Compactification with $U(1)$ magnetic field within Dirac supersymmetry

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We consider Dirac-supersymmetric interactions, which produce CP-conserving separation of positive and negative energy solutions in the Dirac equation in order to investigate an alternative to the Kaluza-Klein mechanism. We review conditions under which separation is possible into free particle and compactified behaviors in different dimensions, with attention to spin degrees of freedom. We show a $U(1)$ constant magnetic field produces such kind of behavior; an explicit treatment is given to the 6-d to 4-d and 4-d to 2-d breaking cases and the spectrum is obtained. A dynamical mass-creation mechanism is suggested from the procedure.

Keywords: Compactification; dimensions; gauge; Dirac; supersymmetry.

Se consideran interacciones supersimétricas de Dirac, las cuales producen la separación de las soluciones de energía positiva y negativa en la ecuación de Dirac que conservan CP, con el objeto de investigar una alternativa para el mecanismo de Kaluza-Klein. Consideramos aquellas condiciones bajo las cuales es posible la separación en partícula libre y comportamientos de compactificación, con atención a grados de libertad de espín. Mostramos que un campo magnético constante produce este tipo de comportamiento; se da un tratamiento explícito a los casos de rompimiento de 6-d a 4-d y 4-d a 2-d y se obtiene el espectro. Este procedimiento sugiere un mecanismo dinámico de creación de masa.

Descriptores: Compactificación, dimensiones, norma, Dirac, supersimetría

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1. Introduction

The Kaluza-Klein idea has been a promising and useful assumption in modern physical research whose aim is the unification of forces. Although no experimental indication exists to presuppose its existence, it remains a useful working hypothesis in various theories. The original Kaluza-Klein idea proposes a fifth dimension that accounts for the electromagnetic interaction in the framework of an extended general relativity, and thus has succeeded in presenting gravity and electromagnetism in a unified picture. Additional dimensions have been proposed to account for the other fundamental interactions. Indeed, this is the underlying aim in applications in supergravity and string theory, where additional compactified dimensions of space have been linked to gauge interactions. In fact, arguments based on the supersymmetry breaking scale [1] and recent developments in string theory [2] have opened up the possibility that unification occurs at the electroweak scale, with the implication that the additional dimensions might be detected through its effect on gravity at millimeter scales.

Within an ampler view, the question why there are four and not any other number of physical dimensions remains unanswered. To investigate this question, one can check whether it is possible to construct a model in which additional dimensions lead consistently to the same four-dimensional physics.

In particular, the idea of higher dimensions should provide also for mechanisms in which an assumed larger dimensional universe transforms into the present one, with an expected explanation of the fate of other dimensions. The most popular related assumption for this is Klein's idea of compactification, which assumes dimensions become unobservable by closing on themselves at a small length. However, the origin, dynamics, and range of applicability of this process have not been thoroughly pursued and clarified, as will be implied from the present work. Indeed, when considering this mechanism, its existence has generally been taken for granted but not the causes leading to it.

In addition, several methods of dimensional reduction and compactification (see Ref. 3 for a review) are known but few account successfully for the particles' spectrum, representations and, in particular, for the demand that chiral fermions be obtained to reproduce the quantum numbers of physical particles [4].

Even when acceptable compactifications are found, there are multiple choices with similar four-dimensional physics, which points at the need of additional feasible restrictions to reduce the number of possibilities and increase the predictability. Gauge fields acting on particles is such a physical mechanism, and it needs to be examined. The choice of these fields is motivated by the already known four-dimensional ones. The self-consistency of these fields should be checked in the next stage of the study of the problem.

To investigate a possible dynamical process that generates compactification is the main object of this work. We will explore the idea that this compactification is generated by the gauge fields themselves, which we think more economical. It is natural to start by considering the Dirac equation, which describes basic spin-1/2 fields. Thus, the mechanism proposed should allow for a description of the fermion fields in the presence of usual and compactifying interactions. We require that the latter should not affect the workings of the first

and the Dirac fields. It is also expected that they conserve the symmetry in the weight of positive and negative energies so that when these are turned on the vacuum should not be altered asymmetrically. It is known that such interactions exist [5] and that they satisfy a restriction related to the presence of the Dirac supersymmetry [6].

In this paper we investigate Dirac-supersymmetric interactions which effectively lead to separation into physical and compactified dimensions and we find indeed a simple instance in which this is possible. This material is organized as follows: In Sec. II we review Dirac supersymmetry which defines some of the conditions for interactions in which a dynamical dimension separation mechanism can be possible. In Sec. III we investigate separation of the Dirac-supersymmetric equation using as example the 3+1 dimensional case. In Sec. IV we consider a dimension separation mechanism in which a $U(1)$ magnetic field is assumed and which produces an effective compactification in some dimensions. This model is considered in both a 4-d to 2-d (1+1) and a 6-d to 4-d (3+1) transitions. In Section V we draw some conclusions from this work.

2. Interactions in Dirac supersymmetry

The basic feature defining Dirac supersymmetric interactions is the possibility of applying a generalized Foldy-Wouthuysen transformation which brings the usual Dirac Hamiltonian to a form in which positive and negative solutions separate with equal weight. Explicitly, we assume the initial Dirac Hamiltonian H can be written in the form

$$H = Q + Q^\dagger + \lambda, \tag{1}$$

where λ is a Hermitian operator and Q is a fermionic operator such that it and its adjoint $Q^\dagger$ satisfy $Q^2 = 0$ and $Q^{\dagger 2} = 0$, and also the anticommutation relations

$$\{Q, \lambda\} = 0, \quad \{Q^\dagger, \lambda\} = 0. \tag{2}$$

After applying the Foldy-Wouthuysen transformation (FWT) H becomes [5]

$$H' = \frac{\lambda}{(\lambda^2)^{1/2}}(\{Q, Q^\dagger\} + \lambda^2)^{1/2}. \tag{3}$$

An extensive analysis of the interactions that take the form in Eq. (1) is given in Ref. 5. The resulting ones are given in the Hamiltonian (in 4+1 dimensions)

$$H = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \alpha_5 \pi_5 + \mu\beta, \tag{4}$$

where

$$\pi_I = p_I + \mathsf{A}_I(\mathbf{x}) + i\beta \mathsf{E}_I(\mathbf{x}) \quad I = 1, 2, 3, 5, \tag{5}$$

α_i are Dirac matrices with $\alpha_i = \gamma_0\gamma^i$, $\alpha_5 = i\beta\gamma_5$, $\mathsf{A}_I(\mathbf{x})$ and $\mathsf{E}_I(\mathbf{x})$ are external arbitrary fields, and μ is the mass constant, which multiplies the corresponding term.

3. Separation of Dirac-supersymmetric equation

Our main concern here is to study dynamical compactification as a dimensional reduction effect and, in particular, the form of those interactions that not only satisfy Dirac-supersymmetric conditions but also allow for such a mechanism. Specifically, we need a mechanism in which an interaction would confine particles in some dimensions and keep free-particle behavior in the unconfined (physical) dimensions. For this purpose we use as illustrative model the $3{+}1 \to 1{+}1$ reduction case, studying separation of variables for H in Eq. (4). This H can be written in the form of Eq. (1) with the association

$$\lambda = \mu\beta, \tag{6}$$

$$Q = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix}, \tag{7}$$

$$Q^\dagger = \begin{pmatrix} 0 & M^\dagger \\ 0 & 0 \end{pmatrix}, \tag{8}$$

where

$$M = \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{C}) - iC_5, \tag{9}$$

$C_I = A_I - iE_I$, $I = 1, 2, 3, 5$ and we use here, unless otherwise stated, the standard representation for the Dirac matrices [7]. The expression for the Hamiltonian in Eq. (4), after application of the FWT and after using Eqs. (6)-(8), is

$$H' = \beta \left[\begin{pmatrix} MM^\dagger + \mu^2 & 0 \\ 0 & M^\dagger M + \mu^2 \end{pmatrix} \right]^{1/2}, \tag{10}$$

where the equation to solve is

$$H'\Psi = E\Psi. \tag{11}$$

The equality between positive and negative spectrum, except for a sign, follows from the equal eigenvalues expected for both terms.

The square-operator form of the upper and lower terms of H' on Eq. (10) suggests a simplification by considering the action of single operators. Clearly, this is permitted when M is hermitian. It is then is possible to reformulate Eq. (11) in terms of another eigenvalue equation as alternative sufficient condition, linear in M, and given by

$$M\psi = E_c\psi, \tag{12}$$

where ψ is the positive energy spinor component of Ψ since, as can be proved, both the upper and lower sides in H' in Eq. (11) have the same eigenvalues. In fact, this equation is valid also in the case M is not hermitian and therefore E_c complex, for we only require $|E_c|^2 = E^2 - \mu^2$.

The linear form of Eq. (12) hints at a possible separation of degrees of freedom corresponding to different dimensions. The separation of this equation (in the stationary case) into independent components describing different dimensions $M = M_1 + M_2$ is limited, for the form of M

implies that even if we use, say, in H_1 potentials with restrained dependence on the coordinates we generally obtain $[M_1, M_2] \neq 0$. Still, we can find a partial separation of the 3+1 Eq. (12) into 1+1 dimensional terms and two additional dimensional terms (1+1 scalar). We choose the 1+1 spatial coordinate as z, with $x_1 = x$, $x_2 = y$, $x_3 = z$, and

$$M_1 = \sigma_z[p_z + C_z(z)], \tag{13}$$

$$M_2 = \boldsymbol{\sigma}_b \cdot [\mathbf{p}_b + \mathbf{C}_b(x, y)], \tag{14}$$

where b represents the $(\hat{x}, \hat{y})$ directions, and $\mathbf{C}_b(x, y) = (C_x(x, y), C_y(x, y))$, with dependences of $\mathbf{C}_b$ and C_z as specified. The Anzats

$$\psi = \begin{pmatrix} g_1(x, y) f_1(z) \\ g_2(x, y) f_2(z) \end{pmatrix} \tag{15}$$

leads to the partial separation of Eq. (11)

$$M_1 \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} E_1 f_1(z) \\ {E_1}' f_2(z) \end{pmatrix}, \tag{16}$$

where we have used the diagonal character (in spin space) of the M_1 component which allows for cancellation of the $g_i(x, y)$ in Eq. (16). We further assume that the upper and lower components share the same solutions, so that $f_1(z) = f_2(z)$. Then, $E_1 = -{E_1}'$. We note the solutions can be interpreted as positive and negative chirality components in 1+1 space. The other part of Eq. (11) has consequently the form

$$M_2 \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix} = \begin{pmatrix} (E_c - E_1) g_1(x, y) \\ (E_c + E_1) g_2(x, y) \end{pmatrix}, \tag{17}$$

and the z dependence can be divided out. The M_2 part plays the role of a scalar interaction in $1 + 1$ space. The absence of the $f_i(z)$ functions implies the eigenvalues E_c depend on the f_i solutions only through E_1. Thus, this separation is only partial yet sufficient for our purposes because it decouples at least one group of dimensions. It is clear that the separation depends on the presence of a diagonalizable component as M_1, and on the specific spatial dependence of the potentials in M_1, M_2, in accordance to the Lorentz index of the α_i matrices. In this case, it is the need to account for the spin degree of freedom that requires the additional condition that M_1 be diagonalizable. We also find that in passing from the higher dimension to the lower, the original spin is reinterpreted and forms the chiral components in the lower dimension.

4. Dirac supersymmetry in higher dimensions

A thorough analysis of the process of Dirac-supersymmetric Hamiltonians breaking into lower dimensional components requires an understanding of a Dirac-matrix construction which exhibits Dirac supersymmetry. This analysis is performed in detail in Ref. [8] and here we reproduce some useful results. In general, a Clifford algebra is defined through the anticommutation relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}, \tag{18}$$

where $i, j = 0, 1, \ldots, d-1$ and d is an integer. The main relation that we will use is a recursive formula that provides the elements of the even-d algebra in terms of the $d-2$ algebra. This is

$$\alpha_0 = \sigma_3 \otimes \hat{1} = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \tag{19}$$

$$\alpha_i = \sigma_1 \otimes \hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\alpha}_i \\ \hat{\alpha}_i & 0 \end{pmatrix}, \quad 1 \leq i \leq d-2, \tag{20}$$

$$\alpha_{d-1} = \sigma_1 \otimes \hat{1} = \begin{pmatrix} 0 & \hat{\alpha}_0 \\ \hat{\alpha}_0 & 0 \end{pmatrix}, \tag{21}$$

$$\alpha_d = \sigma_2 \otimes \hat{1} = \begin{pmatrix} 0 & -i\hat{1} \\ i\hat{1} & 0 \end{pmatrix}, \tag{22}$$

where the caret denotes the lower-dimensional algebra. The latter element α_d is obtained from the definition

$$\alpha_d = e^{-i(d/2)\pi/2} \alpha_0 \ldots \alpha_{d-1}, \tag{23}$$

and it extends the algebra from even d to odd $d + 1$.

The Lorentz generator antisymmetric tensors, generalized to any dimension, can be deduced from Eqs. (19)-(22) and the definition of the γ matrices $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$. They can be shown to be

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} 0 & i\sigma_1 \otimes \hat{\alpha}_i & i\sigma_2 \otimes \hat{1} & 0 \\ & 1 \otimes \hat{\sigma}^{ij} & \sigma_3 \otimes \hat{\alpha}_i & \\ & & 0 & \end{pmatrix}, \quad 1 \leq i \leq d-1, \; d, \tag{24}$$

where the rows' labels are given and the column labels follow the same order (the elements below the diagonal are minus the transpose of those above).

In general, by investigating the general structure of all supersymmetric terms, one can also show all Dirac-supersymmetric interactions have the form

$$Q = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \hat{q}, \tag{25}$$

$$Q^\dagger = \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes \hat{q}^\dagger, \tag{26}$$

where $\hat{q}$ represents any interaction in the $d-2$ space. With this characterization one may proceed from the transformation that departs from Eq. (1), goes through H' in Eq. (3) and leads to Eq. (10), an expression containing of $Q + Q^\dagger$ and λ, and use the preceding section to select interactions that allow for dimensional separation.

5. $U(1)$ magnetic field as Dirac-supersymmetric interaction

A magnetic field derived from a general $U(1)$ interaction satisfies both conditions of Dirac supersymmetry and

separability. It is possible to describe it in any dimension in terms of an expression for $\boldsymbol{\pi}$ of the form of Eq. (5).

5.1. $3+1 \to 1+1$ reduction

In the $3+1$ dimensional case a constant magnetic field along the $\hat{z}$ direction

$$\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B} \tag{27}$$

corresponds to the real elements in Eq. (4) $\mathsf{A}_I = eA_I$, $I = 1,2,3$, where e is the $U(1)$ coupling constant (corresponding to the electric charge) and the $\mathbf{A}$ components are

$$A_1{=}A_x{=}-\frac{1}{2}yB, \quad A_2{=}A_y{=}\frac{1}{2}xB, \quad A_3{=}A_z{=}0. \tag{28}$$

The Hamiltonian H' (Eq. (3)) results in

$$H' = \alpha_0[(\boldsymbol{\alpha}\cdot\boldsymbol{\pi})^2 + \mu^2]^{1/2}. \tag{29}$$

That the magnetic field interaction chosen in Eq. (28) separates in the sense of Eqs. (16) and (17) can be seen from the construction of H' in Eq. (29). H' is composed from a term multiplying the unit matrix and the term

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\pi})^2 = \alpha_i\alpha_j\pi_i\pi_j \tag{30}$$

$$= (\frac{1}{2}\{\alpha_i,\alpha_j\} + \frac{1}{2}[\alpha_i,\alpha_j])\pi_i\pi_j \tag{31}$$

$$= \boldsymbol{\pi}\cdot\boldsymbol{\pi} + i\sigma_{ij}\pi_i\pi_j, \tag{32}$$

where use has been made of Eqs. (18) and (24). The term

$$i\sigma_{ij}\pi_i\pi_j = i1 \otimes \hat{\sigma}_{ij}\pi_i\pi_j, \tag{33}$$

where Eq. (24) has been used, gives rise to a separable equation, which will be shown explicitly in the Appendix.

Squaring of H' of Eq. (29) in Eq. (11) and further reduction lead to the equation

$$(\mathbf{p}\cdot\mathbf{p} - 2ie\mathbf{A}\cdot\boldsymbol{\nabla} + e^2\mathbf{A}\cdot\mathbf{A} + 2eS_zB)\Psi = E^2\Psi, \tag{34}$$

where $S_z = (i/2)\gamma_1\gamma_2$ is the spin along $\hat{\mathbf{z}}$. This equation can also be written in terms of the orbital angular momentum z-component $L_z = -i(\partial_y x - \partial_x y)$ as

$$[-\boldsymbol{\nabla}^2 + \frac{1}{4}e^2B^2\rho^2 + eB(L_z + 2S_z)]\Psi = E^2\Psi, \tag{35}$$

where $\rho^2 = x^2 + y^2$ is the radial cylindrical coordinate. As can be seen in the Appendix, the separation of this equation is manifest within these coordinates. This equation has the well-known non-relativistic (nr) Schrödinger-equation counterpart of a scalar particle in a magnetic field of magnitude B [9]

$$\frac{1}{2\mu}(-\boldsymbol{\nabla}^2 + \frac{1}{4}e^2B^2\rho^2 + eBL_z)\Psi_{nr} = E_{nr}\Psi_{nr}. \tag{36}$$

When Ψ_{nr} is separated into cylindrical coordinates $\Psi_{nr} = u(\rho)e^{ik_zz}e^{im\phi}$ the radial component $u(\rho)$ satisfies

$$u'' + \frac{1}{\rho}u' - \frac{m^2}{\rho^2}u - \frac{e^2B^2}{4}\rho^2 u + \left(2\mu E_{nr} - k_z^2 - eBm\right)u = 0, \tag{37}$$

with corresponding energy

$$E_{nr} = \frac{k_z^2}{2\mu} + \frac{eB}{2\mu}(2n_r + 1 + |m| + m), \qquad n_r = 0,1,\ldots, \tag{38}$$

where m, n_r are quantum numbers related to L_z and the radial motion, respectively. The Landau levels emerge not surprisingly.

The solution of the relativistic Eq. (35) follows from Eq. (36), which differs from the former, up to factors, by the spin operator. Eq. (35) constitutes only a necessary condition and has in fact more freedom in the solutions than the original Eq. (11) (albeit the energies are the same, for the latter equation is obtained from the modified Foldy-Wouthuysen unitary transformation). The eigenfunctions from the original equations are worked out in the Appendix. The results can be obtained by using the eigenfunctions of the total angular momentum component $L_z + S_z$, which leads to equations of the form of the massless ($\mu = 0$) (A.7)-(A.8) or massive (A.25)-(A.28). The energy eigenvalues are

$$E = \sqrt{k_z^2 + 2eB(n_r + m + 1) + \mu^2}, \qquad m \geq 0 \qquad n_r = 0,1,\ldots \tag{39}$$

$$E = \sqrt{k_z^2 + 2eBn_r + \mu^2}, \quad m < 0 \qquad n_r = 1,2,\ldots \tag{40}$$

5.2. $5+1 \to 3+1$ reduction

The procedure we have followed in Eqs. (29)-(34) is valid for any dimensional reduction. In the 5+1 case

$$A_1 = A_x = 0, \quad A_2 = A_3 = 0, \quad A_3 = A_z = 0,$$
$$A_4 = A_u = -\frac{1}{2}vB, \qquad A_5 = A_v = \frac{1}{2}uB, \tag{41}$$

where we have chosen 5-d coordinate labels $(u_1,u_2,u_3,u_4,u_5) = (x,y,z,u,v)$. Generalized terms π_I are then obtained from Eq. (5), using the corresponding A_I terms defined above and the $\mathsf{E}_I = 0$. The Hamiltonian H' [Eq. (3)] results in

$$H' = \alpha_0[(\boldsymbol{\alpha}\cdot\boldsymbol{\pi})^2 + (\alpha_I\pi_I)^2 + \mu^2]^{1/2} \qquad (I \text{ summed over } 4,5). \tag{42}$$

The $5+1 \to 3+1$ case equation, counterpart to Eq. (34), with the magnetic field in Eq. (41) has the form

$$[-\boldsymbol{\nabla}^2 - \partial_u^2 - \partial_v^2 + \frac{1}{4}e^2B^2\rho^2 + eB(L_{45} + 2S_{45})] \times\Psi = E^2\Psi, \tag{43}$$

where now

$$\rho^2 = u^2 + v^2,$$
$$L_{45} = -i(\partial_v u - \partial_u v),$$
$$S_{45} = \frac{i}{2}\gamma_4\gamma_5,$$

and the spectrum has a similar form to Eqs. (39)-(40), with the transformation $k_z^2 \rightarrow \mathbf{k}^2$, $\mathbf{k}$ representing the 3-d momentum, which gives a contribution to the energy as a free-particle kinetic term. This is but a consequence of the *decoupling* caused by the fact that the interaction acts purely on the u, v dimensions but leaves the other free. This in turn results from the form of the interaction which separates the equations corresponding to Eqs. (13) and (14).

Thus, we obtain a tower of states, similarly to the Kaluza-Klein mechanism, but with different associated radii (up to degeneracy), the minimum being occupied by the ground state.

The form of Eqs. (39)-(40) suggests that the magnetic field extra-dimensional parameter may be interpreted as a mass term in "real" dimensions. Therefore, we note that a mass-creation mechanism emerges here, with the masses appearing with a characteristic spectrum. This mechanism is possible only in a reduction from even d to $d-2$ dimensions.

Although other Dirac-supersymmetric interactions lead to compactification, they elude a simple solution treatment as obtained with the magnetic field.

6. Conclusions

In this work we have presented a mechanism for compactification through gauge fields. This mechanism allows for independent behavior in some dimensions but forces motion of particles in the other dimensions to be confined, which amounts to an effective compactification (driven by a physical process). Simplicity and succinctness are gained for compactification can be ascribed to a field rather than being assumed. Also, the familiar gauge fields can produce this mechanism, without need to invoke others. Although in this work we have concentrated on the didactic $3+1 \rightarrow 1+1$ and novel $5+1 \rightarrow 3+1$ cases, this mechanism is applicable to any even-dimension reduction $d+2 \rightarrow d$. In addition, this mechanism is general in the sense that it is valid for fundamental spin-1/2 matter fields. This mechanism is generic in the sense that a homogeneous field with a constant direction in space will lead to the same confining effect, and free behavior in the parallel direction, albeit the orbits may be different from the Landau ones.

The results obtained are general for the compactification presented can be relevant both independently of or in relation to curved space. In the latter case, it is assumed that these interactions could be eventually related to the gravitational field acting in the additional dimensions.

However, this work remains exploratory for it concentrates more in showing such a mechanism is possible and one still needs further generalization to relate it to a universal interaction. Fields generated by the hypercharge interaction together with non-abelian ones as the electroweak interaction or gravitational fields are feasible candidates. Some ideas on possible constraints on allowed dimensions and interactions are found in Ref. 10. In addition, there is room for additional types of interaction producing compactificaction. The $U(1)$ magnetic field chosen here preserves translational symmetry which implies the choice of coordinate around which particles rotate in the extra dimensions is arbitrary. Further characterization of the interaction may lead, for example, to a choice of this point in the compactification plane, which would violate Poincaré invariance in the extra dimensions, without direct influence in the "real" ones, just as occurs for branes in string theory.

Another outcome of this work is a possible mass-creation mechanism. We have shown that a mass of a free particle in 4-d space can be generated through an interaction acting on 5 and 6 dimensions.

The main lesson from this paper is that it is possible to construct a compactifying interaction which features dimensional decoupling, at least for the "real" dimensions, and which may have consequences in terms of parameters as the mass, but otherwise leave the same physics for free physical particles. It should be interesting to consider the presence of interactions inside "real" space.

Further work should then deal with non-abelian fields, consider other separable interactions, additional multipoles of the magnetic field, and gravitation, and try to relate them to cosmological models, for these fields should appear self-consistently.

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Appendix

In this Appendix we solve directly Dirac's equation $H\Psi = E\Psi$ for a particle in a constant magnetic field in the 4-d case as a complement to Eqs. (35) and (43), for the massless and massive cases.

Massless case

We use the chiral representation for the Dirac matrices. Then, the $3+1$ component of the Hamiltonian in Eq. (4) leads to

$$\begin{pmatrix} \boldsymbol{\sigma}\cdot(-i\boldsymbol{\nabla}-\mathbf{A}) & 0 \\ 0 & -\boldsymbol{\sigma}\cdot(-i\boldsymbol{\nabla}-\mathbf{A}) \end{pmatrix}\Psi = E\Psi, \quad \text{(A.1)}$$

where, from Eq. (28), a constant magnetic along $\hat{\mathbf{z}}$ is given by $\mathbf{A} = (1/2)B(-y, x, 0)$. We use the constants of the motion to obtain and classify the solutions. These comprise a

component of the total angular momentum, the momentum, both in the $\hat{\mathbf{z}}$ direction, and, of course, the Hamiltonian. In the massless case the chirality is also a constant of the motion and we use it to separate Eq. (A.1) into its chiral components. From the form of γ_5 in the chiral representation

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{A.2}$$

the upper and lower parts of Ψ

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{A.3}$$

correspond to its positive and negative chirality components, respectively. We now consider the upper, positive chirality part while it is clear that the other part is obtained with the solution interchange $E \to -E$. Given the constants of the motion we propose as Anzats for Ψ_1, using cylindrical coordinates (ρ, ϕ, z), separation into a plane wave along the $\hat{\mathbf{z}}$ direction with momentum k_z, a J_z eigenstate with associated angular momentum $m + 1/2$, and radial functions $f(\rho)$, $g(\rho)$ along the xy plane

$$\Psi_1 = \begin{pmatrix} f(\rho)e^{ik_z z}e^{im\phi} \\ ig(\rho)e^{ik_z z}e^{i(m+1)\phi} \end{pmatrix}. \tag{A.4}$$

With this form for Ψ_1 Eq. (A.1) becomes

$$k_z f + g' + \frac{m+1}{\rho}g + e\frac{B}{2}\rho g = Ef \tag{A.5}$$

$$-k_z g - f' + \frac{m}{\rho}f + e\frac{B}{2}\rho f = Eg. \tag{A.6}$$

From Eq. (A.5) f can be expressed in terms g and its derivative. By substituting this f into Eq. (A.6), and carrying out a similar procedure for g from (A.6), one obtains the decoupled equations

$$f'' + \frac{1}{\rho}f' - \frac{m^2}{\rho^2}f - \frac{e^2B^2}{4}\rho^2 f + \left(E^2 - k_z^2 - eB(m+1)\right) f = 0, \tag{A.7}$$

$$g'' + \frac{1}{\rho}g' - \frac{(m+1)^2}{\rho^2}g - \frac{e^2B^2}{4}\rho^2 g + \left(E^2 - k_z^2 - eBm\right) g = 0. \tag{A.8}$$

The solution of these equations is constructed with generalized Laguerre polynomials of the form

$$L_{n_r}^{|m|}(x^2), \quad x = \sqrt{eB/2}\rho,$$

and each solution leads to the following eigenfunctions and energy eigenvalues [9] (see also Eqs. (37)-(38)): For $m+1 \geq 0$

$$f(\rho) = x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2)$$

$$g(\rho) = \frac{\sqrt{2eB}}{E + k_z}x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2), \qquad n_r = 0, 1, \ldots, \tag{A.9}$$

with energy

$$E = \sqrt{k_z^2 + 2eB(n_r + m + 1)}. \tag{A.10}$$

The global coefficients in Ψ_1 above and the wave functions below are arbitrary. To normalize the wave function one uses the cylindrical radial-component integral

$$\int_0^\infty dy y^m e^{-y} L_n^m(y) L_{n'}^m(y) = \begin{cases} 0 & n \neq n' \\ \Gamma(1+m)\begin{pmatrix} n+m \\ n \end{pmatrix} & n = n'. \end{cases} \tag{A.11}$$

For $m + 1 < 0$

$$f(\rho) = x^{|m|}e^{-x^2/2}L_{n_r-1}^{|m|}(x^2)$$

$$g(\rho) = -\frac{\sqrt{2eB}}{E + k_z}n_r x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2), \qquad n_r = 1, 2, \ldots, \tag{A.12}$$

with energy

$$E = \sqrt{k_z^2 + 2eBn_r} \tag{A.13}$$

We see that in the latter case we have an increased degeneracy on the m values which we ascribe to the cancelling contributions to the energy of the angular motion and its magnetic moment opposite to the magnetic field. We give here the solutions in more detail than in Refs. 11, and 12 in particular for states with negative $m+1$. An additional polarization operator exists that commutes with the Hamiltonian, but it is not different from the chirality in the massless case (see below).

The negative chirality component Ψ_2

$$\Psi_2 = \begin{pmatrix} h(\rho)e^{ik_z z}e^{im\phi} \\ ij(\rho)e^{ik_z z}e^{i(m+1)\phi} \end{pmatrix} \tag{A.14}$$

has the solutions for $m + 1 \geq 0$

$$h(\rho) = x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2),$$

$$j(\rho) = \frac{\sqrt{2eB}}{k_z - E}x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2), \qquad n_r = 0, 1, \ldots, \tag{A.15}$$

with energy

$$E = \sqrt{k_z^2 + 2eB(n_r + m + 1)}. \tag{A.16}$$

and for $m + 1 < 0$

$$h(\rho) = x^{|m|}e^{-x^2/2}L_{n_r-1}^{|m|}(x^2)$$

$$j(\rho) = -\frac{\sqrt{2eB}}{k_z - E}n_r x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2), \qquad n_r = 1, 2, \ldots, \tag{A.17}$$

with energy

$$E = \sqrt{k_z^2 + 2eBn_r} \tag{A.18}$$

Massive case

The massive equation

$$\left[\begin{pmatrix} \boldsymbol{\sigma}\cdot(-i\boldsymbol{\nabla}-\mathbf{A}) & 0 \\ 0 & -\boldsymbol{\sigma}\cdot(-i\boldsymbol{\nabla}-\mathbf{A}) \end{pmatrix} + \mu\beta\right] \times\Psi = E\Psi. \quad \text{(A.19)}$$

can be solved using the same quantum numbers except that the mass term (in the chiral representation)

$$\beta=\gamma_0=\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad \text{(A.20)}$$

mixes the chirality components included in Ψ: Ψ_1 in Eq. (A.4) and Ψ_2 in Eq. (A.4), which leads to the slightly more complicated four coupled radial equations for f, g, h, j

$$k_z f + g' + \frac{m+1}{\rho}g + e\frac{B}{2}\rho g - \mu h = Ef, \quad \text{(A.21)}$$

$$-k_z g - f' + \frac{m}{\rho}f + e\frac{B}{2}\rho f - \mu j = Eg, \quad \text{(A.22)}$$

$$-k_z h - j' - \frac{m+1}{\rho}j - e\frac{B}{2}\rho j - \mu f = Eh, \quad \text{(A.23)}$$

$$k_z j + h' - \frac{m}{\rho}h - e\frac{B}{2}\rho h - \mu g = Ej. \quad \text{(A.24)}$$

These equations decouple into four equations for each function. For example, by solving for g in Eq. (A.22) and for h in Eq. (A.23) and substituting them into Eq. (A.21) an equation only for f is obtained; by using similar procedures for the other functions one gets for all

$$f'' + \frac{1}{\rho}f' - \frac{m^2}{\rho^2}f - \frac{e^2B^2}{4}\rho^2 f + \left(E^2 - k_z^2 - eB(m+1) - \mu^2\right) f = 0, \quad \text{(A.25)}$$

$$g'' + \frac{1}{\rho}g' - \frac{(m+1)^2}{\rho^2}g - \frac{e^2B^2}{4}\rho^2 g + \left(E^2 - k_z^2 - eBm - \mu^2\right) g = 0, \quad \text{(A.26)}$$

$$h'' + \frac{1}{\rho}h' - \frac{m^2}{\rho^2}h - \frac{e^2B^2}{4}\rho^2 h + \left(E^2 - k_z^2 - eB(m+1) - \mu^2\right) h = 0, \quad \text{(A.27)}$$

$$j'' + \frac{1}{\rho}j' - \frac{(m+1)^2}{\rho^2}j - \frac{e^2B^2}{4}\rho^2 j + \left(E^2 - k_z^2 - eBm - \mu^2\right) j = 0. \quad \text{(A.28)}$$

To further classify the solutions we choose the additional operator commuting with the Hamiltonian [12]

$$\frac{1}{\mu}\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}, \quad \boldsymbol{\pi} = -i\boldsymbol{\nabla} + e\mathbf{A}, \quad \boldsymbol{\Sigma} = \gamma_5\gamma_0\boldsymbol{\gamma},$$

which can be calculated from $\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}\Psi=(E\gamma_5-\mu\gamma_5\gamma_0)\Psi$ and eigenvalues $\sqrt{E^2-\mu^2}\Sigma$.

The solutions for $m+1\geq 0$, with $\Sigma=1$ are

$$f(\rho) = x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2), \qquad g(\rho)=\frac{\sqrt{2B\,e}}{k_z+\sqrt{E^2-\mu^2}}x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2),$$

$$h(\rho)=\frac{-E+\sqrt{E^2-\mu^2}}{\mu}x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2),\ j(\rho)=\frac{\sqrt{2Be}\left(-E+\sqrt{E^2-\mu^2}\right)}{\mu\left(k_z+\sqrt{E^2-\mu^2}\right)}x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2),\ n_r=0,1,\ldots, \quad \text{(A.29)}$$

while for $\Sigma=-1$

$$f(\rho) = x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2), \qquad g(\rho) = \frac{\sqrt{2B\,e}}{k_z-\sqrt{E^2-\mu^2}}x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2),$$

$$h(\rho)=-\frac{E+\sqrt{E^2-\mu^2}}{\mu}x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2), \qquad j(\rho)=\frac{\sqrt{2Be}\left(E+\sqrt{E^2-\mu^2}\right)}{\mu\left(-k_z+\sqrt{E^2-\mu^2}\right)}x^{|m|}e^{-x^2/2}L_{n_r}^{|m|}(x^2),\ n_r=0,1,\ldots \quad \text{(A.30)}$$

For $m+1<0$, $\Sigma=1$

$$f(\rho) = x^{|m|}e^{-x^2/2}L_{n_r-1}^{|m|}(x^2),$$

$$g(\rho) = \frac{k_z-\sqrt{E^2-\mu^2}}{\sqrt{2B\,e}}x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2), \qquad h(\rho) = \frac{-E+\sqrt{E^2-\mu^2}}{\mu}x^{|m|}e^{-x^2/2}L_{n_r-1}^{|m|}(x^2),$$

$$j(\rho) = \frac{-E^2+\mu^2+k_z\sqrt{E^2-\mu^2}+E\left(-k_z+\sqrt{E^2-\mu^2}\right)}{\sqrt{2B\,e}\,\mu}x^{|m+1|}e^{-x^2/2}L_{n_r}^{|m+1|}(x^2),\ n_r=1,2,\ldots, \quad \text{(A.31)}$$

and for $\Sigma = -1$

$$f(\rho) = x^{|m|} e^{-x^2/2} L_{n_r-1}^{|m|}(x^2),$$

$$g(\rho) = \frac{k_z + \sqrt{E^2-\mu^2}}{\sqrt{2B\,e}} x^{|m+1|} e^{-x^2/2} L_{n_r}^{|m+1|}(x^2),$$

$$h(\rho) = \frac{-E - \sqrt{E^2-\mu^2}}{\mu} x^{|m|} e^{-x^2/2} L_{n_r-1}^{|m|}(x^2),$$

$$j(\rho) = -\frac{E^2-\mu^2 + k_z\sqrt{E^2-\mu^2} + E\left(k_z + \sqrt{E^2-\mu^2}\right)}{\sqrt{2B\,e}\,\mu} x^{|m+1|} e^{-x^2/2} L_{n_r}^{|m+1|}(x^2), \quad n_r = 1, 2, \ldots, \tag{A.32}$$

with energies as above, except that these are modified with a mass term, and they are given in Eqs. (39) and (40).

This 4-d massive case can also serve to solve the 6-d massless equation with a generalized magnetic field. Indeed, we divide the extended Dirac Hamiltonian equation into the 3- and 4- and 5-d components

$$(\boldsymbol{\alpha}\cdot\mathbf{p} + \alpha_I \pi_I)\Psi = E\Psi \quad (I \text{ summed over } 4, 5), \tag{A.33}$$

where $\pi_I = p_I + \mathsf{A}_I$, and the A_I are obtained from Eq. (41). The 3-d space components appear with bold type. This massless equation can be projected into the two chiral components by $(1/2)(1 \pm \gamma_7)$, with $\gamma_7 = -i\alpha_0\alpha_6$ [Eqs. (19), (21)] being the matrix which anticommutes with all 6-d $\gamma_\mu{}'s$. Each projected equation can be written in terms of 4-d matrices. As these satisfy the same relations as those contained in Eq. (A.19), with the mapping 6-$d \to$ 4-d with

$$\frac{1}{2}(1+\gamma_7)\boldsymbol{\alpha}\cdot\mathbf{p} \to \boldsymbol{\alpha}\cdot\boldsymbol{\pi}, \quad \frac{1}{2}(1+\gamma_7)\alpha_I\pi_I \to \mu\beta,$$

one obtains the corresponding set of equations as (A.21)-(A.24) (and similarly for the other chirality part).

The equations solved here can also be useful to solve the intermediate Eq. (12), and reproduce in fact the separation of variables as described in Sec. 3.

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