Higgs mass and grand unification

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The knowledge of the solution of the Renormalization Group Equation (RGE) for the quartic self-interaction of the Higgs scalar λ_H is crucial for the determination of the energy limits on the Higgs mass. λ_H is also important in order to solve the RGEs at the two loop level for other observables like the quark masses or the Cabibbo-Kobayashi-Maskawa (CKM) matrix. We obtain an analytical and numerical (considering Grand Unification) solution to the one loop renormalization group evolution of the Higgs quartic coupling λ_H in the energy range $[m_t, E_{GU}]$, where m_t is the mass of the top quark and $E_{GU} = 10^{14}$ GeV. We find that depending on the value of $\lambda_H(m_t)$ the solution for $\lambda_H(E)$ may have singularities or zeros and become negative, in which case the Standard Model (SM) becomes inconsistent. We obtain that for $0.65 \leq \lambda_H(m_t) \leq 0.69$ the SM is valid in the whole range $[m_t, E_{GU}]$. These values of $\lambda_H(m_t)$ correspond to the following Higgs mass $198 \leq m_H \leq 205$ GeV.

Keywords: Standard model; Higgs; quartic self-interaction; Riccati equation.

El conocimiento de la solución de la Ecuación del Grupo de Renormalización (EGR) correspondiente al acoplamiento cuártico del escalar de Higgs λ_H es crucial para la determinación de los límites de energía de la masa del Higgs. λ_H es también importante para resolver las EGR al nivel de dos lazos para otras observables tales como las masas de los quarks o la matriz de Cabibbo Kobayashi Maskawa (CKM) en las que juega un papel básico. Obtenemos una solución analítica y numérica (considerando Gran Unificación) para la evolución del grupo de renormalización a un lazo del acoplamiento cuártico de Higgs λ_H en el rango de energías $[m_t, E_{GU}]$ donde m_t es la masa del quark top y $E_{GU} = 10^{14}$ GeV. Encontramos que dependiendo del valor de $\lambda_H(m_t)$ la solución para $\lambda_H(E)$ puede contener singularidades o ceros y hasta puede tomar valores negativos en cuyo caso el Modelo Standard (MS) sería inconsistente. Obtenemos que para $0.65 \leq \lambda_H(m_t) \leq 0.69$ el MS es válido en el rango completo $[m_t, E_{GU}]$. Estos valores para $\lambda_H(m_t)$ corresponden a las siguientes masas del Higgs $198 \leq m_H \leq 205$ GeV.

Descriptores: Modelo standard; Higgs; auto-interacción cuártica; ecuación de Riccati.

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1. Introduction

The question about the mechanisms by which the fundamental particles get their masses is important and very basic to acquire an understanding of the nature of the constituents of matter and forces among them. The Standard Model (SM) is the gauge theory with the gauge group $SU(3) \times SU(2) \times U(1)$ that provides a very precise description of the microscopic interactions, and it is compatible with the present elementary particle data [1, 2]. Higgs found [3] that the parameters in the field equations (Lagrangian) associated with the scalar particle H can be chosen in such a way that in the lowest energy state of that field (in empty space) the value of the field is not zero and it is the so called non zero vacuum expectation value VEV. As the field is not zero in the empty space the particles that interact with it gain mass from that interaction. The Higgs boson H gives the mechanism by which the particles can acquire mass. To constrain these ideas more rigorously and to obtain very significant clues, it is important to find first hand evidence for the Higgs field. The Higgs is a hypothetical particle about whose properties we still have only vague hints, and speculations and its discovery is the most exciting prospect in contemporary particle physics. There are many unanswered questions about the Higgs. The most crucial one, which may guide the experimentalist to its discovery is related with the knowledge of the behavior of the quartic coupling λ_H through which the mass of the Higgs particle m_H is obtained. On the other hand, the idea of Grand Unification (GU) is to look for symmetries in the SM at very high energies. The most notable sign of the presence of GU is the (approximate) convergence of the three gauge couplings to one common value at the energies $10^{14} - 10^{15}$ GeV. [4]. The main tool of the GU models are the RGEs that relate various observables (like couplings or masses) at different energies, and also allow the study of their asymptotic behavior. The term “renormalization” means, together with the redefinition of the mass and coupling constant, the readjustment of the normalization of Green functions by suitable multiplicative factors which may eliminate possible infinities in the Green functions. The finite renormalization of the Green functions constitutes a group called the renormalization group. The physical predictions are invariant under the renormalization group transformations, and an analytic expression of this property is given by the renormalization group equation. The one loop RGEs for the best measured observables g_i's and the Cabibbo-Kobayashi-Maskawa (CKM) matrix are independent of the Higgs quartic coupling. This allowed to derive the running of those observables at the lowest order without the knowledge of the λ_H [5].

In this paper, using previous results for the gauge couplings and for the quark-top Yukawa coupling, we solve the one loop RGE equation [6] for the quartic Higgs coupling λ_H. The equation for the λ_H is of Riccati's type [7]. We show how to solve this equation explicitly. We find that the function $\lambda_H(E)$ has a Landau singularity for values of $\lambda_H(m_t) \geq 0.6915$. For the values of $\lambda_H(m_t) \leq 0.65016$ the solution $\lambda_H(E)$ passes through zero and becomes negative before reaching the E_{GU} energy, and the SM becomes invalid. As it is well known the coupling λ_H is related to the Higgs mass, so our results for λ_H are important for the determination of this mass. With the combination of the RGE with GU we obtain numerical and graphical results for the behavior of the $\lambda_H(E)$ which result in some bounds on the Higgs boson mass.

2. Quartic self-interaction of the Higgs scalar

The scalar part of the SM Lagrangian, after the spontaneous symmetry breaking, in terms of the physical states, contains among other terms: the diagonalized quark Yukawa couplings $Y_{u,d}$ of the up and down quarks, the quartic interaction λ_H of the Higgs scalar H, and a the term with μ which is connected to the (tree-level) Higgs mass:

$$-\mathcal{L} = \ldots + \frac{v}{\sqrt{2}}\overline{u}_L Y_u\ u_R + \frac{v}{\sqrt{2}}\overline{d}_L Y_d\ d_R + \frac{1}{2}\mu^2 H^2 + \frac{1}{8}\lambda_H H^4. \quad (1)$$

The relation between μ and the Higgs mass is: $m_H = \sqrt{-2\mu^2} = \sqrt{2\lambda_H}v$ where v is the vacuum expectation value of the Higgs field [2] (see Appendix A).

In previous papers we have discussed a consistent approximation scheme for the solution of the RGEs that was based on the expansion of the solutions in terms of the powers of λ, where $\lambda \simeq 0.22$ is the absolute value of the $|V_{us}|$ element of the CKM matrix.

In such an approximation, we have obtained the energy dependence in terms of $t = \ln(E/m_t)$ of the gauge couplings $g_i(t)$

$$g_i^2(t) = \frac{g_i^2(t_0)}{1 - \frac{2}{(4\pi)^2} g_i^2(t_0) b_i (t - t_0)}, \quad (2)$$

$$(b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7\right),$$

and the RGE running of the Y_t which is the largest eigenvalue of the up quark Yukata coupling matrix

$$Y_t^2(t) = \frac{Y_t^2(t_0)}{1 - \frac{2\left(\alpha_2^u + \alpha_3^u\right)}{(4\pi)^2} (Y_t^0)^2 \int_{t_0}^{t} r(\tau) d\tau} \cdot \prod_{k=1}^{k=3} \left[\frac{g_k^2(t_0)}{g_k^2(t)}\right]^{\frac{c_k}{b_k}}, \quad (3)$$

where

$$\alpha_2^u = \frac{3}{2}, \quad \alpha_3^u = 3, \quad c_k = (17/20, 9/4, 8).$$

Also, in the same approximation the lowest order RGE for the λ_H, given in Ref. 6, has the form of a Riccati differential equation

$$\frac{d\lambda_H}{dt} = \frac{12}{(4\pi)^2} \left\{ \lambda_H^2 + \left[|Y_t|^2 - \frac{3}{4}\left(\frac{1}{5}g_1^2 + g_2^2\right)\right] \lambda_H + \frac{3}{16}\left(\frac{3}{25}g_1^4 + \frac{2}{5}g_1^2 g_2^2 + g_2^4\right) - |Y_t|^4 \right\}. \quad (4)$$

The behavior of the gauge coupling g_i's is given in Eq. (2) and the explicit energy dependence of $Y_t(t)$ is given in Eq. (3). The g_i's and $Y_t(t)$ as functions of energy have no singularities in the range $[m_t, E_{GU}]$. The solutions of the Riccati's equations can become singular even if the coefficients of the equation are smooth regular functions of the energy.

3. Solution of the Riccati equation

The solution of Eq. (4) is obtained using the substitution of λ_H by the following expression containing the auxiliary function W

$$\lambda_H(t) = -\frac{4\pi^2}{3} \frac{W'(t)}{W(t)} \quad (5)$$

which fulfills the linear second order differential equation

$$W'' + p(t) W' + q(t) W = 0. \quad (6)$$

where

$$p(t) = -\frac{12}{(4\pi)^2} \left[|Y_t(t)|^2 - \frac{3}{4}\left(\frac{1}{5}g_1^2(t) + g_2^2(t)\right)\right], \quad (7)$$

$$q(t) = \left(\frac{12}{(4\pi)^2}\right)^2 \times \left[\frac{3}{16}\left(\frac{3}{25}g_1^4(t) + \frac{2}{5}g_1^2(t) g_2^2(t) + g_2^4(t)\right) - |Y_t(t)|^4\right]. \quad (8)$$

Any solution of Eq. (6) generates the solutions of Eq. (4). Eq. (6) is of the Frobenius type [8], and the solution $W(t)$ is a *regular* function of t in the region where the coefficients of Eq. (6) are *regular*. One can look for the solutions of this equation in terms of an infinite series (see Appendix B). We look for the two independent solutions of this equation with the following properties

$$W_1(t_0) = 1, \quad W_1'(t_0) = 0,$$
$$W_2(t_0) = 0, \quad W_2'(t_0) = 1. \quad (9)$$

The solution for λ_H in terms of the functions $W_1(t)$ and $W_2(t)$ has the following form

$$\lambda_H(t) = -\frac{(4\pi)^2}{12} \frac{W_1'(t) - \frac{12}{(4\pi)^2}\lambda_H(t_0) W_2'(t)}{W_1(t) - \frac{12}{(4\pi)^2}\lambda_H(t_0) W_2(t)}, \quad (10)$$

The most important property of the solution (10) is that the singularities (Landau poles) and the zeros of the solution $\lambda_H(t)$ are determined from the zeros of the denominator and numerator, respectively. One is able to determine precisely the position of the singularities and zeros and their dependence on the initial value of the Higgs quartic coupling $\lambda_H(t_0)$, where $t_0 = \ln(E/m_t)_{E=m_t} = 0$.

The form of the functions $p(t)$ and $q(t)$ is too complicated to be able to solve Eq. (6) in terms of simple functions. To find the solution of this equation we approximate $p(t)$ and $q(t)$ in the range of energies $[m_t, E_{GU}]$ by polynomials of 4-th order and 6-th order in t (see Appendix B). These polynomials approximate perfectly both functions in the whole range of energies and this allows us to find the solution of Eq. (6) as a power series of t .

From Eq. (10) we find the dependence of $\lambda_H(t)$ on the energy and important properties of its behavior. It is very interesting to investigate how $\lambda_H(t)$ depends on the initial values of $\lambda_H(t_0)$ and to find out the range of the validity of the SM. As discussed earlier for the SM to be valid $\lambda_H(t)$ must be positive and cannot be singular. Since $\lambda_H(m_t) > 0$, it means that the SM is valid for the energies between m_t and such a value of the energy E where $\lambda_H(E)$ is zero or becomes singular.

The singularity of $\lambda_H(t)$ can be determined from the denominator of Eq. (10). For the singularity (a simple pole) of $\lambda_H(t)$, we obtain that the value of $\lambda_H^s(t_0)$ at which the singularity occurs is equal $\lambda_H^s(t_0) = ((4\pi)^2/12)W_1(t)/W_2(t)$.

The behavior of the inverse of $\lambda_H^s(m_t)$ is plotted in Fig. 1. In the numerical evaluations we consider

$$g_1(t_{GU}) = g_2(t_{GU}) = g_3(t_{GU}) = 0.5$$

and

$$m_t(t_0) = v(t_0) = 174.1.$$

If we impose the condition that $\lambda_H(t)$ is regular in the whole range of the energies $[m_t, E_{GU}]$ then

$$\lambda_H(t_0) \leq 0.6915. \tag{11}$$

For the SM to be valid, the quartic coupling $\lambda_H(t)$ should not become negative. We use the numerator of Eq. (10) to find the first zero of $\lambda_H(t)$. In Fig. 2, we have drawn the dependence on energy of the $\lambda_H^z(t_0)$ at which the numerator vanishes, given by the relation $\lambda_H^z(t_0) = ((4\pi)^2/12)W_1'(t)/W_2'(t)$ which determines at what energy t occurs the first zero of $\lambda_H(t)$ depending on the value of $\lambda_H(t_0)$. Now, from the condition that $\lambda_H(t)$ should not have zeros in the whole range of the energies $[m_t, E_{GU}]$ we obtain

$$\lambda_H(t_0) \geq 0.6502. \tag{12}$$

We thus see that the consistency of the theory imposes a narrow band on the admissible values of the $\lambda_H(t_0)$.

$$0.6502 \leq \lambda_H(t_0) \leq 0.6915. \tag{13}$$

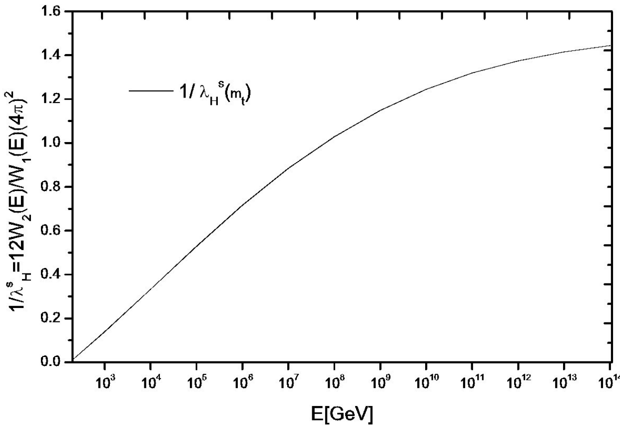

FIGURE 1. Energy dependence of the $\lambda_H(m_t)$ that determines the singularity of $\lambda_H(E)$.

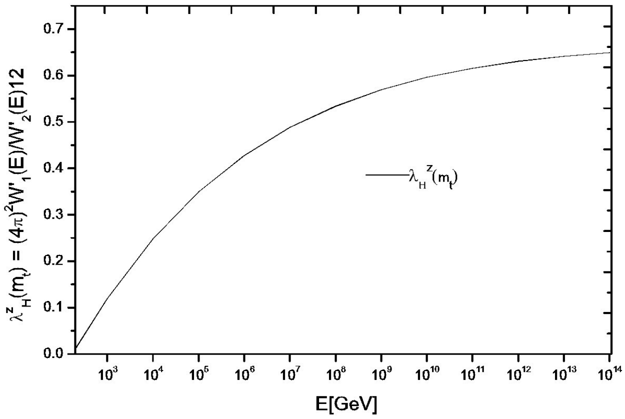

FIGURE 2. Energy dependence of the $\lambda_H(m_t)$ for which $\lambda_H(E) = 0$.

This band of the admissible values of $\lambda_H(m_t)$ can be transformed into the narrow band of the allowed values of the Higgs boson mass at t_0.

$$198 \leq m_H(t_0) \leq 205 \text{ GeV}. \tag{14}$$

This condition has been obtained from the analysis of the RGEs for the Higgs quartic coupling in the SM, and the condition that this model remains a consistent theory in the whole range of energies between the top mass and the energy of grand unification. If the Higgs mass is in the range given in (14) then this would be a strong indication for the grand unification hypothesis.

Finally, in Fig. 3 we show the dependence on the energy of $\lambda_H(E)$ considering $\lambda_H(m_t) = 0.65$ for which we have $\lambda_H(E_{GU}) = 0$. The relation $\lambda_H(E_{GU}) = 0$ can be also used as the condition which defines the grand unification energy E_{GU}. In Fig. 4 the region between the boundaries determined from the zero value and the singularity of $\lambda_H(E)$, *i.e.* the boundaries at which the SM breaks down is displayed.

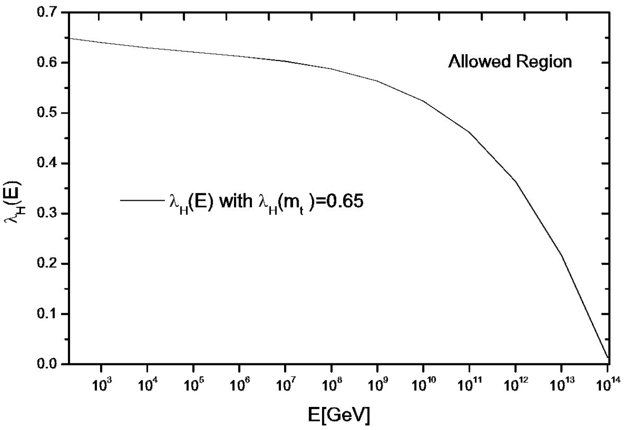

FIGURE 3. Lower boundary of the region at which $\lambda_H(E) > 0$ in the $[m_t, E_{GU}]$ region.

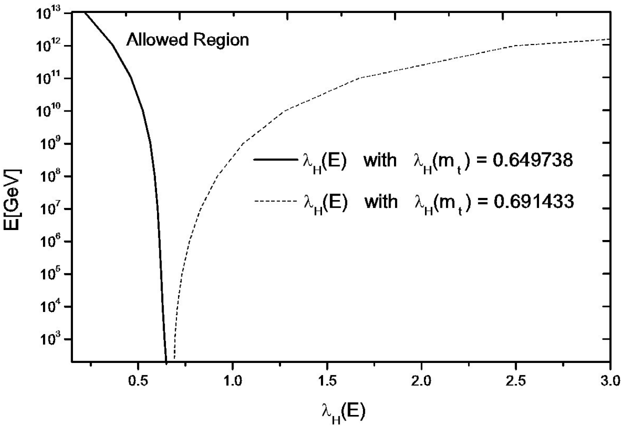

FIGURE 4. Plot of the $\lambda_H(m_t)$ boundaries at which the SM breaks down.

4. Outlook

The knowledge of the λ_H is essential due to its appearance in the RGEs for many observables, like the quark masses or the CKM matrix, having a decisive influence on their behavior. It is also crucial in the determination of the mass of the Higgs particle which is equal to $\sqrt{2\lambda_H(E)}v$, where v is the Higgs field vacuum expectation value.

Let us stress that if $m_H(t_0) = 198$ GeV then the condition $m_H(E_{GU}) = 0$ may be another definition of the E_{GU}. It may also be another indication of a very special behavior of the SM for energies at the grand unification scale 10^{14} GeV.

The results of this paper are consistent with those of Refs. 9 and 10 where the similar problem was considered. In Ref. 9 the authors were using the simplified assumption that the gauge couplings and the top quark Yukawa coupling are constant and do not run according to the RGE. Our treatment is more precise and the simplifying assumptions are not necessary. It is interesting that running of the gauge couplings and the top quark Yukawa coupling has an important influence on the results especially for the low Higgs masses, where the one loop approximation is the best.

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A Higgs potential and its mass

$$
\begin{aligned}
V &= \mu^2\left(\phi\phi^\dagger\right) + \frac{1}{2}\lambda_H\left(\phi\phi^\dagger\right)^2 \\
\phi &= \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right) \to \frac{1}{\sqrt{2}}h, \\
h^2 &\equiv 2v^2 \\
V &= \frac{1}{2}\mu^2 h^2 + \frac{1}{8}\lambda_H h^4, \\
\frac{\partial}{\partial h}V &= \mu^2 h + \frac{1}{2}\lambda_H h^3 = 0 \\
\mu^2 + \frac{1}{2}\lambda_H h^2 &= 0, \qquad h^2 = -\frac{2\mu^2}{\lambda_H} \\
m_H^2 &= \frac{\partial^2}{\partial h^2}V = \mu^2 + \frac{3}{2}\lambda_H h^2 = \lambda_H h^2 \\
&= -2\mu^2 = 2\lambda_H v^2
\end{aligned}
$$

B Series solution of Eq. (6)

$$
\begin{aligned}
& p(t) = \sum_0^6 p_n t^n \qquad q(t) = \sum_0^6 q_n t^n \\
& p_0 = -5.43358\cdot 10^{-2}, \quad p_1 = 1.13094\cdot 10^{-3}, \\
& p_2 = -1.02011\cdot 10^{-4}, \quad p_3 = 3.03248\cdot 10^{-6}, \\
& p_4 = -6.874\cdot 10^{-8}, \qquad p_5 = p_6 = 0. \\
& q_0 = -5.61543\cdot 10^{-3}, \quad q_1 = 2.06607\cdot 10^{-4}, \\
& q_2 = -1.89481\cdot 10^{-5}, \quad q_3 = 1.0073\cdot 10^{-6}, \\
& q_4 = -5.14838\cdot 10^{-8}, \quad q_5 = 1.49736\cdot 10^{-9}, \\
& q_6 = -2.1906\cdot 10^{-11}. \\
& W(t) = \sum w_n t^n,
\end{aligned}
$$

The conditions

$$
\begin{aligned}
& W_1\left(t_0\right) = 1, \quad W_1'\left(t_0\right)\big|_{t_0} = 0, \\
& \Rightarrow\ w_0 = 1, \quad w_1 = 0,
\end{aligned}
$$

and

$$W_2(t_0) = 0, \quad W_2'(t_0)|_{t_0} = 1,$$
$$\Rightarrow \; w_0 = 0, \quad w_1 = 1.$$

The coefficients w_n are obtained using the relations:

$$w_2 = -\frac{1}{2}\left(p_0 w_1 + q_0 w_0\right),$$

$$w_3 = -\frac{1}{6}\left[2p_0 w_2 + (q_0 + p_1) w_1 + q_1 w_0\right],$$

$$w_4 = -\frac{1}{12}\left[3p_0 w_3 + (q_0 + 2p_1) w_2\right. ,$$
$$\left. + (q_1 + p_2) w_1 + q_2 w_0\right]$$

$$w_5 = -\frac{1}{20}\left[4p_0 w_4 + (q_0 + 3p_1) w_3\right. ,$$
$$+ (q_1 + 2p_2) w_2$$
$$\left. + (q_2 + p_3) w_1 + q_3 w_0\right],$$

$$w_6 = -\frac{1}{30}\left[5p_0 w_5 + (q_0 + 4p_1) w_4\right.$$
$$+ (q_1 + 3p_2) w_3 + (q_2 + 2\,p_3) w_2$$
$$\left. + (p_4 + q_3) w_1 + q_4 w_0\right],$$

$$w_7 = -\frac{1}{42}\left[6p_0 w_6 + (q_0 + 5p_1) w_5\right.$$
$$+ (q_1 + 4p_2) w_4 + (q_2 + 3\,p_3) w_3$$
$$\left. + (2p_4 + q_3) w_2 + (q_4 + p_5) w_1 + q_5 w_0\right],$$

$$w_k = -\frac{1}{k} p_0 w_{k-1} - \frac{1}{k(k-1)}\left[q_0 + p_1 (k-2)\right] w_{k-2}$$
$$- \frac{1}{k(k-1)}\left[q_1 + p_2 (k-3)\right] w_{k-3}$$
$$- \frac{1}{k(k-1)}\left[q_2 + p_3 (k-4)\right] w_{k-4}$$
$$- \frac{1}{k(k-1)}\left[q_3 + p_4 (k-5)\right] w_{k-5}$$
$$- \frac{1}{k(k-1)}\left[q_4 + p_5 (k-6)\right] w_{k-6}$$
$$- \frac{1}{k(k-1)}\left[q_5 + p_6 (k-7)\right] w_{k-7}$$
$$- \frac{1}{k(k-1)} q_6 w_{k-8}$$

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