Electroweak radiative corrections to semileptonic τ decays

J. Erler
Instituto de Física, Universidad Nacional Autónoma de México,
04510 México D.F., México,
e-mail: erler@fisica.unam.mx

Recibido el 13 de octubre de 2003; aceptado el 24 de noviembre de 2003

I present an update on the electroweak radiative correction factor to semileptonic τ decays, including a next-to-leading order resummation of large logarithms. My result differs both qualitatively and quantitatively from the one recently obtained by Davier *et al.*. As two consequences, (i) the discrepancy between the predictions for the muon $g-2$ based on τ decay data and e^+e^- annihilation data increases, and (ii) the $g-2$ prediction based on τ decay data appears to be consistent (within about one standard deviation) with the experimental result from BNL.

Keywords: Electroweak radiative corrections; tau decay; perturbative QCD.

Presentamos una actualizacion del factor de correcion radiativo electrodebil al decaimiento semileptonico del τ, incluyendo una resumacion de sugundo orden de logaritmos grandes. Nuestro resultado difiere cualitativa y cuantitativamente del recientemente obtenido por Davier et al. Tenemos dos resultados,(i) la discrepancia entre la preciones para el g-2 del muon basado en los datos del decaimientos del τ y los datos de la aniquilacion e^+e^-, se incrementa y (ii) la prediccion de g-2 basada en los datos del decaimiento del τ parece ser consistente (dentro de una desviacion estandar) con el resultado experimental de BNL.

Descriptores: Correciones radiativas electrodebiles; decaimiento del τ; QCD perturbativa.

PACS: 11.10.Hi,12.15.Lk,13.35.Dx,13.40.Em.

The largest theoretical uncertainty in the Standard Model prediction of the anomalous magnetic moment of the muon, $a_\mu = (g_\mu - 2)/2$, arises from the hadronic two-loop vacuum polarization contribution, $\Delta a_\mu^{\rm had,(2)}$. This contribution is two orders of magnitude larger than the ultimate experimental error anticipated by the Muon $g-2$ Collaboration at BNL [2], so it needs to be controlled at the 1% level or better. While non-perturbative QCD effects prevent a first principles calculation, $\Delta a_\mu^{\rm had,(2)}$ can be rigorously obtained experimentally from a dispersion relation which relates it to an integral over e^+e^- annihilation cross sections. Using the conserved vector current (CVC) hypothesis one can obtain additional information by studying the invariant mass distribution of τ decay hadronic final states. This necessitates a careful assessment of CVC breaking effects, which was done in a recent article by Davier *et al.* [1]. In this note, I present an update of the short distance electroweak radiative corrections to τ decays, representing a particular CVC breaking effect. This update is motivated by two mistakes in one of the formulas of Ref. [1]. Numerically, the corresponding shifts are modest, but not negligible, and have the same sign.

The leading electroweak radiative corrections to τ decays are enhanced by a large logarithm [3,4],

$$S_{\rm EW} = 1 + \frac{3\alpha}{4\pi}(1+2\overline{Q})\ln\frac{M_Z^2}{m_\tau^2} = 1.01878 \qquad (1)$$

where $M_Z = 91.1876(2)$ GeV [5] is the Z boson mass, and $\alpha = \alpha(m_\tau) = 1/133.50(2)$ [6] is the QED coupling at the τ lepton mass, $m_\tau = 1776.99(3)$ MeV [7], evaluated in the $\overline{\rm MS}$ renormalization scheme. $\overline{Q}$ is the hypercharge of the weak doublet produced in the final state. Therefore, $\overline{Q} = 1/6$ for semileptonic decays, $\tau^- \to \nu_\tau \bar{u} d(s)$. Since $\overline{Q} = -1/2$ for leptons, there are no large logarithms for leptonic τ decays.

The remaining (not logarithmically enhanced) corrections at $\mathcal{O}(\alpha)$ have been obtained in Ref. [8] (final state fermion masses are neglected throughout). In the notation of Eq. (17) of Ref. [1] they are,

$$S_{\rm EW}^{\rm sub,had} = 1 + \frac{\alpha(m_\tau)}{\pi}\left(\frac{85}{24} - \frac{\pi^2}{2}\right), \qquad (2)$$

$$S_{\rm EW}^{\rm sub,lep} = 1 + \frac{\alpha(m_\tau)}{\pi}\left(\frac{25}{8} - \frac{\pi^2}{2}\right), \qquad (3)$$

for semileptonic and leptonic decays, respectively. In Ref. [1], however, $S_{\rm EW}^{\rm sub,had}$ was erroneously identified with the ratio,

$$\frac{S_{\rm EW}^{\rm sub,had}}{S_{\rm EW}^{\rm sub,lep}} = 1 + \frac{5}{12}\frac{\alpha(m_\tau)}{\pi} = 1.00099. \qquad (4)$$

This amounts to a double counting of the correction $S_{\rm EW}^{\rm sub,lep} - 1 = -0.00432$: the hadronic spectral functions are normalized relative to the leptonic branching ratio (see Eq. (10) of Ref. [1]) so that the ratio (4) must be included, but it is incorrect to perform an additional division by $S_{\rm EW}^{\rm sub,lep}$.

Since numerically,

$$\frac{\alpha_s}{\pi}\ln\frac{M_Z^2}{m_\tau^2} \sim \mathcal{O}(1), \qquad (5)$$

short distance QCD effects are of similar size as the $\mathcal{O}(\alpha)$ corrections discussed in the previous paragraph. They have been computed in Ref. [9] and modify Eq. (1),

$$S_{\rm EW} = 1 + \frac{3\alpha}{4\pi}\ln\frac{M_Z^2}{m_\tau^2}\left[(1+2\overline{Q}) - 2\overline{Q}\frac{\alpha_s}{\pi}\right]. \qquad (6)$$

The short distance QCD correction corresponding to the term proportional to the strong coupling constant, α_s, has been approximated [9] in order to obtain an analytic result. I have checked that this approximation reproduces the exact $\mathcal{O}(\alpha\alpha_s \ln M_Z^2)$ result within about 1%. Since two scales enter Eq. (6), it is clear that a next-to-leading order renormalization group analysis is in order.

Resummation of the leading order logarithms in Eq. (1) is done using the renormalization group equation (RGE) [10],

$$\left[\mu^2 \frac{\partial}{\partial\mu^2} + \beta_0^{(1)} \frac{\alpha^2}{\pi}\frac{\partial}{\partial\alpha} - \frac{\alpha}{\pi}\right] S(\mu_0,\mu) = 0, \qquad (7)$$

where $\beta_0^{(1)}$ is the lowest order QED β-function coefficient. This RGE is subject to the initial condition, $S(\mu_0,\mu_0) = 1$, and its solution is given by,

$$S(\mu_0,\mu) = \left[1 - \frac{\alpha(\mu_0)}{\pi}\beta_0^{(1)} \ln\frac{\mu^2}{\mu_0^2}\right]^{-\frac{1}{\beta_0^{(1)}}}. \qquad (8)$$

Applied to the case at hand, this is often written as,

$$S(m_\tau, M_Z) = \left[\frac{\alpha(m_b)}{\alpha(m_\tau)}\right]^{\frac{9}{19}} \left[\frac{\alpha(M_W)}{\alpha(m_b)}\right]^{\frac{9}{20}} \left[\frac{\alpha(M_Z)}{\alpha(M_W)}\right]^{\frac{36}{17}} = 1.01937, \qquad (9)$$

where the solution to the one-loop RGE of QED,

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha(\mu) = \beta_0^{(1)} \frac{\alpha^2(\mu)}{\pi}, \qquad (10)$$

has been employed. It should be stressed, that consistency with the RGE demands one-loop evolution of $\alpha(\mu)$ *within* each of the factors in Eq. (9). On the other hand, the values used *across* the various factors, may be related to each other either by one-loop evolution or including higher order running effects, since the difference is of higher order in the RGE (7). The increase of $S(m_\tau, M_Z)$ in Eq. (9) relative to Eq. (1) due to the summation of $\mathcal{O}(\alpha^n \ln^n M_Z^2)$ effects is about 3% of the non-resummed correction.

I will now extend the RGE analysis of the previous paragraph to properly sum up all logarithms[We neglect non-logarithmic and therefore non-enhanced terms of $\mathcal{O}(\alpha\alpha_s)$. This is in accordance with common practice where solutions of an n–loop RGE are supplemented by $(n-1)$–loop threshold (matching) terms of non-logarithmic nature.] of $\mathcal{O}(\alpha\alpha_s^n \ln^n M_Z^2)$. Eq. (7) is to be replaced by,

$$\left[\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} - \frac{\alpha}{\pi}\left(1 - \frac{\alpha_s}{4\pi}\right)\right] S(\mu_0,\mu) = \left[\mu^2 \frac{\partial}{\partial\mu^2} + \beta_0^{(1)}\frac{\alpha^2}{\pi}\frac{\partial}{\partial\alpha} - \beta_0^{(3)}\frac{\alpha_s^2}{\pi}\frac{\partial}{\partial\alpha_s} - \frac{\alpha}{\pi}\left(1 - \frac{\alpha_s}{4\pi}\right)\right] S = 0, \qquad (11)$$

where $\beta_0^{(3)}$ is the lowest order QCD β-function coefficient. With the definitions,

$$\eta_\tau = \frac{\alpha_s(m_\tau)}{4\pi}\left[1 + \frac{75}{76}\frac{\alpha_s(m_\tau)}{\alpha(m_\tau)}\right]^{-1},$$
$$\eta_b = \frac{\alpha_s(m_b)}{4\pi}\left[1 + \frac{69}{80}\frac{\alpha_s(m_b)}{\alpha(m_b)}\right]^{-1},$$
$$\eta_W = \frac{\alpha_s(M_W)}{4\pi}\left[1 + \frac{69}{17}\frac{\alpha_s(M_W)}{\alpha(M_W)}\right]^{-1}, \qquad (12)$$

Eq. (11) is solved by,

$$\begin{aligned} S(m_\tau, M_Z) &= \left[\frac{\alpha(m_b)}{\alpha(m_\tau)}\right]^{\frac{9}{19}(1-\eta_\tau)} \left[\frac{\alpha_s(m_b)}{\alpha_s(m_\tau)}\right]^{\frac{9}{19}\eta_\tau} \\ &\quad \left[\frac{\alpha(M_W)}{\alpha(m_b)}\right]^{\frac{9}{20}(1-\eta_b)} \left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)}\right]^{\frac{9}{20}\eta_b} \\ &\quad \left[\frac{\alpha(M_Z)}{\alpha(M_W)}\right]^{\frac{36}{17}(1-\eta_W)} \left[\frac{\alpha_s(M_Z)}{\alpha_s(M_W)}\right]^{\frac{36}{17}\eta_W} \\ &= 1.01907 \pm 0.00001, \end{aligned} \qquad (13)$$

where I used the solutions to the one-loop RGE of QCD,

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha_s(\mu) = -\beta_0^{(3)} \frac{\alpha_s^2(\mu)}{\pi}, \qquad (14)$$

and of QED[QCD corrections to Eq. (10) are suppressed by an additional factor α_s/π. Their inclusion gives rise to the summation of $\mathcal{O}(\alpha_s\alpha^n \ln^n M_Z^2)$ effects, but the integration cannot be performed analytically. Numerically this summation affects the result at the 10^{-5} level which can safely be neglected.]. The shift, -0.00030, between Eqs. (9) and (13) is somewhat larger than the shift, -0.00022, obtained in Ref. [4], which is in part due to the summation, but mainly due to the inputs. The uncertainty in Eq. (13) is from the current uncertainty in $\alpha_s = 0.120 \pm 0.002$, while other parametric uncertainties are minuscule[What enters Eqs. (9) and (13) is the $\overline{\mathrm{MS}}$ b-quark definition, which is free of renormalon ambiguities and therefore much better known (see Ref. [11] for a recent sub-percent determination) than the b-quark pole mass. Higher order matching corrections are also smaller if one uses the $\overline{\mathrm{MS}}$ mass definition.]. Neglecting two-loop $\mathcal{O}(\alpha^{n+1} \ln^n M_Z^2)$ effects, Eq. (13) simplifies,

$$\begin{aligned} S(m_\tau, M_Z) = &\left[\frac{\alpha(m_b)}{\alpha(m_\tau)}\right]^{\frac{9}{19}} \left[\frac{\alpha(M_W)}{\alpha(m_b)}\right]^{\frac{9}{20}} \left[\frac{\alpha(M_Z)}{\alpha(M_W)}\right]^{\frac{36}{17}} \\ &\left[\frac{\alpha_s(m_b)}{\alpha_s(m_\tau)}\right]^{\frac{3}{25}\frac{\alpha(m_\tau)}{\pi}} \left[\frac{\alpha_s(M_Z)}{\alpha_s(m_b)}\right]^{\frac{3}{23}\frac{\alpha(m_b)}{\pi}}, \end{aligned} \qquad (15)$$

which differs by only $\approx 3 \times 10^{-6}$. Neglecting further the numerically similar three-loop $\mathcal{O}(\alpha_s\alpha^n \ln^n M_Z^2)$ effects, one can expand the second line in Eq. (15) to linear order in α. If one then rewrites the resulting expression in terms of QCD scale parameters, Λ_{QCD}, one encounters the double-logarithmic form originally obtained 20 years ago [9].

To summarize, next-to-leading order effects *reduce* the leading order summation by about 50%, *i.e.*, they are numerically of the same order. Both effects are in turn numerically of order α, so they must be included for a complete $\mathcal{O}(\alpha)$ evaluation. Unknown higher orders are suppressed by at least a factor of α_s/π relative to any of the effects mentioned before. Thus, the uncertainty due to higher order effects is of order $\mathcal{O}(\alpha\alpha_s) \sim 0.0003$.

Following Ref. [1] one can write,

$$S_{\rm EW} \equiv S(m_\tau, M_Z)\frac{S_{\rm EW}^{\rm sub,had}}{S_{\rm EW}^{\rm sub,lep}} = 1.0201 \pm 0.0003, \quad (16)$$

but there $S_{\rm EW} = 1.0267 \pm 0.0027$ is quoted instead. Almost 2/3 of the difference is due to the error pointed out after Eq. (4), and about 5% due to neglecting next-to-leading order contributions to $S(m_\tau, M_Z)$. Another 15% difference is likely due to applying Eq. (9) incorrectly (as discussed above). The remaining 15% can perhaps be traced to use of the on-shell definition of α in place of the $\overline{\rm MS}$ definition as used in the present work. Note, that the derivation of Eq. (9) assumes a *mass-independent* renormalization scheme (such as the $\overline{\rm MS}$ scheme), in which at each fermion threshold the β-function coefficients change by a finite amount: this is the origin of the product form of Eq. (9). Thus, the solution (9) cannot be applied to *mass-dependent* schemes, such as the on-shell renormalization scheme. It is emphasized again, that the numerical difference to Ref. [1] should not be viewed as a scheme-dependence and thus as an estimate of uncalculated higher order corrections (which are much smaller as discussed above). On the contrary, one should expect that a self-consistent treatment within the on-shell scheme will reproduce the result of the present work.

As far as the τ-based analysis of Ref. [1] is concerned, about 77% of the data is affected by $S_{\rm EW}$. Since $S_{\rm EW}$ obtained in this paper differs by about 0.65% from the one in Ref. [1], one expects a 0.5% shift in the extracted $\Delta a_\mu^{\rm had,(2)}$. Including an update of the CKM matrix element $|V_{ud}|$ entering the analysis (the value, $|V_{ud}| = 0.9752 \pm 0.0007$, is replaced by, $|V_{ud}| = 0.97485 \pm 0.00046$, from the fit result of Ref. [7]), this amounts to about one half of the current experimental uncertainty of 0.8 parts per billion [2] for the muon magnetic moment. The τ-based Standard Model prediction would then be consistent with the measurement [2] within about one standard deviation. The discrepancy to the e^+e^- based analysis of Ref. [1] would correspondingly be larger. Furthermore, the smaller errors in Eq. (16) and in $|V_{ud}|$ compared to Ref. [1] should lead to a slight reduction of the overall uncertainty of the τ-based result. As a final remark, the recent determination [11] of α_s from the τ lifetime, when updated with the present next-to-leading order analysis, increases $\alpha_s(M_Z)$ by less than 0.0001.

Acknowledgments

It is a pleasure to thank Eric Braaten and Michael Ramsey-Musolf for their comments.

1. M. Davier, S. Eidelman, A. Höcker, and Z. Zhang, *Confronting Spectral Functions from $e+e-$ Annihilation and τ Decays: Consequences for the Muon Magnetic Moment*, e-print hep-ph/0208177.
2. Muon g-2 Collaboration: G.W. Bennett *et al.*, *Phys. Rev. Lett.* **89** (2002) 101804.
3. A. Sirlin, *Rev. Mod. Phys.* **50** (1978) 573.
4. W.J. Marciano and A. Sirlin, *Phys. Re. Lett.* **61** (1988) 1815.
5. ALEPH, DELPHI, L3, and OPAL Collaborations, LEP Electroweak Working Group and SLD Heavy Flavor and Electroweak Groups: D. Abbaneo *et al.*, *A Combination of Preliminary Electroweak Measurements and Constraints on the Standard Model*, e-print hep-ex/0112021.
6. J. Erler, *Phys. Rev. D* **59** (1999) 054008.
7. Particle Data Group: K. Hagiwara *et al.*, *Phys. Rev. D* **66** (2002) 010001.
8. E. Braaten and C.S. Li, *Phys. Rev. D* **42** (1990) 3888.
9. A. Sirlin, *Nucl. Phys. B* **196** (1982) 83.
10. W.J. Marciano and A. Sirlin, *Phys. Rev. Lett.* **56** (1986) 22.
11. J. Erler and M. Luo, *Precision Determination of Heavy Quark Masses and the Strong Coupling Constant*, e-print hep-ph/0207114.