6) reduce to Eqs. (1) when

$$(\sigma^{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{7}$$

where I denotes the $n \times n$ unit matrix and

$$(x^1, \ldots, x^{2n}) = (q^1, \ldots, q^n, p_1, \ldots, p_n)$$

(in which case Eqs. (5) are trivially satisfied). According to Darboux's theorem, if $(\sigma^{\mu\nu})$ is a nonsingular antisymmetric matrix satisfying Eqs. (5), there exist locally coordinates (defined up to canonical transformations) in terms of which $(\sigma^{\mu\nu})$ takes the form (7) (see, *e.g.*, Ref. 6).

If the $2n \times 2n$ matrix $(\sigma^{\mu\nu})$ is invertible, its inverse, $(\omega_{\mu\nu})$, is also an antisymmetric, invertible matrix and Eq. (5) is equivalent to

$$\frac{\partial\omega_{\mu\nu}}{\partial x^\lambda} + \frac{\partial\omega_{\nu\lambda}}{\partial x^\mu} + \frac{\partial\omega_{\lambda\mu}}{\partial x^\nu} = 0. \tag{8}$$

(Any matrix $(\omega_{\mu\nu})$ satisfying these conditions is said to define a symplectic structure.) Then, the Hamilton Eqs. (6) are equivalent to

$$\omega_{\mu\nu}\frac{dx^\nu}{dt} = \frac{\partial H}{\partial x^\mu}. \tag{9}$$

As pointed out above, the Hamiltonian is usually obtained from the Lagrangian of the mechanical system; however, under the present assumptions, *any* constant of motion can be used as Hamiltonian, with an appropriate definition of the Poisson bracket (or, equivalently, of the symplectic structure). One chooses H and then finds $\omega_{\mu\nu}$ such that Eqs. (8) and (9) are satisfied.

3. Examples

3.1. Particle in a uniform gravitational field

We shall consider the equations of motion

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{p}_x = 0, \quad \dot{p}_y = -mg, \tag{10}$$

corresponding to a particle with mass m in a uniform gravitational field (here g is a positive constant representing the acceleration of gravity). As can be readily seen using Eqs. (10), $p_x p_y/m + mgx$ and p_x^2 are constants of the motion. Then, taking

$$H = \frac{p_x p_y}{m} + mgx + \frac{\lambda}{2m}p_x^2, \tag{11}$$

where λ is an arbitrary real constant, with

$$(x^1, x^2, x^3, x^4) = (x, y, p_x, p_y),$$

Eqs. (9) are given explicitly by

$$\begin{aligned}
\omega_{12}\frac{p_y}{m} + \omega_{14}(-mg) &= mg, \\
-\omega_{12}\frac{p_x}{m} + \omega_{24}(-mg) &= 0, \\
-\omega_{13}\frac{p_x}{m} - \omega_{23}\frac{p_y}{m} + \omega_{34}(-mg) &= \frac{p_y}{m} + \frac{\lambda p_x}{m}, \\
-\omega_{14}\frac{p_x}{m} - \omega_{24}\frac{p_y}{m} &= \frac{p_x}{m},
\end{aligned} \tag{12}$$

where we have made use of Eqs. (10) and of the antisymmetry of $\omega_{\mu\nu}$. By inspection, one finds that a particular solution of Eqs. (12) is

$$\begin{aligned}
&\omega_{12} = 0, \quad \omega_{13} = -\lambda, \quad \omega_{14} = -1, \\
&\omega_{23} = -1, \quad \omega_{24} = 0, \quad \omega_{34} = 0,
\end{aligned} \tag{13}$$

which trivially satisfies conditions (8). Furthermore, for any value of λ, the determinant of the antisymmetric matrix $(\omega_{\mu\nu})$ given by Eqs. (13) is equal to 1. The inverse of $(\omega_{\mu\nu})$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\lambda \\ 0 & -1 & 0 & 0 \\ -1 & \lambda & 0 & 0 \end{pmatrix},$$

which means that the only nonvanishing Poisson brackets among the phase space coordinates are

$$\{x, p_y\} = 1, \quad \{y, p_x\} = 1, \quad \{y, p_y\} = -\lambda. \tag{14}$$

In fact, using Eqs. (14) and (11) in $\dot{x}^\mu = \{x^\mu, H\}$ one recovers Eqs. (10) [see Eqs. (4) and (6)]. Thus, (x, y, p_x, p_y) are not canonical coordinates with respect to the symplectic structure defined by Eqs. (13) but one can readily verify that

$$Q^1 = x, \quad Q^2 = y, \quad P_1 = p_y + \lambda p_x, \quad P_2 = p_x,$$

is a set of canonical coordinates. In terms of these variables, the Hamiltonian (11) is given by

$$H = \frac{P_1 P_2}{m} + mgQ^1 - \frac{\lambda}{2m}P_2^2$$

[*cf.* Eq. (11)].

One can apply the Legendre transformation to find the Lagrangian corresponding to this Hamiltonian and the result is

$$L = P_i\dot{Q}^i - H = P_i\frac{\partial H}{\partial P_i} - H = \frac{m}{2}(2\dot{x}\dot{y} + \lambda\dot{x}^2) - mgx. \tag{15}$$

Substituting this Lagrangian into the Euler–Lagrange equations one obtains Eqs. (10). It may be noticed that (15) does not depend on y and therefore the momentum conjugate to y, $\partial L/\partial\dot{y} = m\dot{x}$, is conserved.

3.2. The two-dimensional isotropic harmonic oscillator

As a second example we shall consider the two-dimensional isotropic harmonic oscillator, which is defined by the equations of motion

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{p}_x = -kx, \quad \dot{p}_y = -ky, \qquad (16)$$

where m and k are constants. This time we shall make use of the usual Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}k(x^2 + y^2), \qquad (17)$$

which corresponds to the total energy; however, as in the preceding example, we might use any other constant of motion as Hamiltonian (see below and Ref. 3). Taking $(x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)$, Eqs. (9) are given explicitly by

$$\begin{aligned}
\omega_{12}\frac{p_y}{m} + \omega_{13}(-kx) + \omega_{14}(-ky) &= kx, \\
-\omega_{12}\frac{p_x}{m} + \omega_{23}(-kx) + \omega_{24}(-ky) &= ky, \\
-\omega_{13}\frac{p_x}{m} - \omega_{23}\frac{p_y}{m} + \omega_{34}(-ky) &= \frac{p_x}{m}, \\
-\omega_{14}\frac{p_x}{m} - \omega_{24}\frac{p_y}{m} - \omega_{34}(-kx) &= \frac{p_y}{m}.
\end{aligned}$$

These equations allow us to express, *e.g.*, ω_{14}, ω_{23} and ω_{34} in terms of ω_{13}, ω_{24} and ω_{12}

$$\begin{aligned}
\omega_{14} &= \frac{p_y}{mky}\omega_{12} - \frac{x}{y}(\omega_{13} + 1), \\
\omega_{23} &= -\frac{p_x}{mkx}\omega_{12} - \frac{y}{x}(\omega_{24} + 1), \\
\omega_{34} &= -\frac{p_x}{mky}(\omega_{13}+1) + \frac{p_y}{mkx}(\omega_{24}+1) + \frac{p_x p_y}{m^2k^2xy}\omega_{12}. \qquad (18)
\end{aligned}$$

If we set $\omega_{13} = -1$, $\omega_{24} = -1$ and $\omega_{12} = 0$, we obtain $\omega_{14} = \omega_{23} = \omega_{34} = 0$, which satisfy Eqs. (9) and correspond to the usual symplectic structure. A different symplectic structure is given by

$$\omega_{13} = -1, \quad \omega_{24} = -1, \quad \omega_{12} = \lambda mkxy, \qquad (19)$$

where λ is an arbitrary real constant. Then Eqs. (18) yield

$$\omega_{14} = \lambda x p_y, \quad \omega_{23} = -\lambda y p_x, \quad \omega_{34} = \lambda\frac{p_x p_y}{mk}, \qquad (20)$$

and a straightforward computation shows that Eqs. (8) are satisfied. The inverse of the matrix $(\omega_{\mu\nu})$ defined by Eqs. (19) and (20) is given by

$$\begin{aligned}
&\{x, y\} = \lambda\frac{p_x p_y}{mk}, \ \{x, p_x\} = 1, \ \{x, p_y\} = -\lambda y p_x, \\
&\{y, p_x\} = \lambda x p_y, \ \{y, p_y\} = 1, \ \{p_x, p_y\} = \lambda mkxy \qquad (21)
\end{aligned}$$

(recall that $\sigma^{\mu\nu} = \{x^\mu, x^\nu\}$). One can verify that

$$\begin{aligned}
Q^1 &= x\cos\left[\frac{\lambda}{4\sqrt{mk}}(p_y^2 + mky^2)\right] + \frac{p_x}{\sqrt{mk}}\sin\left[\frac{\lambda}{4\sqrt{mk}}(p_y^2 + mky^2)\right], \\
Q^2 &= y\cos\left[\frac{\lambda}{4\sqrt{mk}}(p_x^2 + mkx^2)\right] - \frac{p_y}{\sqrt{mk}}\sin\left[\frac{\lambda}{4\sqrt{mk}}(p_x^2 + mkx^2)\right], \\
P_1 &= p_x\cos\left[\frac{\lambda}{4\sqrt{mk}}(p_y^2 + mky^2)\right] - \sqrt{mk}\,x\sin\left[\frac{\lambda}{4\sqrt{mk}}(p_y^2 + mky^2)\right], \\
P_2 &= p_y\cos\left[\frac{\lambda}{4\sqrt{mk}}(p_x^2 + mkx^2)\right] + \sqrt{mk}\,y\sin\left[\frac{\lambda}{4\sqrt{mk}}(p_x^2 + mkx^2)\right], \qquad (22)
\end{aligned}$$

are canonical variables with respect to the Poisson bracket (21). Making use of Eqs. (17) and (22) one finds that

$$H = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{1}{2}k[(Q^1)^2 + (Q^2)^2],$$

which is of the form (17).

The most general symplectic structure that leads to the equations of motion (16) with the Hamiltonian (17) may be obtained substituting Eqs. (18) into Eqs. (8). However, in order to solve these partial differential equations easily, it is convenient to replace the variables (x, y, p_x, p_y) by (r, θ, ρ, ϕ) according to

$$\begin{aligned}
x &= \frac{1}{\sqrt{mk}}r\cos\theta, \qquad p_x = r\sin\theta, \\
y &= \frac{1}{\sqrt{mk}}\rho\cos\phi, \qquad p_y = \rho\sin\phi \qquad (23)
\end{aligned}$$

(note that (r, θ, ρ, ϕ) is not formed by pairs of conjugate variables). Then from Eqs. (16) we obtain

$$\dot{r} = 0, \quad \dot{\theta} = -\sqrt{k/m}, \quad \dot{\rho} = 0, \quad \dot{\phi} = -\sqrt{k/m}, \qquad (24)$$

and the Hamiltonian (17) is expressed as

$$H = \frac{1}{2m}(r^2 + \rho^2). \tag{25}$$

Substituting Eqs. (24) and (25) into Eqs. (9) we find that

$$\begin{aligned} -\sqrt{k/m}(\omega_{12} + \omega_{14}) &= \frac{r}{m}, \\ \omega_{24} &= 0, \\ \sqrt{k/m}(\omega_{23} - \omega_{34}) &= \frac{\rho}{m}, \end{aligned} \tag{26}$$

where now $\omega_{\mu\nu}$ are the components of the symplectic form with respect to the coordinates $(x^1, x^2, x^3, x^4) = (r, \theta, \rho, \phi)$. Thus we can write

$$\omega_{14} = -\omega_{12} - \frac{r}{\sqrt{mk}}, \qquad \omega_{23} = \omega_{34} + \frac{\rho}{\sqrt{mk}} \tag{27}$$

[*cf.* Eqs. (18)]. Substituting Eqs. (27) into Eqs. (8), with $\omega_{24} = 0$ we obtain

$$\begin{aligned} \frac{\partial\omega_{34}}{\partial\phi} + \frac{\partial\omega_{34}}{\partial\theta} &= 0, \qquad \frac{\partial\omega_{34}}{\partial r} + \frac{\partial\omega_{13}}{\partial\phi} + \frac{\partial\omega_{12}}{\partial\rho} = 0, \\ \frac{\partial\omega_{12}}{\partial\theta} + \frac{\partial\omega_{12}}{\partial\phi} &= 0, \qquad \frac{\partial\omega_{12}}{\partial\rho} - \frac{\partial\omega_{13}}{\partial\theta} + \frac{\partial\omega_{34}}{\partial r} = 0. \end{aligned}$$

These equations imply that ω_{12}, ω_{13}, and ω_{34} must depend on r, ρ and $\theta - \phi$ only; that is, making $\alpha \equiv \theta - \phi$,

$$\omega_{12}=F(r,\rho,\alpha), \;\; \omega_{13}=G(r,\rho,\alpha), \;\; \omega_{34}=K(r,\rho,\alpha), \tag{28}$$

where F, G, and K are functions of three variables which must be related by the condition

$$\frac{\partial F}{\partial\rho} - \frac{\partial G}{\partial\alpha} + \frac{\partial K}{\partial r} = 0. \tag{29}$$

Making use of Eqs. (27) and (28) one finds that

$$\det(\omega_{\mu\nu})=\left[\left(F+\frac{r}{\sqrt{mk}}\right)\left(K+\frac{\rho}{\sqrt{mk}}\right)-FK\right]^2. \tag{30}$$

Thus, choosing two functions of three variables, F and K, such that the right-hand side of Eq. (30) does not vanish, the function G is determined by Eq. (29), up to an arbitrary function of r and ρ. Once $(\omega_{\mu\nu})$ is known, one can compute its inverse, whose entries are the Poisson brackets among r, θ, ρ and ϕ, and using the properties of the Poisson bracket one can also find the Poisson brackets among the original variables x, y, p_x and p_y. A more involved problem is that of finding canonical coordinates.

If we choose as Hamiltonian the constant of motion

$$H = \sqrt{\frac{k}{m}}(xp_y - yp_x), \tag{31}$$

which, apart from a constant factor, is the angular momentum, then in terms of the coordinates

$$(x^1, x^2, x^3, x^4) = (r, \theta, \rho, \phi)$$

we have [see Eqs. (23)]

$$H = \frac{r\rho}{m}\sin(\phi - \theta), \tag{32}$$

and, instead of Eqs. (27), we obtain

$$\omega_{14}=-\omega_{12}+\frac{\rho\sin(\theta-\phi)}{\sqrt{mk}}, \;\; \omega_{23}=\omega_{34}-\frac{r\sin(\theta-\phi)}{\sqrt{mk}}, \quad \omega_{24}=\frac{r\rho\cos(\theta-\phi)}{\sqrt{mk}} \tag{33}$$

and Eqs. (8) imply that ω_{12}, ω_{13}, and ω_{34} depend on r, ρ, and $\theta - \phi \equiv \alpha$ only, with

$$\frac{\partial F}{\partial\rho} - \frac{\partial G}{\partial\alpha} + \frac{\partial K}{\partial r} = \frac{\sin\alpha}{\sqrt{mk}}, \tag{34}$$

where F, G, and K are defined as in Eqs. (28). In the present case

$$\det(\omega_{\mu\nu})=\left[\left(F-\frac{\rho\sin\alpha}{\sqrt{mk}}\right)\left(K-\frac{r\sin\alpha}{\sqrt{mk}}\right) -FK+\frac{Gr\rho\cos\alpha}{\sqrt{mk}}\right]^2, \tag{35}$$

therefore, we can choose, *e.g.*, two functions of three variables, F and K, then G is determined by Eq. (34) up to a function of r and ρ, which must be chosen in such a way that the right-hand side of Eq. (35) does not vanish.

It may be remarked that, by contrast with the Hamiltonian (17), which is a non-negative function, the Hamiltonian (31) can take any real value.

4. General propositions

Given a mechanical system with n degrees of freedom, there exist, at least locally, $2n - 1$ functionally independent constants of motion, $x^2, \ldots, x^{2n}$ which can be used as part of a coordinate system in phase space; furthermore, it is possible to find locally a function x^1 such that $\dot{x}^1 = 1$. (For instance, in the case of the equations of motion (24) we can take $x^1 = -\frac{1}{2}\sqrt{m/k}(\theta + \phi)$, $x^2 = \theta - \phi$, $x^3 = r$, $x^4 = \rho$.) Any constant of motion, H, is a function of $x^2, \ldots, x^{2n}$ only. Therefore, by means of a coordinate transformation $(x^2, \ldots, x^{2n}) \mapsto (x^{2'}, \ldots, x^{2n'})$ such that $x^{2'} = H$, from Eqs. (9), dropping the primes, we have $\omega_{\mu\nu}\delta_1^\nu = \delta_\mu^2$ that is,

$$\omega_{12} = -1, \quad \omega_{13} = \omega_{14} = \cdots = \omega_{1,2n} = 0$$

and Eqs. (8) imply that the remaining $(n-1)(2n-1)$ independent components $\omega_{\mu\nu}$ must be functions of $(x^2, \ldots, x^{2n})$ only, such that

$$\frac{\partial\omega_{\mu\nu}}{\partial x^\lambda} + \frac{\partial\omega_{\nu\lambda}}{\partial x^\mu} + \frac{\partial\omega_{\lambda\mu}}{\partial x^\nu} = 0. \tag{36}$$

$(\mu, \nu, \lambda = 2, \ldots, 2n)$ [*cf.* Eq. (29)]. Any solution of Eqs. (36) can be locally expressed in the form

$$\omega_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

($\mu, \nu = 2, \ldots, 2n$), where $A_2, \ldots, A_{2n}$ depend on $(x^2, \ldots, x^{2n})$ only. The only restriction on the functions A_μ comes from the condition $\det(\omega_{\mu\nu}) \neq 0$.

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