

The Lagrangians of a one-dimensional mechanical system

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Starting from the fact that for an arbitrary autonomous mechanical system any constant of motion can be used as Hamiltonian, the expression for the Lagrangians of a one-dimensional mechanical system previously found by other authors is derived.

Keywords: Lagrangians; Hamilton equations.

Partiendo del hecho de que para un sistema mecánico autónomo arbitrario cualquier constante de movimiento puede usarse como hamiltoniana, se deduce la expresión para las lagrangianas de un sistema mecánico unidimensional previamente hallada por otros autores.

Descriptores: Lagrangianas; ecuaciones de Hamilton.

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1. Introduction

For a given one-dimensional autonomous dynamical system in classical mechanics, there are infinitely many Lagrangians; in fact, from each constant of motion of the system a Lagrangian can be obtained. Specifically, if $K(q, \dot{q})$ is a constant of motion of the system, a Lagrangian is explicitly given by [1-4]

$$L(q, \dot{q}) = \dot{q} \int \frac{K(q, y)}{y^2} dy \quad (1)$$

and, as can be readily verified, the corresponding Hamiltonian is $H(q, p) = K(q, \dot{q}(q, p))$, assuming that the relation between the canonical momentum and \dot{q} can be inverted, *i.e.*, the constant of motion employed in the construction of the Lagrangian is essentially the Hamiltonian.

In a recent paper [5] a somewhat similar result has been established. For an autonomous system with n degrees of freedom, and forces derivable from a potential, any constant of motion can be employed as Hamiltonian of the system, provided that the Poisson bracket is suitably defined, and if $n > 1$, there are infinitely many suitable Poisson brackets for each Hamiltonian. The aim of this paper is to show that in the case where $n = 1$ these two approaches are equivalent to each other; following the procedure given in Ref. 5 we derive Eq. (1).

Expression (1) is also closely related to the results of Ref. 6, where it is shown that for a one-dimensional, possibly time-dependent, mechanical system, two Lagrangians yield the same equations of motion if and only if they are related by means of a constant of motion (see Eq. (2) below). One can easily derive the main result of Ref. 6, in the case of an autonomous system. Indeed, a straightforward computation using Eq. (1) yields

$$\mathcal{E}(L) \equiv \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{\partial K}{\partial q} + \frac{\dot{q}}{\dot{q}} \frac{\partial K}{\partial \dot{q}} = \frac{1}{\dot{q}} \frac{dK}{dt},$$

which is equal to zero if K is a constant of motion. A second Lagrangian, L' , will also have the form (1) with K replaced

by another constant of motion, K' , which must be some function of K , $K' = f(K)$, since for a mechanical system with one degree of freedom there is only one functionally independent constant of motion; hence

$$\mathcal{E}(L') = \frac{1}{\dot{q}} \frac{dK'}{dt} = \frac{1}{\dot{q}} \frac{df(K)}{dK} \frac{dK}{dt} = \frac{df(K)}{dK} \mathcal{E}(L), \quad (2)$$

the factor $df(K)/dK$, being a function of K only, is a constant of motion. (Note also that the relation between L and L' that follows from Eq. (1) is much simpler than that found in Ref. 6.)

In Sec. 2 we summarize the results of Ref. 5, applying them to the specific case of a one-dimensional mechanical system. In Sec. 3 we show that in the case of a one-dimensional system, a simple expression for the momentum canonically conjugate to a given coordinate can be obtained, which allows us to find explicitly the Lagrangian corresponding to a given Hamiltonian. A simple example is given in the Appendix which, at the same time, shows that the formalism is also applicable to dissipative systems (see also Ref. 7).

2. Hamiltonians and Poisson brackets

The Hamilton equations expressed in terms of canonical coordinates q, p for a one-dimensional mechanical system are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad (3)$$

where H is the Hamiltonian function. Instead of canonical coordinates we can employ arbitrary coordinates x^1, x^2 , in terms of which the Hamilton equations are given by [see Eqs. (3)]

$$\begin{aligned} \frac{dx^\mu}{dt} &= \frac{\partial x^\mu}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial x^\mu}{\partial p} \frac{\partial H}{\partial q} \\ &= \left(\frac{\partial x^\mu}{\partial q} \frac{\partial x^\nu}{\partial p} - \frac{\partial x^\mu}{\partial p} \frac{\partial x^\nu}{\partial q} \right) \frac{\partial H}{\partial x^\nu} = \{x^\mu, x^\nu\} \frac{\partial H}{\partial x^\nu} \end{aligned} \quad (4)$$

$(\mu, \nu, \dots = 1, 2)$, where $\{, \}$ is the Poisson bracket. Letting $\sigma \equiv \{x^1, x^2\}$, the Hamilton equations in an arbitrary coordinate system (4) amount to

$$\frac{dx^1}{dt} = \sigma \frac{\partial H}{\partial x^2}, \quad \frac{dx^2}{dt} = -\sigma \frac{\partial H}{\partial x^1}. \tag{5}$$

The equations of motion of a given one-dimensional autonomous mechanical system have the form

$$\frac{dx^1}{dt} = f(x^1, x^2), \quad \frac{dx^2}{dt} = g(x^1, x^2), \tag{6}$$

where f and g are some functions of two variables only. As shown in Ref. 5, we can take H in Eqs. (5) as any constant of motion and then look for the function σ that satisfies Eqs. (5). In fact, according to Eqs. (6), a function $H(x^1, x^2)$ is a constant of motion if and only if

$$0 = \frac{\partial H}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial H}{\partial x^2} \frac{dx^2}{dt} = f \frac{\partial H}{\partial x^1} + g \frac{\partial H}{\partial x^2}, \tag{7}$$

while from Eqs. (5) and (6) we find that σ must be such that

$$\sigma \frac{\partial H}{\partial x^2} = f, \quad \sigma \frac{\partial H}{\partial x^1} = -g. \tag{8}$$

These equations for σ are compatible as a consequence of Eq. (7).

3. Canonical coordinates and Lagrangians

Starting from arbitrary coordinates x^1, x^2 in the phase space we can look for a system of canonical coordinates formed by x^1 , and another coordinate p_1 to be determined. Thus, we require that $\{x^1, p_1\} = 1$ (i.e., p_1 will be a momentum canonically conjugate to x^1). From

$$1 = \{x^1, p_1\} = \{x^1, x^2\} \frac{\partial p_1}{\partial x^2} = \sigma \frac{\partial p_1}{\partial x^2}$$

it follows that

$$p_1 = \int_{x^1}^{x^2} \frac{dy}{\sigma(x^1, y)}, \tag{9}$$

which gives the canonical momentum p_1 as a function of x^1, x^2 . If $\partial f / \partial x^2 \neq 0$ then the first equation in (6) can be inverted, at least locally, to give x^2 as a function of x^1 and \dot{x}^1 .

The Lagrangian corresponding to a Hamiltonian H is given by the usual expression $L(x^1, \dot{x}^1) = p_1 \dot{x}^1 - H$; therefore, making use of Eq. (9), the first equation in (8) and inte-

grating by parts, we obtain

$$\begin{aligned} L(x^1, \dot{x}^1) &= \dot{x}^1 \int_{x^1}^{x^2(x^1, \dot{x}^1)} \frac{dy}{\sigma(x^1, y)} - H \\ &= \dot{x}^1 \int_{x^1}^{x^2(x^1, \dot{x}^1)} \frac{1}{f(x^1, y)} \frac{\partial H(x^1, y)}{\partial y} dy - H \\ &= \dot{x}^1 \left[\frac{H(x^1, y)}{f(x^1, y)} \Big|_{x^1}^{x^2(x^1, \dot{x}^1)} \right. \\ &\quad \left. + \int_{x^1}^{x^2(x^1, \dot{x}^1)} \frac{H(x^1, y)}{(f(x^1, y))^2} \frac{\partial f(x^1, y)}{\partial y} dy \right] - H. \tag{10} \end{aligned}$$

Using again the first equation in (6), and a change of variable from Eq. (10) we obtain

$$L(x^1, \dot{x}^1) = \dot{x}^1 \int \frac{K(x^1, u)}{u^2} du, \tag{11}$$

where $K(x^1, u) \equiv H(x^1, x^2(x^1, u))$, which is just Eq. (1).

4. Concluding remarks

In the case of mechanical systems with a number of degrees of freedom greater than 1, the main obstacle to find an expression analogous to Eq. (1) following the procedure employed in this paper comes from the difficulty in finding an expression for the canonical momenta, analogous to Eq. (9); nevertheless, in each case where the Hamiltonian and the Poisson bracket have been chosen, canonical coordinates and the Lagrangian can be obtained, at least in principle (see the examples in Ref. 5).

An important point that perhaps needs to be stressed is that the evolution of a mechanical system in the phase space (or in the configuration space) does not depend on the coordinates employed to describe it. The different choices for the Hamiltonian or the Lagrangian of a given mechanical system lead to different definitions for the momenta canonically conjugate to the coordinates in the configuration space, but the curves traced by the evolution of the mechanical system in the phase space do not depend on these choices. Thus, it is erroneous to claim that, making use of two different Hamiltonians for the one-dimensional harmonic oscillator, “the dynamics must be different since the generalized momentum for the Hamiltonian H_2 is much more complex and has different units than the one related to H_1 ” (Ref. 4, Sec. 3). Basically, this error in Ref. 4 comes from calling always ‘ p ’ the momentum canonically conjugate to x , without realizing that with each Lagrangian one obtains a possibly different momentum conjugate to x .

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A Appendix

As a simple example we consider a particle subjected to a frictional force proportional to its velocity. If x^1 represents the position of the particle, $d^2x^1/dt^2 = -kdx^1/dt$, where k is a constant; hence, taking $x^2 \equiv dx^1/dt$, we have the system of equations

$$\frac{dx^1}{dt} = x^2, \quad \frac{dx^2}{dt} = -kx^2, \quad (\text{A.1})$$

which is of the form (6). Equations (A.1) imply that $H \equiv kx^1 + x^2$ is a constant of motion and Eqs. (8) yield

$\sigma = x^2$. Substituting into Eq. (9) it follows that we can take $p_1 = \ln x^2$, i.e., $x^2 = \exp p_1$. Then, the Hamiltonian H , expressed in terms of these canonical coordinates, is given by

$$H = kx^1 + \exp p_1.$$

Since $\dot{x}^1 = x^2 = \exp p_1$, the corresponding Lagrangian is

$$L(x^1, \dot{x}^1) = p_1 \dot{x}^1 - H = \dot{x}^1 \ln \dot{x}^1 - kx^1 - \dot{x}^1. \quad (\text{A.2})$$

On the other hand, substituting $K(x^1, \dot{x}^1) = kx^1 + \dot{x}^1$, which is obtained from H eliminating p_1 in favor of \dot{x}^1 , into Eq. (1) one obtains $L(x^1, \dot{x}^1) = -kx^1 + \dot{x}^1 \ln \dot{x}^1$, which differs from (A.2) by the total derivative with respect to time of a function of x^1 only and, therefore, yields the same equations of motion.

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