Heat conduction and near-equilibrium linear regime

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A comparison of experiment on heat conduction in a rod with a calculation shows that if the conditions for the near-equilibrium linear regime are fulfilled, the differences between the rigorous solution of the minimum entropy production problem and its linearized version are small. They usually fall within the limits of experimental error. Hence, it may well illustrate that the Prigogine's theorem of minimum entropy production, despite its well-known limits recalled in a recent discussion, may serve as a useful approximation to the problem in question.

Keywords: Nonequilibrium thermodynamics; heat conduction.

Una comparación del experimento en la conducción del calor en una barra del aluminio con un cálculo muestra, que si las condiciones para el régimen lineal del equilibrio cercano están cumplidos, las diferencias entre la solución exacta del problema de la producción mínima de la entropía y su versión linealizada son pequeñas. Las diferencias citadas caen generalmente dentro de los límites del error experimental. Por lo tanto, pueden mostrar bien que el teorema de Prigogine de la producción mínima de la entropía, a pesar de sus límites bien conocidos, recordados en una discusión reciente, puede servir como una aproximación útil al problema bajo consideración.

Descriptores: Termodinámica del desequilibrio; conducción del calor.

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1. Introduction

In Ref. 1 it has been shown that measurements performed with the apparatus described in classical textbooks and originally applied for determination of the heat conduction coefficient κ, can be used for a classroom illustration of the Prigogine's theorem [2]. It says that in the near-equilibrium linear regime (which includes, if not implies, small temperature differences), the total entropy production in a system subject to flow of heat, reaches a minimum value at the nonequilibrium stationary state, characterized in our example by a constant heat flow along a rod and a simple position-independent temperature gradient. Hence, at the nonequilibrium stationary state the dependence of temperature on distance is linear and satisfies the Fourier's law [2]. It has been remarked in note [3] that a variational approach to the problem of heat flow in a metal rod placed between two reservoirs kept at different temperatures leads to an exponential profile of temperature $T(x)$ rather than a linear one, as presented in Refs. 1 and 2. However, the exponential solution for $T(x)$ is not stationary since $\partial T/\partial t \sim \partial^2 T/\partial x^2$ is nonzero [4]. It is known that Prigogine's theorem has a limited applicability: Hoover [4] has invoked two counterexamples to the validity of this theorem, an old one due to Klein (*cf.* Ref. 5), and a more recent one related to the Rayleigh-Bérnard flow.

This discussion [3,5] might put in doubt the sense of relating the experiment with a constant flow of heat as originally sketched in Fig. 17.1, p. 386 of Ref. 2 combined with Fourier's law to give Eq. 17.1.5 therein, with minimizing entropy production in the linear regime (Eq. 17.2.44 in Ref. 2). We argue that if one takes care to work in the linear regime of the nonequilibrium stationary states, the Prigogine's theorem holds in many cases with sufficient practical accuracy, if not strictly. This line of thought follows that of Kondepudi [6] who argued that "the difference between the actual steady state ... and the state ... that minimizes the entropy production ... gets smaller as we approach the state of equilibrium". Hence, it makes sense to present its application to the students, as suggested in Ref. 1. It may be chosen as a part of a physicist's *curriculum vitae* enabling him or her to practise dealing with physical rules with limited applicability, but known limits. In the present case this concerns the Prigogine's theorem. Another, widely known example is the linear Ohm law, taught at all levels of physics teaching despite of, and together with, a wealth of nonlinear current-voltage characteristics observed in physics and applied in devices.

To show the accuracy with which the Prigogine's theorem is applicable to our experiment, we first recall a detail related to Ref. 5, and follow by discussing the exponential solution for $T\ (x)$ [3] more closely.

Most of the details of the Klein's counterexample to the general applicability of Prigogine's theorem do not matter in our case. However, Klein has remarked that a numerical analysis of his example revealed that the steady state and the state with minimum entropy production are practically undistinguishable from each other for certain values of parameters [5]. In the following, we shall check if the same applies to the heat flow in a rod. Namely, we shall verify if the linear dependence obtained in our experiment can serve as a good approximation of the exponential dependence [3]. In order to perform this verification we have assembled our own measuring setup of the type described in Ref. 7 and measured temperature $T(x,t)$ as a function of position x and time t . The results did not essentially differ from those in Ref. 7. Also, we have calculated a linear approximation of the exponential dependence given in Ref. 3. Finally, we have fitted the linear and exponential dependencies to the results of our measurements and compared the fits.

The measuring set and results will be presented in Sec. 2. In Sec. 3 it will be shown that Fourier's linear dependence of temperature on distance [1,2] represents the linear approximation of the exponential dependence [3]. It is shown that the difference between the linear and the exponential fits to our experimental data falls within the limits of the accuracy of the experiment and that under well-defined conditions the Fourier's law of heat conduction does not exclude the approximate applicability of the theorem of minimum entropy production.

2. Experimental

The experimental setup (Fig. 1) consists of an aluminum rod of 45 cm in length and 4 cm in diameter. One end of the rod is placed in a cold thermal reservoir of a double-wall cylindrical vessel C. A stream of water of a steady temperature $T_c = 290K$ flows in the direction indicated by arrows in Fig. 1 between the walls of the vessel C. The other end of the rod is placed in a hot thermal reservoir maintained at a temperature T_h by an electric heater (Fig. 1). The numbers 1 - 6 (Fig. 1) mark the points at which temperature is measured. The lateral surface of the rod is thermally insulated.

Figure 2 presents temperature $T(x,t)$ as a function of position x and time t , where t is the time from the moment when the rod was put in contact with the two thermal reservoirs. The numbers to the right of the curves are the distances of the six thermometers from the hot thermal reservoir. Table I gives the data of $T(x)$ for $t > 65$ min. The device to measure κ is usually set so that heating and cooling of the rod at each end occurs along some finite segment of the rod, as shown in Fig. 1, rather than at the base surfaces of the cylinder [Fig. 3(b)] having coordinates $x_h = 0$ at the hot thermal reservoir and $x_c = L$ at the cold thermal reservoir. In practice, the determination of temperature at the coordinate $x_h = 0$ and the coordinate $x_c = L$ is based on using the quantities extrapolated from the plots in Fig. 3(a), solid line; these extrapolated quantities are marked (*) in Table I. The experimental error (standard deviation S_T) of temperature $T(x)$ measurement is found to be 1 K. Namely, the measurement of voltage U during the gauge procedure of the thermoelements was preformed with error $S_U = 0.01mV$. It resulted in linear dependence $T = kU$ with k = 24.1 K/mV and standard regression S k = 0.4 K/mV. To get the standard deviation S_T, the values of S_U , S_k , as well as the error $S_x = 0.03$ cm in determining the position x of a thermoelement are taken.

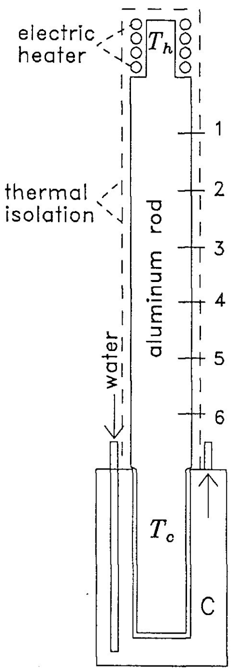

FIGURE 1. Experimental setup. C - double-wall cylindrical vessel. The arrows indicate the direction of the flow of water between the walls. The numbers 1 - 6 mark the points at which the temperature is measured.

TABLE I. x-distances of the six particular thermometers from the hot thermal reservoir (h). $T(x)$ - temperature at a distance x from the hot thermal reservoir. The asterisk (*) marks the extrapolated quantities.

x[cm]	0 (x_h)	4	8	12	16	20	24	27.77*(x_c)
T(x)[K]	343*	335	327	320	312	304	296	290

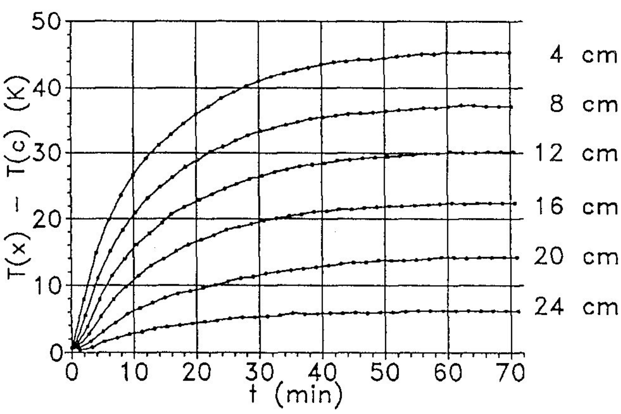

FIGURE 2. Difference between the temperatures $T(x)$ of the six thermometers and the temperature T (c) of the cold thermal reservoir as a function of time t . For each curve the distance x = const. The numbers to the right of the curves mark the distances of the six particular thermometers from the hot thermal reservoir.

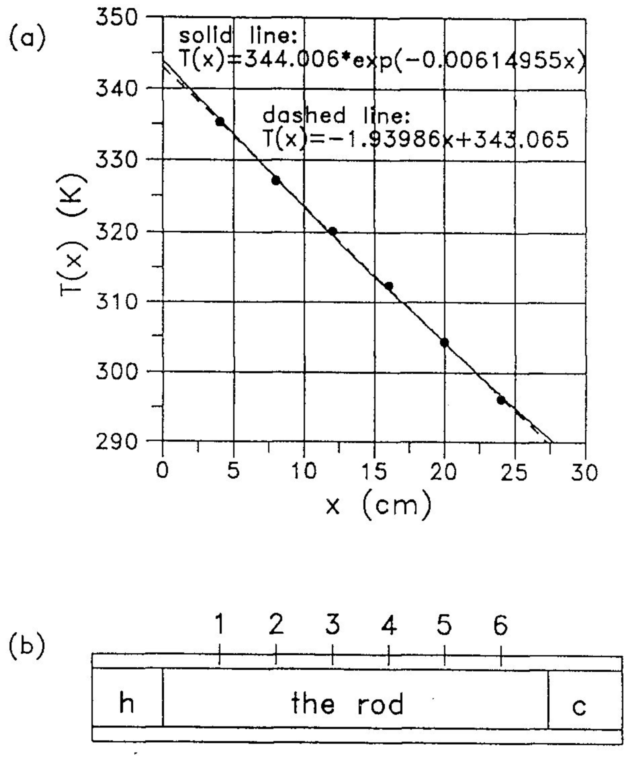

FIGURE 3. (a) The temperatures $T(x)$ of the six thermometers as a function of their distance x from the hot thermal reservoir. The points mark the steady temperatures $T(x)$ observed after $t > 65$ min. Solid line represents the exponential fit and the dashed line - the linear fit to the data. (b) Schematic drawing of the experimental setup. The hot thermal reservoir T (h) maintained at a temperature T (h) and the cold thermal reservoir (c) maintained at a temperature T (c) are shown. The numbers 1 - 6 denote the points at which the temperature is measured.

3. Near-equilibrium linear regime

A comparison of the Fourier's law of heat conduction in the rod,

$$J_q = -\kappa \frac{\partial T}{\partial x}, \tag{1}$$

where T is the absolute temperature, κ is the heat conductivity and J_q - is the heat flow, with the following relation between the flow of heat and the forces [2]

$$J_q = -T^{-2} L_{qq} \frac{\partial T}{\partial x}, \tag{2}$$

where L_{qq} is constant and called a phenomenological coefficient, leads to the identification

$$L_{qq} = \kappa T^2. \tag{3}$$

In Ref. 2 the so-called near-equilibrium linear regime is defined. Within this regime L_{qq} [Eq. (3)] may be treated as a constant. "Since $T(x)$ is a function of position, such an assumption is strictly not valid. It is valid only in the approximation that the change in T from one end of the system to another is small, when compared to the average T, *i.e.*, if the average temperature is T_{avg}, then $|T(x) - T_{avg}| \ll 1$ for all x . Hence, we may approximate $T^2 \approx T_{avg}^2$ and use κT_{avg}^2 in place of κT^2" (Ref. 2, p. 359). The author of Ref. 3 has contrasted the linear approximation and the rigorous solution of the problem of minimum entropy production during heat conduction in the rod, proving that the exponential dependence of temperature on distance minimized the entropy production, although it concerned a non-stationary state [4].

It will be shown that in the setup described in Ref. 1 and used in this work the near-equilibrium linear regime is achieved, the linear approximation is applicable and the contrast between the rigorous (exponential) and linear approximation solutions is smaller than the accuracy of the experiment. Our data are shown in Fig. 2. The local gradients become increasingly uniform with time an the system tends to a final steady (time-independent) state (Table I, Fig. 3), characterized by a constant heat flow and a simple position-independent temperature gradient

$$\frac{\partial T(x)}{\partial x} = \frac{T_c - T_h}{L}, \tag{4}$$

where $L = x_c - x_h$ and $x_h(c)$ denotes the coordinate of the hot (cold) end of the rod and $T_{h(c)} = T(x_{h(c)})$. This linear temperature distribution satisfies the Fourier law as follows:

$$T_f(x) = T_h - (T_h - T_c)\frac{x}{L}. \tag{5}$$

However, as noted by Peter Palffy-Muhoray [3], in general it is the exponential temperature distribution along the rod which minimizes entropy production:

$$T_M(x) = T_h \exp\left(-\frac{x}{L}\ln\frac{T_h}{T_c}\right). \tag{6}$$

[See Appendix A for a detailed derivation of Eq. (6)]. Now, we shall seek for a linear approximation of Eq. (6). To this aim we re-write it in the form

$$T_M(x) = T_h \exp\left(\frac{x}{L}\ln\frac{T_c}{T_h}\right). \tag{7}$$

We expand the functions ln and exp in Eq. (7) into a power series and drop all terms but the linear ones. Since $0 < T_c/T_h < 1$, we obtain

$$\ln\frac{T_c}{T_h} \approx \frac{T_c - T_h}{T_h} \tag{8}$$

On applying the approximation (8) the function exp in Eq. (7) expanded in a power series with only linear terms left, takes the form

$$\exp\left(\frac{x(T_c - T_h)}{LT_h}\right) \approx 1 + \frac{x(T_c - T_h)}{LT_h}. \tag{9}$$

Introducing (9) into (7) one obtains the form equivalent to Eq. (5):

$$T_M(x) = T_h \exp\left(\frac{x}{L}\ln\frac{T_c}{T_h}\right) \approx T_h\left(1 + \frac{x(T_c - T_h)}{LT_h}\right)$$
$$= T_h - (T_h - T_c)\frac{x}{L} = T_f(x).$$

This shows that the linear temperature distribution [given in Eq. (5)] and following the Fourier's law represents a linear approximation to the rigorous distribution [given in Eq. (6)], which minimizes entropy production [3].

Under the externally maintained temperature gradient $\partial T(x)/\partial x$ the system cannot relax to equilibrium. Immediately after placing the rod in contact with two thermal reservoirs heat starts to flow and consequently an inhomogeneous, time-dependent temperature distribution appears in the rod. The aim of the measurements undertaken was to find the character of the time-independent temperature distribution observed at $t > 65$ min. To the experimental data (Table I, Fig. 3) both the exponential [solid line, *cf.* Eq. (6)] and linear [dashed line, *cf.* Eq. (5)] fits are applied. The maximum discrepancies between the solid and the dashed lines are found to be less than 1 K. Hence, the maximum discrepancy is comparable to the experimental error. An improved accuracy of the experiment could reveal the differences between the linear and exponential $T\ (x)$ profiles. However, it is easy to show that after appropriately diminishing the difference $T_h - T_c$ the discrepancies between both profiles become again negligible. This shows that the theorem of minimum entropy production may serve as a useful approximation to the problem in question.

4. Conclusion

The steady (time-independent) state observed in this work (and in Ref. 1) with temperature varying very nearly linearly with distance along a rod, can be practically undistinguishable, within the limits of experimental error, from the exponential dependence of temperature on distance representing the state of minimum entropy production. This situation strongly resembles that discussed by Klein [5].

The linear dependence of temperature on distance, which is in agreement with the well-established Fourier's law of heat conduction, in fact indicates that the Prigogine's theorem of minimum entropy production, at least in the case of thermal conduction with temperature independent thermal conductivity, represents a useful approximation of satisfactory accuracy if the conditions for near-equilibrium linear regime are fulfilled. This approximation improves on approaching the equilibrium, as noted long ago by Kondepudi [6].

Appendix A

In this Appendix we shall derive the differential equation for $T(x)$, stemming from the condition of minimum entropy production, namely

$$\left[\frac{\partial T(x)}{\partial x}\right]^2 = T(x)\frac{\partial^2 T(x)}{\partial x^2} \tag{A.1}$$

(see Ref. 3, Eq. (6) therein) and solve it arriving at the exponential solution given above [our Eq. (6)]. We start with the expression for the total entropy production P in the rod given in our previous work (Ref. 1, Eq. (7) therein):

$$P = \int_{x_h}^{x_c} \frac{\kappa}{[T(x)]^2}\left(\frac{\partial T(x)}{\partial x}\right)^2 dx. \tag{A.2}$$

In the following we apply the following short-hand notation;

$$\frac{\partial T(x)}{\partial x} = T'; \quad \frac{\partial^2 T(x)}{\partial x^2} = T'';$$

$$(T(x))^{-2}\left[\frac{\partial T(x)}{\partial x}\right]^2 = T^{-2}(T')^2 = F(T, T'.)$$

The question is: when P is an extremum? This is equivalent to say that among all curves $T = T(x)$ going through the points (x_c, T_c) and (x_h, T_h) one seeks such for which

$$\frac{P}{\kappa} = \int_{x_h}^{x_c} F(T, T')dx \tag{A.3}$$

is an extremum (we assume κ = const.). This condition can be re-written as δP = 0, where δP is the first variation of P [8], or

$$\int_{x_h}^{x_c}\left(F_T - \frac{d}{dx}F_{T'}\right)\delta T dx = 0, \tag{A.4}$$

where $F_T = \partial F(T, T')/\partial T$ etc.

The expression in (A.4) is the variation of the functional F. It is zero for an arbitrary δT if the expression in the brackets is zero:

$$F_T - \frac{d}{dx}F_{T'} = 0. \tag{A.5}$$

Equation (A.5) represents the Euler equation of our problem. More explicitly, Eq. (A.5) can be written as

$$F_T - F_{TT'}T' - F_{T'T'}T'' = 0 \tag{A.6}$$

Note that since F does not depend explicitly on x, this equation can be solved by quadratures. Let us calculate in some detail the terms in Eq. (A.6). We obtain

$$F_T = \frac{\partial F(T, T')}{\partial T} = \frac{\partial}{\partial T}[T^{-2}(T')^2] = -2T^{-3}(T')^2;$$

$$F_{TT'} = \frac{\partial}{\partial T'}F_T = -4T^{-3}T'; F_{T'T'} = \frac{\partial}{\partial T'}F'_T = 2T_{-2}.$$

Substitution to Eq. (A.6) leads to:

$$-2T^{-3}(T')^2 - [-4T^{-3}T']T' - 2T^{-2}T'' = 0. \tag{A.7}$$

Finally,

$$T^{-2}[T^{-1}(T')^2 - T''] = 0. \tag{A.8}$$

For finite T, the first factor in Eq. (A.8) is nonzero, and we arrive at

$$T^{-1}(T')^2 - T'' = 0, \tag{A.9}$$

which is the same as Eq. (A.1). It remains to solve Eq. (A.1). The standard method of solving differential equations of the form $T''=f(T,T')$ with no x in the argument of f is to substitute $T'=z$ and $T''=z(dz/dT)$. Doing this in Eq. (A.9) leads to a separation of the variables: $dz/z=dT/T$. The integration of both sides of the latter equation leads to $\ln z=\ln T+const$. Coming back to old variables we obtain $T(x)=A\exp(bx)$, which is the equation of the curve, corresponding to the extremum of P. Since it has to go through the points $T_c=T(x_c)=T(0)$ and $Th=T(x_h)=T(L)$, one finally obtains $A=T_h$ and $b=-L_{-1}\ln(Th/Tc)$, which ends the derivation of our Eq. (6).

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