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Ermakov–Lewis invariants for a class of parametric anharmonic oscillators

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In this letter, we investigate a general class of damped anharmonic oscillators with time-dependent coefficients. The model is a second-order ordinary differential equation in which the driving is a general function of the solution and time. Several well-known equations of mathematical physics are generalized by our model. The equation is presented then as a generalized Ray–Reid system, and an invariant of the Ermakov–Lewis type is derived next. Particular forms of this invariant are obtained for the classical harmonic oscillator and the Ermakov equation. In this form, this work opens the investigation on the determination of Ermakov–Lewis invariants of anharmonic systems.

Keywords: Damped anharmonic oscillator; generalizations of Ray–Reid systems; Ermakov–Lewis invariants.

En este artículo investigamos una clase general de osciladores anarmónicos con coeficientes dependientes del tiempo. El modelo es una ecuación diferencial ordinaria de segundo orden en el cual el término forzado es una función general de la solución y el tiempo. Varias ecuaciones bien conocidas de la física matemática son generalizadas por nuestro modelo. La ecuación es presentada como un sistema Ray–Reid generalizado y posteriormente es obtenido un invariante del tipo Ermakov–Lewis. Las formas particulares de este invariante son obtenidas para el oscilador armónico clásico y la ecuación de Ermakov. De esta manera, se abre una línea de investigación en la determinación de invariantes de Ermakov–Lewis para sistemas anarmónicos.

Descriptores: Oscilador anarmónico amortiguado; generalizaciones de sistemas Ray–Reid; invariantes de Ermakov–Lewis.

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1. Introduction

The determination of invariants of physical systems is an important topic of research in view of the fact that preserved quantities may provide additional information about the models themselves [1,2], including novel information on the particular geometry of the physical systems [3]. For the class of harmonic oscillators with time-dependent coefficients, the Ermakov-Lewis invariants for Ray-Reid systems have been particularly useful in the elucidation of useful new properties of nonlinear regimes [4–7].

From a historical point of view, it is important to mention that the Ermakov-Lewis invariants were investigated independently by Ermakov in 1880 (see [8] for an English translation of the original work), and Lewis in 1967 and 1968 (see [9] and [10], respectively). More precisely, Ermakov and Lewis investigated the system of ordinary differential equations of second-order described by

$$\begin{cases} \ddot{y}(t) + \omega^2(t) y(t) = 0, \\ \ddot{\rho}(t) + \omega^2(t)\rho(t) = \dfrac{1}{\rho^3(t)}. \end{cases} \tag{1}$$

In these expressions, a dot represents the derivative with respect to t, and two dots denote the respective second derivative. Using different approaches, Ermakov and Lewis showed that the following quantity is conserved through time:

$$I = \frac{1}{2}\left[\left(y(t)\dot{\rho}(t) - \rho(t)\dot{y}(t)\right)^2 + \left(\frac{y(t)}{\rho(t)}\right)^2 \right]. \tag{2}$$

Later on, in 1979, Ray and Reid [11] established the existence of an infinite number of invariants for a generalized form of the coupled system (1). More precisely, they assumed that f and g were arbitrary integrable functions, and considered the coupled system of undamped driven harmonic oscillators

$$\begin{cases} \ddot{y}(t) + \omega^2(t) y(t) = \dfrac{g(\rho(t)/y(t))}{\rho(t) y^2(t)}, \\ \ddot{\rho}(t) + \omega^2(t)\rho(t) = \dfrac{f(y(t)/\rho(t))}{y(t)\rho^2(t)}. \end{cases} \tag{3}$$

Applying a procedure similar to that employed by Ermakov, Ray and Reid obtained the conserved quantity

$$\begin{aligned} I = \frac{1}{2}\Big[&\left(y(t)\dot{\rho}(t) - \rho(t)\dot{y}(t)\right)^2 \\ &+ \phi(y(t)/\rho(t)) + \theta(\rho(t)/y(t))\Big], \end{aligned} \tag{4}$$

where

$$\phi(v) = 2F(v), \tag{5}$$

$$\theta(v) = 2G(v). \tag{6}$$

The functions F and G in these formulas are arbitrary anti-derivatives of the functions f and g, respectively.

There are already various approaches to obtain dynamical invariants for harmonic and anharmonic time-dependent oscillators. Many of these approaches are based on the Hamiltonian formalism. For instance, Leach [12, 13] employed canonical transformations. Meanwhile, Korsch [14], Takayama [15] and Maamache [16] used the dynamical algebra generated by the Hamiltonian associated to the oscillator of the Ermakov system. In the present work, we propose an alternative non-Hamiltonian technique in order to take into account those cases in which the Hamiltonian formalism imposes constraints.

From a practical point of view, the Ermakov–Lewis invariants of systems of coupled harmonic oscillators have found a vast range of interesting applications. For instance, in recent years, various applications have been proposed to the geometric investigation of photonic lattices of waveguide arrays which are classical analogs of quantum harmonic oscillators [17]. They have also been employed in the numerical investigation of the effects of additive [6] and multiplicative [5] noise on some Ray–Reid systems. Among many other applications, general multi-component Ermakov systems in a two-layer fluid with a circular paraboloidal bottom topography have been investigated using variational methods [18]. In particular, various properties of driven harmonic oscillators have been elucidates. However, it is interesting to notice that the investigation on the existence of Ermakov-Lewis invariants for anharmonic systems has been left aside by researchers in the area, in part due to the complexity of this task.

The purpose of the present work is to propose Ermakov–Lewis invariants for a damped anharmonic oscillator with time-dependent coefficients. As in the case of the seminal paper by Ray and Reid [11], we will establish here an infinite number of invariant for our anharmonic system in analytic form. However, contrary to the harmonic case, the Ermakov-Lewis invariants of the anhamonic systems considered here cannot be expressed in exact form in general. Particular forms of these invariants will be calculated for some specific models of mathematical physics, namely, the harmonic oscillator and the Ermakov equation.

The present letter is organized as follows. In Sec. 2, we introduce the general system under investigation in this work. As we will note, the system is a damped anharmonic oscillator with time-dependent coefficients that generalizes the damped and driven harmonic oscillator. A suitable change of variables leads then to a simplification of the problem. An analytic class of Ermakov–Lewis invariants are obtained next using the Ray–Reid approach. In Sec. 3, we calculate invariants for the harmonic oscillator and the Ermakov system. Finally, we close this work with a section of concluding remarks and directions for future investigation.

2. Anharmonic oscillators

Let $I = (a, b)$ be an open interval of time, and suppose that $u : I \to \mathbb{R}$ is a function which has continuous derivatives up to the second order. Assume that g_i and h are real functions which are Riemann-integrable on each closed subinterval of I, for each $i = 1, 2$. For practical purposes, one may think of g_i and h as piecewise continuous functions on the real line for each i. Throughout, we will suppose that u satisfies the ordinary differential equation of second-order

$$\ddot{u}(t) + g_1(t)\dot{u}(t) + g_2(t)u(t) + h(t, u(t)) = 0. \qquad (7)$$

Obviously, the model (7) is a generalized anharmonic oscillator that extends many well-known models from physics. Among those models, we may readily count the driven harmonic oscillator as well as anharmonic models such as the driven Duffing oscillator [19] and the Lane–Emden equation [20]. To that end, the function h must adopt the form

$$h(t, u(t)) = g_3(t)u^n(t) + g_4(t), \qquad (8)$$

for some value of $n \in \mathbb{R}$, and suitable Riemann-integrable real functions g_3 and g_4 defined on $\mathbb{R}$.

The existence and uniqueness of the solutions of various forms of (7) has been established under a wide variety of assumptions (see [21–23], for instance). So, let u be a solution of the anhamornic Eq. (7). For any integrable function f on an interval I, we will employ the symbols

$$\int f(\tau)d\tau$$

and

$$\int^t f(\tau)d\tau$$

to represent an anti-derivative of f as a function of the variable t. Consider the function

$$y(t) = u(t)\exp\left(\frac{1}{2}\int g_1(\tau)d\tau\right), \qquad (9)$$

for each $t \in I$. A simple change of variable transforms the original anharmonic equation into the new and equivalent ordinary differential equation

$$\ddot{y}(t) + \omega^2(t)y(t) + H(t, y(t)) = 0, \qquad (10)$$

where the new coefficients and the function H are defined by

$$\omega^2(t) = g_2(t) - \frac{g_1'(t)}{2} - \left(\frac{g_1(t)}{2}\right)^2, \qquad (11)$$

$$A(t) = \exp\left(\frac{1}{2}\int g_1(\tau)d\tau\right), \qquad (12)$$

$$H(t, y(t)) = A(t)h\left(t, y(t)e^{-\frac{1}{2}\int g_1(\tau)d\tau}\right). \qquad (13)$$

It is worth noticing that the new model (10) is simpler than the original one in the sense that the term of damping is not present any more.

Suppose now that there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ which are integrable in each closed subinterval of real numbers, such that the following conditions are simultaneously satisfied:

$$\begin{cases} \ddot{y}(t) + \omega^2(t)y(t) = \dfrac{g(\rho(t)/u(t))}{\rho(t)y^2(t)}, \\ \ddot{\rho}(t) + \omega^2(t)\rho(t) = \dfrac{f(u(t)/\rho(t))}{y(t)\rho^2(t)}, \\ g(\rho(t)/y(t)) = -\rho(t)y^2(t)H(t, y(t)). \end{cases} \tag{14}$$

The system (14) is clearly an extension of the Ray–Reid system for harmonic oscillators, and it may be called simply a *generalized Ray–Reid system*. Obviously, each Ray–Reid system is a special case of (14), including the classical driven harmonic oscillator and the well-known Ermakov equation [24].

Suppose that F and G are anti-derivatives of f and g, respectively. Additionally, let ϕ and θ be given as in Eqs. (5) and (6), respectively. Under these assumptions, the Ermakov–Lewis invariants for the system (14) are given by the expression (4). More precisely, the following quantity is conserved through time:

$$I = \frac{1}{2}\left(y(t)\dot{\rho}(t) - \rho(t)\dot{y}(t)\right)^2 + \int^{y(t)/\rho(t)} f(\tau)d\tau + \int^{\rho(t)/y(t)} g(\tau)d\tau. \tag{15}$$

Before closing this section, it is important to note that a further convenient reduction of the third constraint of the system (14) is readily at hand, when we multiply and divide the right-hand side of that equation by $y(t)$. It is obvious then that such algebraic constraint is equivalent to requiring the existence a function $\widehat{g}$ such that

$$\widehat{g}(\rho(t)/y(t)) = y^3(t)H(t, y(t)) \tag{16}$$

holds for each $t \geq 0$.

3. Particular models

3.1. Damped harmonic oscillator

Let α and δ be real functions which are integrable on each closed interval of $\mathbb{R}$. The parametric damped harmonic oscillator is the ordinary differential equation

$$\ddot{u}(t) + \delta(t)\dot{u}(t) + \alpha(t)u(t) = 0, \tag{17}$$

for $t \geq 0$. Obviously, this is a particular form of the model (7) obtained when we let

$$g_1(t) = \delta(t), \tag{18}$$

$$g_2(t) = \alpha(t), \tag{19}$$

$$h(t, u(t)) = 0, \tag{20}$$

for each $t \geq 0$. Under these circumstances, the transformation (9) up to a multiplicative constant assumes the form

$$y(t) = u(t)\exp\left(\frac{1}{2}\int \delta(\tau)d\tau\right), \tag{21}$$

for each $t > 0$.

According to the discussion of the previous section, the resulting regime in terms of the new variable y is governed by the ordinary differential equation (10), with coefficients

$$\omega^2(t) = \alpha(t) - \frac{\dot{\delta}(t)}{2} - \left(\frac{\delta(t)}{2}\right)^2, \tag{22}$$

$$A(t) = \exp\left(\frac{1}{2}\int \delta(\tau)d\tau\right), \tag{23}$$

$$H(t, y(t)) = 0, \tag{24}$$

up to multiplicative constants. Observe then that the last constraint of (14) is satisfied. In fact, the function g is identically equal to zero. As a consequence, is f and ρ are functions satisfying the second equation of the generalized Ray–Reid system, then the quantity

$$I = \frac{1}{2}\left(y(t)\dot{\rho}(t) - \rho(t)\dot{y}(t)\right)^2 + \int^{y(t)/\rho(t)} f(\tau)d\tau \tag{25}$$

is conserved through time. Obviously, a further simplification of this formula arises when we let f be identically equal to zero.

3.2. Damped Ermakov equation

Let g_i be an integrable function over $\mathbb{R}$, for each $i = 1, 2, 3$. The generalized damped Ermakov equation that we will consider in the present subsection is described by the nonlinear regime

$$\ddot{u}(t) + g_1(t)\dot{u} + g_2(t)u(t) + \frac{g_3(t)}{u^3(t)} = 0, \tag{26}$$

for every $t \geq$, and for functions such that $u(t) \neq 0$ for each t. This is a generalization of the system investigated by Pinney in 1950 [25] and Lewis [9, 10], and also a particular case of the anharmonic model (7) with

$$h(t, u(t)) = \frac{g_3(t)}{u^3(t)}. \tag{27}$$

Using the transformation (9), the equivalent model (10) readily results, where ω^2 and A are the functions of time which are given as in the previous section. Moreover, in this case, the function H adopts the form

$$H(t, y(t)) = \frac{g_3(t)}{y^3(t)}\exp\left(2\int g_1(\tau)d\tau\right). \tag{28}$$

Thus,

$$-\rho(t)y^2(t)H(t,y(t)) = -\frac{\rho(t)g_3(t)}{y(t)}\exp\left(2\int g_1(\tau)d\tau\right). \quad (29)$$

From this, we note that a necessary and sufficient condition for the right-hand side of (29) to be a function of the variable $\rho(t)/y(t)$ is that there exists a real constant C with the property that

$$g_3(t) = C\exp\left(-2\int g_1(\tau)d\tau\right). \quad (30)$$

Under these circumstances, for any pair of functions ρ and f satisfying the second condition of (14), the Ermakov–Lewis invariant is preserved.

4. Conclusions and perspectives

In this work, we considered a damped anharmonic oscillator with time-dependent coefficients that generalizes the classical harmonic oscillator. By using a suitable transformation and an associated undamped harmonic oscillator with similar frequency, we obtained a system in the form of a generalized Ray–Reid model which includes an additional algebraic constraint. A family of invariants of the Ermakov–Lewis type were derived next in analytic form. Unfortunately, the expression of the invariants cannot be written in an exact manner. As examples, some particular models appearing in physics were considered next, and the associated Ermakov-Lewis invariants were determined analytically.

Of course, several avenues of research still remain open after this letter. An interesting question that immediately arises is whether the analytical form of the Ermakov–Lewis invariant can be expressed exactly or, at least, in an alternative form. Also, it would be interesting to determine exact forms of these conserved quantities, at least for simple (though meaningful) forms of parametrized anharmonic oscillators. This task seems to be a mere mathematical task. However, the work may be complemented by proposing relevant physical applications in those cases. The development or numerical techniques for anharmonic oscillators that preserve the Ermakov–Lewis invariant in the discrete domain, is also an interesting avenue of research that possesses a pragmatic interest.

To conclude the present letter, we must point out that this is one of the first efforts in order to extend the approach proposed by Ermakov to produce conserved quantities in anharmonic oscillators. Our approach requires to impose an algebraic constraint on the driving contribution of the anharmonic oscillator. In view of this fact, a natural direction for future works would be to provide more flexible conditions to establish the existence of invariants of the Ermakov-Lewis type in anharmonic ordinary differential equations with time-dependent parameters.

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