Schrödinger–Pauli equation for spin-3/2 particles

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A non-relativistic equation for spin-3/2 particles is proposed and the gyromagnetic ratio for charged spin-3/2 particles is determined.

Keywords: Spin-3/2 particles; gyromagnetic ratio.

Se propone una ecuación no relativista para partículas de espín 3/2 y se determina la razón giromagnética para partículas de espín 3/2 cargadas.

Descriptores: Partículas de espín 3/2; razón giromagnética.

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1. Introduction

In the quantum-mechanical description of particles, there are various, relativistic or non-relativistic wave equations whose form depends on the spin of the particles. The usual Schrödinger equation applies to the spin-0 particles in the non-relativistic domain, while the Klein–Gordon equation is the relativistic equation appropriate for spin-0 particles. The spin-1/2 particles are governed by the relativistic Dirac equation which, in the non-relativistic limit, leads to the Schrödinger–Pauli equation (see, *e.g.*, Refs. 1–3). In the case of particles with spin 1 or higher, only relativistic equations are usually considered (see, *e.g.*, Ref. 4).

A charged particle with non-zero spin couples to an external magnetic field as if, in addition to its electric charge, it had a magnetic dipole moment. In the case of a spin-1/2 charged particle, the relation between the magnitudes of the magnetic dipole moment and of the intrinsic angular momentum given by the Dirac or the Schrödinger–Pauli equation does not coincide with that of a uniformly charged rotating body given by classical physics, but somewhat surprisingly it does coincide with that of a rotating charged black hole in the Einstein–Maxwell theory (see, *e.g.*, Refs. 5, 6).

In this paper we propose a non-relativistic wave equation for spin-3/2 particles directly by analogy with the Schrödinger–Pauli equation, to obtain the gyromagnetic ratio of a charged spin-3/2 particle. We find that the relation between the greatest eigenvalue of the magnetic dipole moment, and the charge to mass ratio has a common value for spin-3/2 and spin-1/2 particles. In the relativistic case, there exist several acceptable wave equations for spin-3/2 fields (see, *e.g.*, Ref. 7 and the references cited therein), but we are not studying their non-relativistic limits.

In Sec. 2 the Schrödinger–Pauli equation for spin-1/2 particles is written making use of the Pauli matrices and of the two-component spinor notation which is employed in Sec. 3 to write the proposed equation for spin-3/2 particles. The notation and conventions used throughout this paper are summarized in Sec. 2; further details can be found in Refs. 8, 9.

2. Spin-1/2 particles

The usual Schrödinger equation for a spin-0 particle of mass M in a potential $V(\mathbf{r})$,

$$-\frac{\hbar^2}{2M}\nabla^2\psi + V(\mathbf{r})\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{1}$$

can be obtained from the classical Hamiltonian $H = \mathbf{p}^2/2M + V$, using the fact that the momentum operator in the coordinate representation is given by $-i\hbar\nabla$. In the case of a spin-1/2 particle, the wave function is not a complex-valued function but a *two-component spinor*

$$\psi(\mathbf{r},t) = \begin{pmatrix} \psi^1(\mathbf{r},t) \\ \psi^2(\mathbf{r},t) \end{pmatrix}, \tag{2}$$

which under a rotation through an angle α about the axis defined by a unit vector $\mathbf{n}$ transforms into (see, *e.g.*, Refs. 10, 9)

$$\psi' = \left(\cos\tfrac{1}{2}\alpha\, I - i\sin\tfrac{1}{2}\alpha\,\mathbf{n}\cdot\boldsymbol{\sigma}\right)\psi, \tag{3}$$

where I is the identity 2×2 matrix and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is formed by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Thus, for an infinitesimal rotation,

$$\psi' \simeq \psi - i\alpha\,\tfrac{1}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\,\psi,$$

which means that $\mathbf{n}\cdot\mathbf{S} = (1/2)\hbar\mathbf{n}\cdot\boldsymbol{\sigma}$ is the operator corresponding to the component of the spin angular momentum operator along $\mathbf{n}$.

The Pauli matrices satisfy

$$\sigma_i\sigma_j = \delta_{ij}I + i\varepsilon_{ijk}\sigma_k, \tag{5}$$

where ε_{ijk} is totally antisymmetric with $\varepsilon_{123} = 1, i, j, \ldots = 1, 2, 3$, and there is summation over repeatedindices. Since the entries of the Pauli matrices are constant, making use of Eq. (5),

$$(\boldsymbol{\sigma}\cdot\nabla)^2 = \sigma_i\frac{\partial}{\partial x_i}\sigma_j\frac{\partial}{\partial x_j} = \sigma_i\sigma_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{pmatrix},$$

where the x_i are Cartesian coordinates; therefore, the equation

$$-\frac{\hbar^2}{2M}(\boldsymbol{\sigma}\cdot\nabla)^2\psi + V(\mathbf{r})\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{6}$$

for the two-component spinor (2) implies that each component of ψ (ψ^1 and ψ^2) satisfies the Schrödinger equation (1), and conversely. However, when there is a magnetic field present this equivalence disappears and Eq. (6) leads to a coupling between the two components of the spinor ψ.

The standard procedure to take into account the interaction of a particle of electric charge q with an electromagnetic field consists in replacing the partial derivatives $\partial/\partial x_i$ and $\partial/\partial t$ by $\partial/\partial x_i - iqA_i/(\hbar c)$ and $\partial/\partial t + iq\phi/\hbar$, respectively, where $\mathbf{A} = (A_1, A_2, A_3)$, and ϕ are potentials of the electromagnetic field. In this manner, for a spin-0 charged particle, from Eq. (1) one obtains (see, *e.g.*, Ref. 10)

$$-\frac{\hbar^2}{2M}\left[\nabla^2\psi - \frac{2iq}{\hbar c}\mathbf{A}\cdot\nabla\psi - \frac{iq}{\hbar c}(\nabla\cdot\mathbf{A})\,\psi - \left(\frac{q}{\hbar c}\right)^2\mathbf{A}^2\psi\right] + V(\mathbf{r})\psi + q\phi\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{7}$$

and, similarly, making use of Eqs. (5), Eq. (6) yields

$$-\frac{\hbar^2}{2M}\left[\nabla^2\psi - \frac{2iq}{\hbar c}\mathbf{A}\cdot\nabla\psi - \frac{iq}{\hbar c}(\nabla\cdot\mathbf{A})\psi - \left(\frac{q}{\hbar c}\right)^2\mathbf{A}^2\psi + \frac{q}{\hbar c}\mathbf{B}\cdot\boldsymbol{\sigma}\psi\right] + V(\mathbf{r})\psi + q\phi\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{8}$$

When $\mathbf{B} = 0$, Eq. (8) reduces to two independent equations of the form (7), one for each component of the spinor ψ. However, when $\mathbf{B} \neq 0$, the components of ψ are coupled through the term

$$-\mathbf{B}\cdot\frac{q}{Mc}\frac{1}{2}\hbar\boldsymbol{\sigma}\psi.$$

Recalling that the energy of a magnetic dipole moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ is equal to $-\boldsymbol{\mu}\cdot\mathbf{B}$, it follows that a charged spin-1/2 particle obeying Eq. (8) *behaves* as if it had a magnetic dipole moment represented by the operator

$$\boldsymbol{\mu} = \frac{q}{Mc}\frac{1}{2}\hbar\boldsymbol{\sigma} = \frac{q}{Mc}\mathbf{S}. \tag{9}$$

(By contrast with the electric charge, which is a "c-number", the magnetic dipole moment associated with the particle is an operator.) Equation (9) shows that the ratio of the magnetic dipole moment to the intrinsic angular momentum is equal to

$$\frac{q}{Mc}. \tag{10}$$

Before considering an analog of Eq. (6) applicable to spin-3/2 particles, it will be convenient to write Eq. (6) making use of the two-component spinor notation that will be employed in the treatment of spin-3/2 particles (see also Refs. 8 and 9).

The entries of the Pauli matrices (4) will be denoted by ${\sigma_i}^A{}_B$ ($A, B, \ldots = 1, 2$), so that ${\sigma_i}^A{}_B$ stands for the entry in the A-th row and B-th column of the matrix σ_i. The spinor indices, such as those of the spinor (2), and of the Pauli matrices, will be lowered or raised following the convention

$$\phi_A = \varepsilon_{AB}\phi^B, \qquad \phi^A = \phi_B\varepsilon^{BA}, \tag{11}$$

where

$$(\varepsilon_{AB}) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv (\varepsilon^{AB}). \tag{12}$$

(Thus, $\phi_1 = \phi^2$, $\phi_2 = -\phi^1$.) Hence, ${\varepsilon^A}_B = \delta^A_B$ and

$$\phi_A\psi^A = \varepsilon_{AB}\phi^B\psi^A = -\phi^B\varepsilon_{BA}\psi^A = -\phi^B\psi_B = -\phi^A\psi_A. \tag{13}$$

Any tensor with Cartesian components $t_{ij\cdots k}$ has a *spinor equivalent* defined by

$$t_{ABCD\cdots MN} \equiv \frac{1}{\sqrt{2}}{\sigma^i}_{AB}\frac{1}{\sqrt{2}}{\sigma^j}_{CD}\cdots \times \frac{1}{\sqrt{2}}{\sigma^k}_{MN}t_{ij\cdots k}, \tag{14}$$

where, following the conventions stated above, $\sigma_{iAB} = \varepsilon_{AC}{\sigma_i}^C{}_B$. (Since we are considering here Cartesian coordinates only, the tensor indices are lowered or raised by means of the metric tensor δ_{ij} and its inverse δ^{ij}; hence, $\sigma_{iAB} = {\sigma^i}_{AB}$.) An explicit computation shows that

$$\sigma_{iAB} = \sigma_{iBA}. \tag{15}$$

Furthermore, since the Pauli matrices have a vanishing trace, from Eq. (5) we obtain $\mathrm{tr}\,(\sigma_i\sigma_j) = 2\delta_{ij}$, *i.e.*, ${\sigma_i}^A{}_B{\sigma_j}^B{}_A = 2\delta_{ij}$ or, equivalently [see Eq. (13)]

$$\sigma_i{}^{AB}\sigma_{jAB} = -2\delta_{ij}. \tag{16}$$

Hence, from Eqs. (14) and (16) we find that, if t_{AB} and s_{AB} are the spinor equivalents of t_i and s_i, respectively

$$t^{AB}s_{AB} = -t^is_i. \tag{17}$$

According to the definition (14), we shall write

$$\partial_{AB} = \frac{1}{\sqrt{2}} \sigma^i{}_{AB} \partial_i, \tag{18}$$

where $\partial_i \equiv \partial/\partial x^i$. Thus, the Schrödinger–Pauli equation (6) can be expressed as

$$-\frac{\hbar^2}{M} \partial^A{}_C \partial^C{}_B \psi^B + V(\mathbf{r}) \psi^A = i\hbar \frac{\partial \psi^A}{\partial t}. \tag{19}$$

We can see that Eq. (19) is equivalent to two decoupled Schrödinger equations using the fact that if $\phi_{AB} = -\phi_{BA}$, then

$$\phi_{AB} = \tfrac{1}{2} \phi^R{}_R \varepsilon_{AB}. \tag{20}$$

Indeed, any 2×2 antisymmetric matrix must be proportional to (ε_{AB}) [see Eq. (12)] and, as can be readily seen,

$$\begin{aligned} \phi_{AB} &= \phi_{12}\, \varepsilon_{AB} = \frac{1}{2}(\phi_{12} - \phi_{21})\, \varepsilon_{AB} \\ &= \frac{1}{2}(\phi^2{}_2 + \phi^1{}_1)\, \varepsilon_{AB} = \frac{1}{2} \phi^R{}_R\, \varepsilon_{AB}. \end{aligned}$$

Owing to Eq. (13), and the fact that $\partial_{AB} = \partial_{BA}$ [see Eqs. (15) and (18)]

$$\partial_{AC} \partial^C{}_B = -\partial_A{}^C \partial_{CB} = -\partial_{CB} \partial_A{}^C = -\partial_{BC} \partial^C{}_A,$$

hence [see Eqs. (20), (13), and (17)],

$$\begin{aligned} \partial_{AC} \partial^C{}_B = \tfrac{1}{2} \varepsilon_{AB} \partial^R{}_C \partial^C{}_R &= -\tfrac{1}{2} \varepsilon_{AB} \partial^{RC} \partial_{RC} \\ &= \tfrac{1}{2} \varepsilon_{AB} \nabla^2, \end{aligned}$$

which is equivalent to

$$\partial^A{}_C \partial^C{}_B = \tfrac{1}{2} \delta^A_B \nabla^2, \tag{21}$$

so that, in effect, Eq. (19) amounts to

$$-\frac{\hbar^2}{2M} \nabla^2 \psi^A + V(\mathbf{r}) \psi^A = i\hbar \frac{\partial \psi^A}{\partial t}.$$

When there is an electromagnetic field present, we replace $\partial^A{}_B$ by $\partial^A{}_B - (iq/\hbar c) A^A{}_B$ and $\partial/\partial t$ by $\partial/\partial t + (iq/\hbar)\phi$ in Eq. (19), and we obtain

$$\begin{aligned} -\frac{\hbar^2}{M} \left(\partial^A{}_C - \frac{iq}{\hbar c} A^A{}_C \right) \left(\partial^C{}_B - \frac{iq}{\hbar c} A^C{}_B \right) \psi^B & \\ + V(\mathbf{r}) \psi^A + q\phi \psi^A = i\hbar \frac{\partial \psi^A}{\partial t}, & \end{aligned}$$

which is equivalent to

$$\begin{aligned} -\frac{\hbar^2}{M} \bigg[\partial^A{}_C \partial^C{}_B \psi^B - \frac{iq}{\hbar c} (\partial^A{}_C A^C{}_B) \psi^B & \\ -\frac{iq}{\hbar c} A^C{}_B \partial^A{}_C \psi^B - \frac{iq}{\hbar c} A^A{}_C \partial^C{}_B \psi^B & \\ - \left(\frac{q}{\hbar c} \right)^2 A^A{}_C A^C{}_B \psi^B \bigg] + V(\mathbf{r}) \psi^A + q\phi \psi^A & \\ = i\hbar \frac{\partial \psi^A}{\partial t}. & \end{aligned} \tag{22}$$

In order to reduce this last expression we begin by noticing that [see Eq. (20)]

$$\begin{aligned} \partial_{AC} A^C{}_B &= \tfrac{1}{2} (\partial_{AC} A^C{}_B + \partial_{BC} A^C{}_A) \\ &\quad + \tfrac{1}{2} (\partial_{AC} A^C{}_B - \partial_{BC} A^C{}_A) \\ &= \partial_{(A|C|} A^C{}_{B)} + \varepsilon_{AB} \partial^R{}_C A^C{}_R, \end{aligned}$$

where the parenthesis denotes symmetrization on the indices enclosed (*e.g.*, $M_{(AB)} = (1/2)(M_{AB} + M_{BA})$), and the indices between bars are excluded from the symmetrization. The first term in the right-hand side of the last equality is the spinor equivalent of $(i/\sqrt{2}) \nabla \times \mathbf{A}$, which follows from the fact that the spinor equivalent of the Levi-Civita symbol ε_{ijk} is $\varepsilon_{ABCDEG} = (i/\sqrt{2})(\varepsilon_{AC}\varepsilon_{BE}\varepsilon_{DG} + \varepsilon_{BD}\varepsilon_{AG}\varepsilon_{CE})$ [8,9], while the last term is equal to $(1/2)\varepsilon_{AB} \nabla \cdot \mathbf{A}$ [see Eqs. (13) and (17)]. Making use of Eqs. (13) and (20) we find that

$$\begin{aligned} A^C{}_B \partial_{AC} + A_{AC} \partial^C{}_B &= A^C{}_B \partial_{AC} - A^C{}_A \partial_{BC} \\ &= \varepsilon_{BA} A^{CR} \partial_{RC} = \varepsilon_{AB} \mathbf{A} \cdot \nabla. \end{aligned}$$

Finally, by analogy with Eq. (21), $A^A{}_C A^C{}_B = (1/2) \delta^A_B \mathbf{A}^2$. Thus, Eq. (22) can be also be written as

$$\begin{aligned} -\frac{\hbar^2}{2M} \bigg[\nabla^2 \psi^A + \frac{\sqrt{2}\, q}{\hbar c} B^A{}_B \psi^B - \frac{iq}{\hbar c} (\nabla \cdot \mathbf{A}) \psi^A & \\ -\frac{2iq}{\hbar c} \mathbf{A} \cdot \nabla \psi^A - \left(\frac{q}{\hbar c} \right)^2 \mathbf{A}^2 \psi^A \bigg] + V(\mathbf{r}) \psi^A & \\ + q\phi\, \psi^A = i\hbar \frac{\partial \psi^A}{\partial t}, & \end{aligned} \tag{23}$$

where B_{AB} denotes the spinor equivalent of $\mathbf{B}$, and one can verify that this expression coincides with Eq. (8).

3. Spin-3/2 particles

A spin-3/2 particle is described by a totally symmetric three-index spinor field, ψ^{ABC} [see Eq. (25) below], which under rotations transforms according to

$$\psi'^{ABC} = U^A{}_R U^B{}_S U^C{}_T \psi^{RST},$$

where $(U^A{}_B)$ is the SU(2) matrix appearing in Eq. (3), namely

$$U^A{}_B = \cos \tfrac{1}{2}\alpha\; \delta^A_B - i\sqrt{2}\, \sin \tfrac{1}{2}\alpha\; n^A{}_B \tag{24}$$

and n_{AB} is the spinor equivalent of the unit vector $\mathbf{n}$. Hence, for an infinitesimal rotation,

$$\begin{aligned} \psi'^{ABC} &\simeq \left(\delta^A_R - \frac{i\alpha}{\sqrt{2}} n^A{}_R \right) \left(\delta^B_S - \frac{i\alpha}{\sqrt{2}} n^B{}_S \right) \\ &\quad \times \left(\delta^C_T - \frac{i\alpha}{\sqrt{2}} n^C{}_T \right) \psi^{RST} \\ &\simeq \psi^{ABC} - \frac{3i}{\sqrt{2}} \alpha\, n^{(A}{}_R \psi^{BC)R}, \end{aligned}$$

which implies that the operator $\mathbf{n}\cdot\mathbf{S}$ given by

$$(\mathbf{n}\cdot\mathbf{S}\,\psi)^{ABC} = \frac{3\hbar}{\sqrt{2}} n^{(A}{}_R \psi^{BC)R} \tag{25}$$

corresponds to the component of the spin along $\mathbf{n}$.

By analogy with the Schrödinger–Pauli equation (19), for a spin-3/2 particle we propose the equation

$$-\frac{\hbar^2}{M}\partial^{(A}{}_R\partial^{|R}{}_S\psi^{S|BC)} + V(\mathbf{r})\psi^{ABC} = i\hbar\frac{\partial\psi^{ABC}}{\partial t} \tag{26}$$

which, owing to Eq. (21), means that, in the absence of an electromagnetic field, each component of ψ^{ABC} satisfies the Schrödinger equation (1). (It should be noticed that, even in the absence of an electromagnetic field, the components of the spinor field ψ^{ABC} may be coupled among themselves if a non-Cartesian basis is employed, in which case the partial derivatives appearing in Eq. (26) have to be replaced by covariant derivatives [8, 9].) Instead of the symmetrized second derivatives $\partial^{(A}{}_R\partial^{|R}{}_S\psi^{S|BC)}$ appearing in Eq. (26), one could also consider $\partial^{(A}{}_R\chi^{|R|BC)}$, with $\chi^{RBC} \equiv \partial^{(R}{}_S\psi^{|S|BC)}$; however, in the latter case, each component ψ^{ABC} would not satisfy the Schrödinger equation.

By combining Eq. (26) and its complex conjugate one obtains the continuity equation

$$\frac{\partial}{\partial t}(\widehat{\psi}_{ABC}\psi^{ABC}) + \frac{\hbar}{iM}\partial_{RS}(\psi^{SBC}\partial^{AR}\widehat{\psi}_{ABC} + \widehat{\psi}^{SBC}\partial^{AR}\psi_{ABC}) = 0,$$

where $\widehat{\psi}_{ABC} \equiv \overline{\psi^{ABC}}$ [9]; hence, $\widehat{\psi}_{ABC}\psi^{ABC}$ is real and positive.

As in the case of the relativistic description of spin-3/2 particles, instead of three-index spinors, one can employ wave functions with one spinor index and one tensor index [4]. In the present case, we can define

$$\psi_{iA} \equiv -\frac{1}{\sqrt{2}}\sigma_i{}^{BC}\psi_{ABC},$$

then the symmetry of ψ_{ABC} in its three indices is expressed by the condition $\sigma^{iAB}\psi_{iA} = 0$.

When there is an electromagnetic field present, we replace $\partial^A{}_B$ by $\partial^A{}_B - (iq/\hbar c)A^A{}_B$ and $\partial/\partial t$ by $\partial/\partial t + (iq/\hbar)\phi$, in Eq. (26), which yields

$$-\frac{\hbar^2}{M}\left(\partial^{(A}{}_R - \frac{iq}{\hbar c}A^{(A}{}_R\right)\left(\partial^{|R}{}_S - \frac{iq}{\hbar c}A^{|R}{}_S\right)\psi^{S|BC)} + V(\mathbf{r})\psi^{ABC} + q\phi\psi^{ABC} = i\hbar\frac{\partial\psi^{ABC}}{\partial t}. \tag{27}$$

Following the same steps as in Eq. (22) we obtain

$$-\frac{\hbar^2}{2M}\Bigg[\nabla^2\psi^{ABC} + \frac{\sqrt{2}\,q}{\hbar c}B^{(A}{}_S\psi^{BC)S} - \frac{iq}{\hbar c}(\nabla\cdot\mathbf{A})\psi^{ABC} - \frac{2iq}{\hbar c}\mathbf{A}\cdot\nabla\psi^{ABC} - \left(\frac{q}{\hbar c}\right)^2\mathbf{A}^2\psi^{ABC}\Bigg] + V(\mathbf{r})\psi^{ABC} + q\phi\,\psi^{ABC} = i\hbar\frac{\partial\psi^{ABC}}{\partial t}. \tag{28}$$

Thus, taking into account Eq. (25), one concludes that in the present case there is an interaction with the magnetic field of the form $-\boldsymbol{\mu}\cdot\mathbf{B}$, with

$$\boldsymbol{\mu} = \frac{q}{3Mc}\mathbf{S} \tag{29}$$

[*cf.* Eq. (9)] that corresponds to the gyromagnetic ratio

$$\frac{q}{3Mc}. \tag{30}$$

Owing to the difference between the gyromagnetic ratios (10) and (30), in both cases, the greatest eigenvalue of the operator $\boldsymbol{\mu}$ is given by

$$\mu_{\max} = \frac{|q|}{Mc}\frac{\hbar}{2}. \tag{31}$$

Equations (19) and (26) can be readily generalized for any value of the spin. A spin-s particle would be represented by a totally symmetric 2s-index spinor field, $\psi^{AB\cdots L}$, satisfying

$$-\frac{\hbar^2}{M}\partial^{(A}{}_R\partial^{|R}{}_S\psi^{S|B\cdots L)} + V(\mathbf{r})\psi^{AB\cdots L} = i\hbar\frac{\partial\psi^{AB\cdots L}}{\partial t} \tag{32}$$

and if the particle has electric charge q, the gyromagnetic ratio would be given by

$$\frac{q}{2sMc}.$$

As pointed out above, under the rotation corresponding to the SU(2) matrix $(U^A{}_B)$, the Cartesian components of a spinor $\psi^{AB\ldots L}$ transform according to

$$\psi'^{AB\ldots L} = U^A{}_P U^B{}_Q \cdots U^L{}_R \psi^{PQ\ldots R}.$$

Since each term in Eq. (32) transforms in the same manner as $\psi^{AB\ldots L}$, the validity of Eq. (32) in a given Cartesian frame implies its validity in any Cartesian frame obtained from the original one by means of a rotation. This can be seen as a consequence of the fact that a contraction of the form $\partial^A{}_R\psi^{RB\ldots L}$ transforms as a $2s$-index spinor since

$$U_{RM}U^R{}_N = \det(U^A{}_B)\,\varepsilon_{NM} = \varepsilon_{NM}$$

[see Eq. (20)] and therefore

$$\begin{aligned}
&\partial'^{A}{}_{R}\psi'^{RB\ldots L} \\
&= (U^{A}{}_{P}U_{RM}\partial^{PM})\, U^{R}{}_{N}U^{B}{}_{Q}\cdots U^{L}{}_{S}\psi^{NQ\ldots S} \\
&= U_{RM}U^{R}{}_{N}U^{A}{}_{P}U^{B}{}_{Q}\cdots U^{L}{}_{S}\partial^{PM}\psi^{NQ\ldots S} \\
&= \varepsilon_{NM}U^{A}{}_{P}U^{B}{}_{Q}\cdots U^{L}{}_{S}\partial^{PM}\psi^{NQ\ldots S} \\
&= U^{A}{}_{P}U^{B}{}_{Q}\cdots U^{L}{}_{S}\partial^{P}{}_{N}\psi^{NQ\ldots S}.
\end{aligned}$$

On the other hand, under the Galilean transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t, \qquad t' = t,$$

where $\mathbf{v}$ is constant, using the chain rule, one finds that

$$\frac{\partial}{\partial x'^{i}} = \frac{\partial}{\partial x^{i}}, \qquad \frac{\partial}{\partial t'} = v^{i}\frac{\partial}{\partial x^{i}} + \frac{\partial}{\partial t'}$$

then, assuming that

$$\psi'^{AB\ldots L} = \psi^{AB\ldots L}\exp\left(-i\frac{m\mathbf{v}\cdot\mathbf{r}}{\hbar} + i\frac{mv^{2}t}{2\hbar}\right),$$

a straightforward computation, making use of Eq. (21), shows that Eq. (32) is form-invariant under Galilean transformations.

4. Concluding remarks

The preceding equations show that a genuine spin-3/2 charged particle would behave in a different way than an assembly of three charged spin-1/2 particles in a spin-3/2 state, since in the latter case one would have a gyromagnetic ratio equal to that of a single spin-1/2 particle, which differs from (30).

The origin and physical significance of the coincidence of the gyromagnetic ratios of a spin-1/2 charged particle, and of a rotating charged black hole, mentioned in the Introduction are not evident; and the fact that the gyromagnetic ratio derived from Eq. (32) depends on the spin of the particle suggests that this coincidence is not a straightforward consequence of some basic principle.

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