Gauge-invariant formulation of the electromagnetic interaction in Hamiltonian mechanics

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Making use of the fact that for an arbitrary autonomous mechanical system any constant of motion can be used as Hamiltonian, the equations of motion of a charged particle in an electromagnetic field are written in Hamiltonian form without introducing potentials for the electromagnetic field. It is shown that the Hamiltonian and the Poisson bracket obtained here coincide with those appearing in the standard Hamiltonian formulation.

Keywords: Hamilton equations; electromagnetic interaction.

Haciendo uso del hecho de que para un sistema mecánico autónomo arbitrario cualquier constante de movimiento puede usarse como hamiltoniana, las ecuaciones de movimiento de una partícula cargada en un campo electromagnético se escriben en forma hamiltoniana sin introducir potenciales para el campo electromagnético. Se muestra que la hamiltoniana y el paréntesis de Poisson obtenidos aquí coinciden con los que aparecen en la formulación hamiltoniana estándar.

Descriptores: Ecuaciones de Hamilton; interacción electromagnética.

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1. Introduction

As it is well-known, the equations of motion of a charged particle in a given electromagnetic field can be formulated making use of the Lagrangian formalism, introducing a velocity-dependent potential that contains the potentials of the field, and also using the Hamiltonian formalism (see, *e.g.*, Ref. 1). The fact that the electromagnetic potentials appear in the Lagrangian, and not the electromagnetic fields themselves, is a drawback because the potentials are not uniquely determined by the electromagnetic fields; if the potentials possess some symmetry, the electromagnetic fields also have that symmetry, but if the electromagnetic fields possess some symmetry, the potentials may not share it. Hence, in the standard formulation, the Lagrangian or the Hamiltonian might not exhibit all the symmetries of the system. For instance, the uniform magnetic field $\mathbf{B} = (0, 0, B_3)$, with $B_3 = \text{const.}$, is invariant under all translations in space, but the vector potential of a nonvanishing magnetic field cannot be invariant under all translations.

As shown in Ref. 2, the equations of motion of any autonomous mechanical system can be written in Hamiltonian form (*i.e.*, $df/dt = \{f, H\}$, for any differentiable function, f, defined on the phase space) with the Hamiltonian being *any* constant of motion (not necessarily the total energy) provided that the Poisson bracket is suitably defined. Furthermore, if the number of degrees of freedom of the system is greater than 1, for each choice of the Hamiltonian function, there are infinitely many suitable Poisson brackets (see also Refs. 3 and 4).

Using the fact that, in the framework of Newtonian mechanics, the kinetic energy of a charged particle is not modified by a magnetostatic field, we show that the equations of motion of a charged particle in a magnetostatic field can be written in Hamiltonian form, with the usual Hamiltonian function of a free particle and the magnetic field embodied in the Poisson bracket, without having to introduce auxiliary quantities such as the vector potential. We also show that the Hamiltonian and the Poisson bracket employed here coincide with those appearing in the standard Hamiltonian formulation for a particle in a magnetic field. The general case of a charged particle in an arbitrary electromagnetic field is dealt with in a similar way, considering the corresponding relativistic equations of motion.

In Sec. 2 we give a summary of the Hamiltonian formalism. In Sec. 3 we consider the equations of motion, according to the Newtonian mechanics, of a charged particle in a magnetostatic field and in Sec. 4 we consider the equations of motion of a charged particle in an arbitrary electromagnetic field in the framework of relativistic mechanics.

2. Hamilton's equations

Hamilton's equations are usually written in the form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{1}$$

($i = 1, 2, \ldots, n$), where H is the Hamiltonian function, $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a set of canonical coordinates on the phase space and n is the number of degrees of freedom. Hamilton's equations (1) are equivalent to the formula

$$\frac{df}{dt} = \{f, H\}, \tag{2}$$

for any differentiable function, f, defined on the phase space that does not depend explicitly on the time, where $\{\ ,\ \}$ denotes the Poisson bracket, defined by

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \qquad (3)$$

(Here, and in what follows, there is summation over repeated indices.) In the case of an autonomous system, we can assume that H does not depend explicitly on the time and Eqs. (1) imply that H is a constant of motion.

The form of Eqs. (1) and (3) is invariant under the replacement of q^i, p_i by new coordinates q'^i, p'_i, if the latter are obtained from q^i, p_i by means of a canonical transformation; however, Eq. (2) is applicable in any coordinate system. In terms of an arbitrary coordinate system in the phase space, $(x^1, x^2, \ldots, x^{2n})$, the Poisson bracket (3) is expressed as

$$\begin{aligned}\{f, g\} &= \left(\frac{\partial x^\mu}{\partial q^i}\frac{\partial x^\nu}{\partial p_i} - \frac{\partial x^\mu}{\partial p_i}\frac{\partial x^\nu}{\partial q^i}\right)\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} \\ &= \sigma^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} \qquad (4)\end{aligned}$$

($\mu, \nu, \ldots = 1, 2, \ldots, 2n$), where we have introduced the definition

$$\sigma^{\mu\nu} \equiv \{x^\mu, x^\nu\}. \qquad (5)$$

The Poisson bracket (3) satisfies the Jacobi identity,

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0;$$

therefore,

$$\{\{x^\mu, x^\nu\}, x^\lambda\} + \{\{x^\nu, x^\lambda\}, x^\mu\} + \{\{x^\lambda, x^\mu\}, x^\nu\} = 0$$

which, according to Eqs. (4) and (5) amounts to

$$\sigma^{\rho\lambda}\frac{\partial \sigma^{\mu\nu}}{\partial x^\rho} + \sigma^{\rho\mu}\frac{\partial \sigma^{\nu\lambda}}{\partial x^\rho} + \sigma^{\rho\nu}\frac{\partial \sigma^{\lambda\mu}}{\partial x^\rho} = 0. \qquad (6)$$

Conversely, if a set of functions $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ satisfies Eqs. (6), then the bracket

$$\{f, g\} = \sigma^{\mu\nu}\left(\frac{\partial f}{\partial x^\mu}\right)\left(\frac{\partial g}{\partial x^\nu}\right)$$

satisfies the Jacobi identity. From Eqs. (2) and (4) it follows that the Hamilton equations in an arbitrary coordinate system are

$$\frac{dx^\mu}{dt} = \sigma^{\mu\nu}\frac{\partial H}{\partial x^\nu}. \qquad (7)$$

The equations of motion of a given autonomous system expressed in terms of an arbitrary coordinate system can be written in the Hamiltonian form (7) in infinitely many ways; one just has to choose some constant of motion and use it as Hamiltonian in Eqs. (7), which, together with Eqs. (6), determine functions $\sigma^{\mu\nu}$. If $n > 1$, there are infinitely many ways of choosing the the functions $\sigma^{\mu\nu}$ satisfying Eqs. (6), (7), and the conditions $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ [2,4].

3. Particle in a magnetostatic field

The equations of motion for a charged particle of mass m in a magnetostatic field with induction $\mathbf{B}$, and possibly in a force field derivable from a potential, are given by

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B} - \nabla U, \qquad (8)$$

where e is the electric charge of the particle, $\mathbf{v}$ denotes its velocity, c is the speed of light in vacuum and U is some potential that depends only on the position of the particle. The vector $\mathbf{p}$ is the usual linear momentum

$$\mathbf{p} = m\mathbf{v}. \qquad (9)$$

From Eqs. (8) it follows that $(\mathbf{p}^2/2m) + U$ is a constant of motion. Hence, we can choose

$$H = \frac{\mathbf{p}^2}{2m} + U. \qquad (10)$$

Then, letting $(x^1, \ldots, x^6) = (q^1, q^2, q^3, p_1, p_2, p_3)$, where (q^1, q^2, q^3) are the Cartesian coordinates of the particle and (p_1, p_2, p_3) are the Cartesian components of $\mathbf{p}$, and making use of Eqs. (8) and (9) one finds that Eqs. (7) can be written explicitly in matrix form as

$$\begin{bmatrix} \dfrac{p_1}{m} \\ \dfrac{p_2}{m} \\ \dfrac{p_3}{m} \\ \left(\dfrac{e}{mc}\right)(B_3 p_2 - B_2 p_3) - \dfrac{\partial U}{\partial q^1} \\ \left(\dfrac{e}{mc}\right)(B_1 p_3 - B_3 p_1) - \dfrac{\partial U}{\partial q^2} \\ \left(\dfrac{e}{mc}\right)(B_2 p_1 - B_1 p_2) - \dfrac{\partial U}{\partial q^3} \end{bmatrix} = (\sigma^{\mu\nu}) \begin{bmatrix} \dfrac{\partial U}{\partial q^1} \\ \dfrac{\partial U}{\partial q^2} \\ \dfrac{\partial U}{\partial q^3} \\ \dfrac{p_1}{m} \\ \dfrac{p_2}{m} \\ \dfrac{p_3}{m} \end{bmatrix}. \qquad (11)$$

By inspection, one finds that an antisymmetric matrix $(\sigma^{\mu\nu})$ that satisfies this last equation is

$$(\sigma^{\mu\nu}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \dfrac{eB_3}{c} & -\dfrac{eB_2}{c} \\ 0 & -1 & 0 & -\dfrac{eB_3}{c} & 0 & \dfrac{eB_1}{c} \\ 0 & 0 & -1 & \dfrac{eB_2}{c} & -\dfrac{eB_1}{c} & 0 \end{bmatrix} \qquad (12)$$

i.e.,

$$\{q^i, q^j\} = 0, \ \{q^i, p_j\} = \delta^i_j, \ \{p_i, p_j\} = \frac{e}{c}\varepsilon_{ijk}B_k, \qquad (13)$$

and it can be readily verified that this bracket satisfies the Jacobi identity [substituting Eqs. (12) into Eqs. (6) or directly using Eqs. (13)] as a consequence of the fact that the divergence of $\mathbf{B}$ vanishes.

For an arbitrary mechanical system, any differentiable function, $G(x^\mu)$, is the infinitesimal generator of a local one-parameter group of canonical transformations whose orbits are the solutions of the system of differential equations

$$\frac{dx^\mu}{ds} = \sigma^{\mu\nu} \frac{\partial G}{\partial x^\nu} \tag{14}$$

[*cf.* Eq. (7)] or, equivalently,

$$\frac{dx^\mu}{ds} = \{x^\mu, G\}. \tag{15}$$

The function $G(x^\mu)$ is a constant of motion if and only if the Hamiltonian is invariant under the canonical transformations defined by G [*i.e.*, if and only if H is constant along the solutions of (14)]. Conversely, given a local one-parameter group of canonical transformations, $x^\mu = x^\mu(s)$, Eq. (14) defines up to an additive constant its infinitesimal generator G and, if the Hamiltonian is invariant under these transformations, G is a constant of motion. (The integrability conditions of Eqs. (14) for G are the conditions for the transformations $x^\mu = x^\mu(s)$ to be canonical.)

Owing to the presence of the components of $\mathbf{B}$ in the functions $\sigma^{\mu\nu}$ [or, equivalently, in the Poisson brackets (13)], it turns out that a translation or rotation is a canonical transformation *if and only if the magnetic field is invariant under that transformation.*

Taking, for simplicity, $U = 0$, *any* translation or rotation leaves the Hamiltonian $H = \mathbf{p}^2/2m$ invariant; however, not all of them lead to the existence of constants of motion. Only the translations or rotations that leave the magnetic field invariant correspond to canonical transformations that leave the Hamiltonian invariant, thus implying the existence of constants of motion. (Note that these conditions involve only the magnetic field itself, without making reference to the vector potential which is defined up to gauge transformations.)

For example, if the magnetic field is invariant under translations along the q^1-axis, then

$$\frac{\partial B_i}{\partial q^1} = 0,$$

for $i = 1, 2, 3$, and from $\nabla \cdot \mathbf{B} = 0$ it follows that

$$\frac{\partial B_2}{\partial q^2} + \frac{\partial B_3}{\partial q^3} = 0,$$

which means that $B_3 dq^2 - B_2 dq^3$ is an exact differential, at least locally; hence, there exists a function $\Phi(q^2, q^3)$ such that

$$B_3 dq^2 - B_2 dq^3 = -d\Phi. \tag{16}$$

Then, making use of Eqs. (13) and (16), one finds that

$$\{q^i, p_1 + \frac{e}{c}\Phi\} = \delta^i_1, \qquad \{p_i, p_1 + \frac{e}{c}\Phi\} = 0 \tag{17}$$

and by comparing with Eq. (15) one concludes that $p_1 + (e/c)\Phi$ is a generating function of translations along the q^1-axis, which is a constant of motion as a consequence of the invariance of H under these translations. It should be remarked that Φ is defined up to an additive constant only.

According to Darboux's theorem, Eqs. (6) imply the local existence of canonical coordinates (*i.e.*, coordinates Q^i, P_i, such that $\{Q^i, Q^j\} = 0 = \{P_i, P_j\}$, $\{Q^i, P_j\} = \delta^i_j$). If the vector field $\mathbf{A}$ is a vector potential for $\mathbf{B}$, that is, $\mathbf{B} = \nabla \times \mathbf{A}$, then from Eqs. (13) one finds that

$$\{q^i, p_j + \frac{e}{c}A_j\} = \delta^i_j, \quad \{p_i + \frac{e}{c}A_i, p_j + \frac{e}{c}A_j\} = 0, \tag{18}$$

which means that $Q^i \equiv q^i$, $P_i \equiv p_i + \frac{e}{c}A_i$ are canonical coordinates. If the Hamiltonian (10) is written in terms of these coordinates one obtains the standard expression

$$H = \frac{1}{2m}(\mathbf{P} - \frac{e}{c}\mathbf{A})^2 + U. \tag{19}$$

It should be remarked that from Eqs. (18) it follows that P_k is the infinitesimal generator of translations along the q^k-axis *if and only if* $\partial A_i/\partial q^k = 0$, for $i = 1, 2, 3$. In spite of this fact, it is customary in quantum mechanics to replace the canonical momenta P_k by the differential operators $-i\hbar\partial/\partial q^k$, which is the infinitesimal generator of translations along the q^k-axis in the coordinate representation.

Another example, that perhaps illustrates this point more clearly, is provided by the magnetic field

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3}, \tag{20}$$

which would be produced by a magnetic monopole placed at the origin (though this field does not satisfy the condition $\nabla \cdot \mathbf{B} = 0$ at the origin). The magnetic field (20) is invariant under rotations about any axis passing through the origin, but there is no spherically symmetric vector potential for this field.

The infinitesimal generator of a local one-parameter group of canonical transformations can be derived from Eq. (14). Denoting by $(\omega_{\mu\nu})$ the inverse of the matrix $(\sigma^{\mu\nu})$, from Eq. (14) we have

$$dG = \omega_{\mu\nu}\frac{dx^\nu}{ds}dx^\mu. \tag{21}$$

In the present case, from Eq. (12) one finds that

$$(\omega_{\mu\nu}) = \begin{bmatrix} 0 & \frac{eB_3}{c} & -\frac{eB_2}{c} & -1 & 0 & 0 \\ -\frac{eB_3}{c} & 0 & \frac{eB_1}{c} & 0 & -1 & 0 \\ \frac{eB_2}{c} & -\frac{eB_1}{c} & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

and for rotations about the q^i-axis,

$$\frac{dq^j}{ds} = \varepsilon_{jim} q^m$$

and

$$\frac{dp_j}{ds} = \varepsilon_{jim} p_m,$$

where s is the angle of the rotation. Thus, if L_i denotes the infinitesimal generator of rotations about the q^i-axis, from Eq. (21) we obtain

$$\begin{aligned} dL_i &= \left(\varepsilon_{jkl} \frac{eB_l}{c} \varepsilon_{kim} q^m - \varepsilon_{jim} p_m \right) dq^j + \varepsilon_{jim} q^m dp_j \\ &= d(\varepsilon_{ijk} q^j p_k) + \frac{e}{c}(B_i q^j dq^j - B_j q^j dq^i) \qquad (23) \end{aligned}$$

and substituting Eq. (20) it follows that

$$L_i = \varepsilon_{ijk} q^j p_k - \frac{eg}{c}\frac{q^i}{r}, \qquad (24)$$

which does not coincide with $\varepsilon_{ijk} q^j P_k$, the infinitesimal generator of rotations of the canonical variables. With the aid of Eqs. (13) and (20), one verifies that $\{L_i, L_j\} = \varepsilon_{ijk} L_k$.

4. Particle in an electromagnetic field

In order to express the equations of motion of a charged particle in an arbitrary electromagnetic field in a form similar to that presented in the foregoing section, it is convenient to consider them from the point of view of relativistic mechanics. Since the derivations are almost identical to those given in the previous case, we shall omit some details in what follows. In this section the lower case Greek indices take the values 0, 1, 2, 3.

Let q^μ be the Cartesian coordinates of a charged particle (with respect to some inertial frame) with rest mass m_0 and electric charge e. If τ denotes the particle proper time,

$$p^\mu = m_0 \frac{dq^\mu}{d\tau} \qquad (25)$$

is the usual four-momentum (see, *e.g.*, Ref. 5). The relativistic equations of motion for the particle in a given electromagnetic field are given by

$$\frac{dp^\mu}{d\tau} = \frac{e}{c} F^\mu{}_\nu \frac{dq^\nu}{d\tau} = \frac{e}{m_0 c} F^\mu{}_\nu p^\nu, \qquad (26)$$

where the $F_{\mu\nu}$ are the components of the electromagnetic field tensor. (The tensor indices are lowered or raised with the aid of $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ and its inverse in the usual way, *e.g.*, $p_\mu = \eta_{\mu\nu} p^\nu$, $p^\mu = \eta^{\mu\nu} p_\nu$.) From Eqs. (26) and the antisymmetry of $F_{\mu\nu}$ it follows that

$$H \equiv \frac{p_\mu p^\mu}{2m_0} \qquad (27)$$

is a constant of motion. It can be readily verified that Eqs. (25) and (26) can be written as

$$\frac{dq^\mu}{d\tau} = \{q^\mu, H\}, \quad \frac{dp^\mu}{d\tau} = \{p^\mu, H\}$$

if

$$\{q^\mu, q^\nu\} = 0, \quad \{q^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{p_\mu, p_\nu\} = \frac{e}{c} F_{\mu\nu} \qquad (28)$$

[*cf.* Eqs. (13)]. The Jacobi identity is satisfied by virtue of the Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial q^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial q^\mu} + \frac{\partial F_{\lambda\mu}}{\partial q^\nu} = 0.$$

Thus, a translation, rotation or boost is a canonical transformations if and only if it leaves the electromagnetic field invariant.

If A_μ denotes a four-potential for the electromagnetic field (*i.e.*, $F_{\mu\nu} = \partial A_\nu/\partial q^\mu - \partial A_\mu/\partial q^\nu$) then from Eqs. (28) one finds that $Q^\mu = q^\mu$ and $P_\mu = p_\mu + eA_\mu/c$ are canonical coordinates but, again, this *does not* mean that P_μ is the infinitesimal generator of translations along the q^μ-axis, unless $\partial A_\nu/\partial q^\mu = 0$ for $\nu = 0, 1, 2, 3$.

5. Concluding remarks

The canonical coordinates are distinguished by the fact that in terms of them the Hamilton equations (7) and the Poisson bracket (4) reduce to the simpler forms (1) and (3), respectively, but in the case of particles interacting with the electromagnetic field it is preferable to employ noncanonical coordinates, which may have clearer meaning, and without having to introduce gauge-dependent quantities. The Poisson bracket obtained here as the simplest solution of Eqs. (6) and (11) is precisely the one found in the standard Hamiltonian formulation [2].

As pointed out above, the usual canonical momentum P_i or P_μ need not be the infinitesimal generator of translations along the q^i- or q^μ-axis.

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