

Relation between the field quadratures and the characteristic function of a mirror

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We analyze the possibility of measuring the state of a movable mirror by using its interaction with a quantum field. We show that measuring the field quadratures allows us to reconstruct the characteristic function corresponding to the mirror state.

Keywords: Movable mirrors; displacement operator.

Analizamos la posibilidad de medir el estado cuántico de un espejo móvil usando su interacción con un campo cuantizado. Mostramos que midiendo las cuadraturas del campo permite la reconstrucción de la función característica correspondiente al espejo móvil.

Descriptores: Espejos móviles; operador de desplazamiento.

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1. Introduction

Cavities with moving mirrors have attracted the attention of researchers because of the great possibilities they have to produce non-classical states of both, the mirror and field states [1]. The possibilities to generate non-classical states, and in particular a superposition of coherent states of the quantized field are because of the Kerr like Hamiltonians that may be obtained in these systems [1]. Indeed, it was shown several years ago [2] that an empty cavity with a moving mirror in its steady state may mimic a Kerr medium when it is illuminated with coherent light. This effect is completely because of the radiation pressure force. Bistable behaviour analogous to that produced by an $\chi^{(3)}$ nonlinear medium in a cavity was experimentally demonstrated in the optical [3], as well as in the microwave domains [4]. More recently, we have studied bistability in this system by considering not the semiclassical behaviour [2], but a completely quantum-mechanical treatment [5]. The fact that the motion of the mirror is quantized, allows the possibility to generate non-classical states of the mirror [1], *i.e.* of a quantum state of a macroscopic object. It is therefore very interesting to try to measure such non-classical states.

The reconstruction of a quantum state is a central topic in quantum optics and related fields [6, 7]. During the past years, several techniques have been developed, for instance the direct sampling of the density matrix of a signal mode in multiport optical homodyne tomography [8], tomographic reconstruction by unbalanced homodyning [9], reconstruction via photocounting [10], cascaded homodyning [11] to cite some. There have also been proposals to measure electromagnetic fields inside cavities [12, 13] and vibrational states in ion traps [12, 14]. In fact the full reconstruction of non-classical states of the electromagnetic field [15], and of (motional) states of an ion [16] have been experimentally accomplished. The quantum state reconstruction in cavities is usually achieved through a finite set of selective measurements of atomic states [12] that makes it possible to construct

quasiprobability distribution functions such as the Wigner function, that constitutes an alternative representation of a quantum state of the field.

Recently, there has been interest in the production of superposition states of macroscopic systems such as a moving mirror [18]. It is therefore of interest to have schemes to measure the non-classical states that may be generated for the moving mirror. Here we will propose a method to relate the quadratures of the field to the characteristic function associated to the density matrix of the mirror.

2. The Hamiltonian of the model

We follow Mancini and Tombesi [19], and consider a cavity with two perfectly reflecting mirrors, one of them fixed and the other one can move, undergoing harmonic oscillations. The cavity resonances are calculated in the absence of the impinging field. Given L to be the equilibrium cavity length, the resonant angular frequencies of the cavity are

$$\omega = k\pi \frac{c}{L} \quad (1)$$

where k is an arbitrary integer number and c the speed of light. We assume that the retardation effects due to the oscillating mirror may be neglected. We will use a field intensity such that the correction to the radiation pressure force, due to the Doppler frequency shift of the photons [20] may also be neglected. Therefore we are able to write the relevant Hamiltonian as [19, 21, 22]

$$H = \hbar\omega a^\dagger a + \frac{p^2}{2m} + \frac{m\Omega^2 x^2}{2} + H_{\text{int}} \quad (2)$$

where a and a^\dagger are the annihilation and creation operators for the cavity field, respectively. The field frequency is ω , the mirror oscillates at a frequency Ω , p , and x are the momentum and displacement from the equilibrium position operators of the oscillating mirror with mass m , and H_{int} accounts for the interaction between the cavity mode and the oscillating mirror. Because we have assumed no retardation effects, we may

simply write

$$H_{\text{int}} = -\hbar g a^\dagger a x, \tag{3}$$

where the coupling constant is

$$g = \frac{\omega}{L} \sqrt{\frac{\hbar}{2m\Omega}}, \tag{4}$$

with m the mass of the movable mirror. H_{int} represents the effect of the radiation pressure force $F_R = (\hbar\omega/L)a^\dagger a$ that causes the instantaneous displacement x of the mirror [19].

We can therefore rewrite the Hamiltonian (2) in the form

$$H = \hbar(\omega a^\dagger a + \Omega b^\dagger b - g a^\dagger a (b^\dagger + b)) \tag{5}$$

b and b^\dagger are the annihilation and creation operators for the mirror.

It is convenient to write the Hamiltonian (5) with the help of displacement operators [1]

$$H = D_m(\eta a^\dagger a) (\omega a^\dagger a + \Omega b^\dagger b - \epsilon (a^\dagger a)^2) D_m^\dagger(\eta a^\dagger a) \tag{6}$$

where $\epsilon = g\eta$ with $\eta = g/\Omega$ and the displacement operator is given by

$$D_m(\beta) = e^{\beta b^\dagger - \beta^* b}, \tag{7}$$

with $N = a^\dagger a$. Then the unitary evolution operator is simply

$$U(t) = e^{-\frac{iHt}{\hbar}} D_m(\eta N) e^{-it(\omega N + \Omega b^\dagger b - \epsilon N^2)} D_m^\dagger(\eta N) \tag{8}$$

We will consider the initial state of the field to be in a coherent state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{9}$$

and the initial state of the mirror to be arbitrary and denoted by the density matrix ρ_m , so that the total density matrix at a time t is given by

$$\rho(t) = U(t)|\alpha\rangle\langle\alpha| \otimes \rho_m U^\dagger(t). \tag{10}$$

Once having the evolved density matrix we may calculate the average of any operator, A by the *total* trace:

$$\langle A \rangle = \text{Tr}\{\rho(t)A\} = \text{Tr}\{|\alpha\rangle\langle\alpha| \otimes \rho_m U^\dagger(t) A U(t)\} \tag{11}$$

where we have substituted (10) into the above equation and have made use of the invariance under permutations of the trace.

We can now calculate $\langle a \rangle$ in the form

$$\langle a \rangle = \alpha e^{-i(\omega+\epsilon)t} \text{Tr} \left[\rho_m D_m(\eta e^{i\Omega t}) D_m(-\eta) |\alpha e^{2i(\epsilon t - \eta^2 \sin \Omega t)}\rangle\langle\alpha| \right] \tag{12}$$

where we have used several times the properties of permutation under the trace symbol. By using that

$$D_m(\eta e^{i\Omega t}) D_m(-\eta) e^{i\eta^2 \sin \Omega t} = D_m(\eta(e^{i\Omega t} - 1)) \tag{13}$$

we may finally write

$$\langle a \rangle = \alpha e^{-i(\omega+\epsilon)t} e^{-i\eta^2 \sin \Omega t} e^{-|\alpha|^2(\epsilon t - \eta^2 \sin \Omega t)} \times \chi_m(\eta(e^{i\Omega t} - 1)) \tag{14}$$

where $\chi_m(\eta(e^{i\Omega t} - 1)) = \text{Tr}\{\rho_m D_m[\eta(e^{i\Omega t} - 1)]\}$ is the characteristic function associated to the density matrix ρ_m . Therefore, by measuring the quadratures of the field (see for instance [7]) $\langle X \rangle = \langle (a + a^\dagger) \rangle / \sqrt{2}$ and $\langle Y \rangle = -i \langle (a - a^\dagger) \rangle / \sqrt{2}$ we may obtain the average value for the annihilation operator, and hence information about the state of the mirror through its characteristic function. The argument of the characteristic function may be changed in some range of parameters as $\omega \sim 10^{16} \text{ s}^{-1}$, $\Omega \sim 1 \text{ kHz}$, $L \sim 1 \text{ m}$ and $m \sim 10 \text{ mg}$ [2,3,22]. One could use the present method to reconstruct the quantum superpositions of a mirror state recently proposed by Marshall *et al.* [18] around the origin to look for a negative Wigner function in this region.

3. Wigner function in terms of characteristic function

We now write the characteristic function in terms of the average value of the annihilation operator

$$\chi_m(\eta(e^{i\Omega t} - 1)) = \frac{\langle a \rangle}{\alpha e^{-i(\omega+\epsilon)t} e^{-i\eta^2 \sin \Omega t} e^{-|\alpha|^2(\epsilon t - \eta^2 \sin \Omega t)}}, \tag{15}$$

from which we can obtain the set of s -parametrized quasiprobability distributions, and in particular the Wigner function for the mirror state [23]:

$$W_m(\xi) = \frac{1}{\pi^2} \int d^2\beta e^{\xi\beta^* - \xi^*\beta} \chi_m(\beta), \tag{16}$$

where we have defined $\beta = \eta(e^{i\Omega t} - 1)$. Note that a value of β defines only one point in the dual phase space, or in other words, a particular set of parameters, such as interaction time, mirror-field interaction constant, etc., define only one value of β . The transformation of the characteristic function above requires an infinite set of points (in general, a continuous set of values from minus infinity to infinity). Therefore the precision of the method pointed out here is related with the amount of times the experiment has to be repeated. However, this is a 'problem' related to all reconstruction schemes [7]. The Wigner function has a one to one correspondence with the density matrix, in fact they are related by an integral Fourier transform [7, 23]. Detailed forms to measure the quadratures of the field may be found in reference [7]. There, the methods to reconstruct the Wigner function are based on tomography and Radon transforms. Therefore, measurement of field

quadratures are required for different angles (again, the quality of the reconstruction depends on the 'distance' of these angles, and therefore the number of times the experiment has to be repeated).

4. Conclusions

In conclusion, we have shown that by measuring field quadratures one may be able to reconstruct the characteristic function for the density matrix of the mirror. We have given an explanation on why it is possible to reconstruct the characteristic function by using an analogy with the atom field, interaction with counter-rotating terms.

What makes it possible to obtain information about the mirror state is the initial coherence of the field and the form

of the Hamiltonian that has the term

$$b + b^\dagger. \quad (17)$$

Wilkins and Meystre [24] had shown that for the Jaynes-Cummings Model (JCM) [25], it was possible to obtain information about the characteristic function of the field only if the system interacted with an extra (classical) field to allow several absorptions (a^k) or emissions [$(a^\dagger)^k$] such that moments of a and/or a^\dagger could be obtained, *i.e.* the characteristic function reconstructed. Here, the multiple absorptions/emissions are given by the term in Eq. (17), being this term responsible for the possibility of the reconstruction.

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