Complex functions as lumps of energy

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We present an application of the basic mathematical concept of complex functions as topological solitons, a most interesting area of research in physics. Such application of complex theory is virtually unknow outside the community of soliton researchers.

Keywords: Soliton; complex function; topology.

Presentamos una aplicación del concepto matemático de funciones complejas como solitones topológicos, una interesante área de investigación en física. Dicha aplicación de la teoría compleja es prácticamente desconocida fuera del círculo de investigación solitónica.

Descriptores: Solitón; función compleja; topología.

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1. Introduction

The complex variable $z = x + iy$, where $x, y \in \Re$ and $i = \sqrt{-1}$, is one of the most familiar and useful concepts in mathematics, with a very large number of well-documented applications in science.

Over the past few years some interesting nonlinear models in physics have received a lot of attention, models bearing the so-called solitons or energy ”lumps”. Some of these models exemplify yet another important application of complex functions, with functions as simple as $f(z) = z$ describing a soliton configuration. Unfortunately, despite the vast literature dealing with complex analysis plus applications, one finds no mention of the starring role of z as a soliton. Reference to such an extraordinary role is found only in highly specialized research books and journals, hence the existence of z as a soliton field is practically unknown outside the group of specialists in the area.

Using the nonlinear sigma $O(3)$ (or CP^1) model in two spatial dimensions, the present work illustrates the context in which z stands for a lump of energy. This fact is most remarkable and, given the growing importance of solitons in physics, we believe that more physicists should know about it. They will find this fresh utility of complex variables quite appealing.

2. Complex theory

Complex theory is a very important branch of mathematics. As a brush-up we just recall that many integrals given in real form are easily evaluated by relating them to complex integrals and using the powerful method of contour integration based on Cauchy's theorem. In fact, the basis of transform calculus is the integration of functions of a complex variable. And intersections between lines and circles, parallel or orthogonal lines, tangents, and the like usually become quite simple when expressed in complex form.

Familiarity with the complex numbers starts early, when at high school the basics of z are taught. Then in college algebra/calculus one learns some more about complex variables, with immediate applications to problems in both physics and engineering like electric circuits and mechanical vibrating systems. Later on complex holomorphic (analytic) functions are introduced, and then applied to a variety of problems: heat flow, fluid dynamics, electrostatics and magnetostatics, to name but few.

The concept of analyticity is extremely important. Many physical quantities are represented by functions $f(x, y)$, $g(x, y)$ connected by the relations $\partial_x f = \partial_y g$, $\partial_y f = -\partial_x g$, where $\partial_x f = \partial f/\partial x$, *etc*. It turns out that f and g may be considered as the real and imaginary parts of a holomorphic function h of the complex variable z:

$$h(z) = f + ig. \tag{1}$$

The equations linking f and g are the Cauchy-Riemann conditions for $h(z)$ being holomorphic, and can be written compactly as

$$\partial_x h = -i\partial_y h. \tag{2}$$

When h is a function of $\bar{z} = x - iy$, the complex conjugate of z, the condition (2) reads $\partial_x h = i\partial_y h$, and $h(\bar{z})$ is said to be anti-holomorphic [1].

We hereby show how functions of the type (1) describe solitons, giving yet another fundamental, if little known, application of analytic complex functions.

3. Solitons

Nonlinear science has developed strongly over the past 40 years, touching upon every discipline in both the natural and social sciences. Nonlinear systems appear in mathematics, physics, chemistry, biology, astronomy, metereology, engineering, economics and many more [2, 3].

Within the nonlinear phenomena we find the concept of 'soliton'. It has got some working definitions, all amounting to the following picture: a travelling wave of semi-permanent lump-like form. A soliton is a non-singular solution of a nonlinear field equation whose energy density has the form of a lump localised in space. Although solitons arise from nonlinear wave-like equations, they have properties that resemble those of a particle, hence the suffix *on* to covey a corpuscular picture to the *soli*tary wave. Solitons exist as density waves in spiral galaxies, as lumps in the ocean, in plasmas, molecular systems, protein dynamics, laser pulses propagating in solids, liquid crystals, elementary particles, nuclear physics…

According to whether the solitonic field equations can be solved or not, solitons are said to be integrable or nonintegrable. Given the limitations to analitycally handle nonlinear equations, it is not surprising that integrable solitons are generally found only in one dimension. The dynamics of integrable solitons is quite restricted; they usually move undistorted in shape and, in the event of a collision, they scatter off undergoing merely a phase shift.

In higher dimensions the dynamics of solitons is far richer, but now we are in the realm of nonintegrable models. In this case analytical solutions are practically restricted to static configurations and Lorentz transfomations thereof. (The time evolution being studied via numerical simulations and other approximation techniques.) A trait of nonintegrable solitons is that they carry a conserved quantity of topological nature, the topological charge –hence the designation *topological solitons*. Entities of this kind exhibit interesting stability and scattering processes, including soliton annihilation which can occur when lumps with opposite topological charges (one positive, one negative) collide. For areas like nuclear/particle physics such dynamics is of great relevance.

Using the simplest model available, below we illustrate the emergence of topological solitons and their representation as complex functions.

4. The planar $O(3)$ model

Models in two dimensions have a wide range of applications. In physics they are used in topics that include Heisenberg ferromagnets, the quantum Hall effect, superconductivity, nematic crystals, topological fluids, vortices and solitons. Some of these models also appear as low dimensional analogues of forefront non-abelian gauge field particle theories in higher dimensions, an example being the Skyrme model of nuclear physics [4, 5].

One of the simplest such systems is the $O(3)$ or CP^1 sigma model in (2+1) dimensions (2 space, 1 time). It involves three real scalar fields ϕ_j (j=1,2,3) functions of the space-time coordinates (t, x, y) [6, 7]. The model is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{4}\sum_{j=1}^{3}[(\partial_t\phi_j)^2 - (\partial_x\phi_j)^2 - (\partial_y\phi_j)^2], \tag{3}$$

where the fields, compactly written as the vector in field space $\vec{\phi} \equiv (\phi_1, \phi_2, \phi_3)$, are constraint to lie on the unit sphere

$$S_2^{(\phi)} = \{\vec{\phi} : \vec{\phi}^2 = 1\}. \tag{4}$$

The Euler-Lagrange field equation derived from (3)-(4) has no known analytical solutions except for the static case, which equation reads

$$\nabla^2\vec{\phi} - (\vec{\phi}.\nabla^2\vec{\phi})\vec{\phi} = \vec{0} \quad [\nabla^2 \equiv \partial_x^2 + \partial_y^2]. \tag{5}$$

The CP^1 solitons are non-singular solutions of (5). Without the constraint (4) the said equation would reduce to $\nabla^2\vec{\phi} = \vec{0}$, whose only non-singular solutions are constants. The condition (4) leads to the second term in (5), equation which does yield non-trivial non-singular solutions as we will later see.

Solitons must also be finite-energy configurations. From (3) we readily get the static energy

$$E = \int \frac{1}{4}\sum_{j=1}^{3}[(\partial_x\phi_j)^2 + (\partial_y\phi_j)^2]\, d^2x = \int \frac{1}{4}\sum_{j=1}^{3}(\nabla\phi_j)(\nabla\phi_j)\, d^2x [\nabla \equiv (\partial_x, \partial_y)]$$

$$= \frac{1}{4}\int (\nabla\vec{\phi}).(\nabla\vec{\phi})\, r dr d\theta \quad \text{(in polar coordinates } r, \theta). \tag{6}$$

We ensure the finiteness of E by taking the boundary condition

$$\lim_{r\to\infty} \vec{\phi}(r,\theta) \to \vec{\phi}_0 \quad \text{(a constant unit vector independent of } \theta), \tag{7}$$

since the integrand in (6) will thus tend to zero at spatial infinity:

$$\lim_{r\to\infty} r|\nabla\vec{\phi}| = \lim_{r\to\infty} r\sqrt{(\partial_r\vec{\phi})^2 + (\frac{1}{r}\partial_\theta\vec{\phi})^2} \to 0. \quad (8)$$

4.1. The complex plane

We are thus considering the model in the $x - y$ plane with a point at infinity, *i.e.*, the extended complex plane which is topologically equivalent to the two-sphere $S_2^{(x)}$. The finite energy configurations are therefore fields $\vec{\phi}$ defined on $\Re_2 \cup \{\infty\} \cong S_2^{(x)}$ and taking values on $S_2^{(\phi)}$. In other words, our finite-energy fields are harmonic maps of the form $S_2^{(x)} \to S_2^{(\phi)}$ [8].

We may imagine the coordinate space $S_2^{(x)}$ as made of *rubber* and the field space $S_2^{(\phi)}$ as made of *marble*; the map $\vec{\phi}$ constrains the rubber to lie on the marble. Then with each point (x, y) in the rubber we have a quantity

$$\vec{\tau} = \nabla^2\vec{\phi} - (\vec{\phi}.\nabla^2\vec{\phi})\vec{\phi}$$

at the point $\vec{\phi}$ in the marble representing the tension in the rubber at that point. Thus the map is harmonic if and only if $\vec{\phi}$ constrains the rubber to lie on the marble in a position of elastic equilibrium, $\vec{\tau} = \vec{0}$, which is just Eq. (5). These are our finite-energy configurations, of which the soliton solutions are a subset.

4.2. Topological charge

In general, as the coordinate $z = (x, y)$ ranges over the sphere $S_2^{(x)}$ once, the coordinate $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ ranges over $S_2^{(\phi)}$ N times. This winding number is called the topological charge in soliton parlance, and classifies the maps $S_2^{(x)} \to S_2^{(\phi)}$ into sectors (homotopy classes); maps within one sector are equivalent in that they can be obtained from each other by continuos transformations.

An expression for the topological charge is obtained by expanding the coordinates ϕ_j of the area element of $S_2^{(\phi)}$ in terms of coordinates (x, y) in $S_2^{(x)}$, and integrating off. In plainer language, from the college formula that computes the flux of a vector $\vec{A}$ through a region D of a surface S :

$$\int_D \vec{A}.\hat{\mathbf{n}}dS \quad [\hat{\mathbf{n}} \text{ a normal unit vector}], \quad (9)$$

the topological charge N follows by calculating the flux of $\vec{A} = (N)/(4\pi a^2)\vec{\phi}$ through the sphere $S_2^{(x)}$ of radius $a = 1$:

$$\int_D \vec{A}.\hat{\mathbf{n}}dS \to \int_{S_2^{(x)}} \frac{N}{4\pi a^2}\vec{\phi}.\vec{\phi}\,dS = N.$$

Notably, a field with topological charge N describes precisely a system of N solitons.

In order to actually find charge-N finite-energy *solutions*, it is convenient to express the model in terms of one independent complex field, W, related to $\vec{\phi}$ via the stereographic projection

$$W = \frac{\phi_1 + i\phi_2}{1 - \phi_3}. \quad (10)$$

In this formulation, the topological charge is given by

$$N = \frac{1}{\pi}\int_{S_2^{(x)}} \frac{|\partial_z W|^2 - |\partial_{\bar{z}} W|^2}{(1 + |W|^2)^2}\, d^2x, \quad N \in \mathcal{Z}, \quad (11)$$

connected with the energy (6) through

$$E \geq 2\pi|N|. \quad (12)$$

5. Lumps

The solitonic solutions we seek correspond to the equality in (12) [9–11]. That is, in a given topological sector N the static solitons of the planar CP^1 model are the configurations whose energy E is an absolute minimum. Combining (11) with $E = 2\pi|N|$ we find that solutions carrying positive or negative topological charge satisfy, respectively,

$$\partial_{\bar{z}} W = 0 \to \partial_x W = -i\partial_y W, \quad (13)$$

$$\partial_z W = 0 \to \partial_x W = i\partial_y W. \quad (14)$$

But recalling Eq. (2) we immediately recognise the above equations as the Cauchy-Riemann conditions for W being a holomorphic function of z or $\bar{z}$. This is most remarkable.

For instance, a single-soliton solution ($N = 1$) may be described by

$$W = z \qquad [\text{note that this satisfies equation (13)}]; \quad (15)$$

its energy density distribution is given by

$$\mathcal{E} = \frac{2}{1 + |z|^2}. \quad (16)$$

Plots of (16) reveal a lump of energy localised in space, as shown in Fig. 1. The same energy corresponds to $W = \bar{z}$, which has $N < 0$ and sometimes is referred to as an anti-soliton.

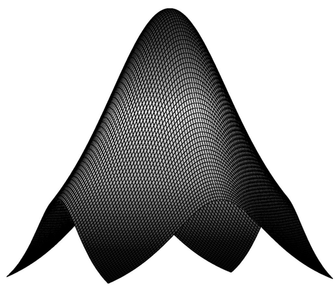

FIGURE 1. The energy distribution corresponding to the soliton W=z.

A more general $N = 1$ solution is given by a rational function $W = \lambda(z-a)/(z-b)$, which we should note is non-singular: $W(z = b) = \infty$ corresponds to $\phi_3 = 1$, the north pole of $S_2^{(\phi)}$ according to (10). A prototype solution for arbitrary $N > 0$ is $\lambda(z-a)^N$.

The dynamics of these structures is studied by numerically evolving the full time-dependent equation derived from (3), with the fields $W(z)$ as initial conditions [12, 13].

Sigma CP^1-type models have several applications, noteworthy among them being the Skyrme model in (3+1) dimensions where the topological solitons stand for ground states of light nuclei, with the topological charge representing the baryon number.

The role of complex functions as topological solitons deserves widespread attention and should not be missing from the modern literature dealing with complex theory and its applications.

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