Comments on area spectra in loop quantum gravity

A. Corichi
Instituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México
Apartado Postal 70-543, México D.F. 04510, México
e-mail: corichi@nuclecu.unam.mx

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We examine and compare different area spectra that have been recently considered in Loop Quantum Gravity (LQG). In particular we focus our attention on an Equally Spaced (ES) spectrum operator introduced recently by Alekseev *et al* that has gained some attention. We show that such operator is not well defined within the LQG framework, and comment on the issues regarding area spectra and QNM frequencies.

Keywords: Quantum gravity; Area spectrum; spin networks.

Se examinan y comparan diferentes espectros del operador de área considerados recientemente dentro del formalismo cuántico de lazos. En particular se considera el operador con un espectro uniformemente espaciado introducido recientemente por Alekseev *et al.* y que ha recibido cierta atención. Se muestra que dicho operador no esta bien definido dentro del formalismo cuántico de lazos y se comenta sobre la relación entre gravedad de lazos y los modos cuasinormales de hoyos negros.

Descriptores: Gravedad cuántica; espectro de área; redes de espín.

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1. Introduction

Loop quantum gravity (LQG) has become in the past years a serious candidate for a non-perturbative quantum theory of gravity [1]. Its most notable predictions are the quantization of geometry [2] and the computation of black hole entropy [3]. One of its peculiarities is the existence of one parameter family of inequivalent quantum theories labelled by the Immirzi parameter γ [4]. The black hole entropy calculation was proposed as a way of fixing the Immirzi parameter γ (and thus the spectrum of the geometric operators), when a systematic approach to quantum black hole entropy was available [3]. This was used to fix the value of the Immirzi parameter γ to the value $\gamma_0 = \ln(2)/(\pi\sqrt{3})$ [3]. Recently, Dreyer made the suggestion that there is an independent way of fixing the Immirzi parameter [5]. The new approach is based on a conjecture by Hod, where the *real* part of the quasinormal mode frequencies $\omega_{\rm QNM}$, for large n has an asymptotic behavior given by [6] $M\omega_{\rm QNM} = \ln 3/(8\pi)$, where M is the mass of the black hole. This conjecture was proved analytically by Motl [7]. These modes have an imaginary part that goes to infinity as n grows, therefore, these are highly damped oscillatory modes. The Hod's conjecture for the limit of the real part of the frequency was within the (quantum) framework pioneered by Bekenstein, in which the area spectrum is assumed to be equally spaced [8]. Dreyer showed that in order to have consistence between the BH entropy calculation and QNM frequencies, one had to assume that the minimum value of j of the spin network piercing the horizon and contributing significantly to the entropy had to be $j = 1$. With this choice, the resulting Immirzi parameter is given by the new value $\gamma_{\rm d} = \ln(3)/(2\pi\sqrt{2})$. Furthermore, Dreyer suggested that if the gauge group of the theory was changed from $SU(2)$ to $SO(3)$ then this requirement would be immediately satisfied. After this observation, there have been several attempts to suggest different scenarios. One could classify these attempts in two categories: those that try to explain the $j = 1$ appearance by means of extra requirements [9], but without changing the geometric operator; and those attempts that suggest modifying the area spectra [10–12].

In this note we shall focus our attention to this later proposal, where the Equally-Spaced (ES) area operator is employed [10–12]:

$$\hat{A}(S)_{\rm ES} \cdot \Psi = 8\pi l_P^2\, \gamma \sum_v (j_v + 1/2)\, \Psi. \qquad (1)$$

This is to be contrasted to the standard Rovelli-Smolin spectrum (for simple intersections),

$$\hat{A}(S)_{\rm RS} \cdot \Psi = 8\pi l_P^2\, \gamma \sum_v \sqrt{j_v(j_v + 1)}\, \Psi. \qquad (2)$$

The ES spectrum has been argued to be relevant for explaining the $j = 1$ contribution while keeping $SU(2)$ as the gauge group [11, 12]. Probably the most important property of the ES-area operator (1) is that it assigns a quantum of area ($4\pi l_P^2\gamma$) to all edges that pierce the surface and carry a $j = 0$ label.

In this note, we shall consider the operator (1) within the framework of LQG. We shall give two different (but related) arguments to show that the operator is not well defined in the theory. The first argument will use the 'old' language of Wilson loops and the second argument uses spin networks and graphs. As we will show, the fact that the operator assigns area to $j = 0$ label is what makes it senseless. We hope that this note will help to settle the issue of this particular operator (or any operator that 'sees' $j = 0$ edges for that matter).

The structure of this note is as follows. In Sec. 2. we consider the C^*-holonomy algebra to show that the ES-operator does not respect the (Hoop) equivalence classes. In Sec. 3. we consider the operator within the graph perspective and show that cylindrical consistency is violated by the operator. In Sec. 4. we comment on the result. Loop Quantum Gravity experts may safely skip the remainder of the note.

2. Holonomy Algebras

In this section we shall analyze the ES area operator as seen from the perspective of Holonomy Algebras (HA) and the GNS construction of the (kinematical) Hilbert space $\mathcal{H} = L^2(\overline{\mathcal{A}/\mathcal{G}}, \mathrm{d}\mu_{\mathrm{AL}})$ of the theory. It is now well understood, that there are several ways of characterizing the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ of gauge invariant generalized connections and of the Hilbert space. Historically, the first construction made use of the fact that one could define an Abelian C^*-algebra of configuration observables, the so-called *Holonomy Algebra* $\mathcal{HA}$ [13]. The Gelfand-Naimark theory tells us that the $\mathcal{HA}$ can be seen as the space of continuous functions $C(\Delta)$ on the spectrum Δ of the algebra $\mathcal{HA}$. This is the quantum configurations space. Furthermore, the Hilbert space can be constructed via the GNS procedure for a properly defined positive functional (the so called Ashtekar-Lewandowski state). Since the elements of the Hilbert space are to be built out of elements $h_\alpha \in \mathcal{HA}$ of the holonomy algebra, then it better be that any operator $\hat{\mathcal{O}}$ in $\mathcal{H}$ respects the algebraic properties of the algebra $\mathcal{HA}$ if it is to be well defined.

Why should there be any problem? The reason for the existence of consistency conditions to be met is that the elements of $\mathcal{HA}$ are *equivalence classes* of loops (closely related to Wilson loops), where two loops α and β are equivalent if the holonomies along them are the same *for all* connections. Furthermore, in order to define $\mathcal{HA}$, one needs to quotient the original algebra by an ideal that takes care of the so-called Mandelstam identities arriving at a new equivalence class K (for details see [13]).

Thus, there are loops that are K-equivalent to the zero loop, and the corresponding algebra element $[h_{\alpha=0}] = \mathrm{Id}$ corresponds to the unit element. The unit element of the algebra, as its name indicates, can be multiplied freely and the resulting state is the same in the GNS construction.

Now, how do we make contact with the operator (1), and the $j = 0$ spin networks? A closed loop is a particular case of a closed graph, and there we can define a spin network by assigning representation of $SU(2)$ to it, labelled by j. If one chooses $j = 0$, one has the trivial (identity) function, and therefore the unit element $h_{\alpha=0}$. That is, the zero-j spin networks correspond to an element of the algebra equivalent to the zero-loop, or, in other words the unit element of the algebra. This means that we can add or remove closed loops with zero-j for free to a state, and get the "same physical state". The ES-area operator (1) endows with different eigen-values for the area to the state, each time one adds or removes a $j = 0$ loop that crosses the surface. Thus, the fact that the ES operator counts zero-j spin networks and assigns area to them means that its action depends on the representative of the equivalence class $[h_\alpha]$. The operator does not respect the K-equivalence classes and is, therefore, not well defined on the Hilbert space $\mathcal{H}$ of the theory.

3. Graphs and Spin Networks

There are alternative ways of characterizing the Hilbert space $\mathcal{H}$ and the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ of the theory. Of particular relevance are the so-called projective techniques that make use of graphs families and projective families (for a nice review see Ref. 14). The basic idea is to define a family of quantum theories that live on closed graphs Γ, corresponding, roughly speaking, to a lattice gauge theory on the graph. The continuum theory is recovered by taking the projective limit of the *largest* graph.

To be concrete, if we have a graph Γ and a spin network $\Psi_{(\Gamma,\vec{j})}(A)$ on it, we can define a unique function $\Psi'_{(\Gamma',\vec{j})}(A)$ on a larger graph $\Gamma' > \Gamma$ as follows: If $\Gamma' > \Gamma$ is such that can be obtained by Γ by adding artificial vertices to already existing edges, define the new function by trivial composition [14]. Here, by artificial vertices, we simply mean declaring a point p lying on the edge e_J (different from the one of the original vertices) to be a vertex. The original edge e_J is divided now in two edges. If the graph $\Gamma' > \Gamma$ contains new edges, then the new function $\Psi'_{(\Gamma',\vec{j})}(A)$ is obtained by assigning the identity function to each new edge. This means defining a new spin network with $j_I = 0$ *for all new edges* e_I.

Thus, for each spin network on Γ, there exists an infinity of spin network states defined on any larger graph $\Gamma' > \Gamma$, with lots of $j = 0$ edges. A function on the full Hilbert space is made of the collection of all these functions that are part of the 'cylindrical family'.

Any operator $\hat{\mathcal{O}}$ of the full theory needs to satisfy what is called *cylindrical consistency*, which means that its action should be the same for all elements of the family. Now, we come back to the $j = 0$ spin networks. If the operator $\hat{\mathcal{O}}_\Gamma$ is able to see the $j = 0$ edges of the graph Γ, then its action will depend on the element of the cylindrical family and therefore will not be consistent. We can then state that any operator that acts non-trivially on a given graph on $j = 0$ edges, will not be part of a consistent cylindrical family, of operators, and will *not* define an operator on the continuum. The ES-area operator (1) is clearly an example of this class of operators and is, therefore, not well defined.

As may be expected, the reason why the operator does not exist is simple to understand, and can be seem from these two (slightly different) perspectives. In fact, the langauge of loops or closed graphs is only a matter of convenience, but they are equivalent. Every graph Γ can be decomposed into N independent loops α_i, $i = 1, \ldots, N$. On the other hand, the graph might have M edges e_I, $I = 1, \ldots, M$, with

$M \geq N$, and therefore any cylindrical function is a function $f_\Gamma : G^M \mapsto C$ from M copies of the gauge group G to the complex numbers. On the other hand, one has N Wilson loops $W[\alpha_i, A] = (1/2)\mathrm{Tr}\mathcal{P}\exp(\oint_{\alpha_i} A)$ that are complex valued functions. The statement is that any Spin network $\Psi_{(\Gamma,\vec{j},\vec{m})}(A)$ on Γ can be written as a polynomial of degree given by the maximum value of the labels j_I as follows,

$$\Psi_{(\Gamma,\vec{j},\vec{m})}(A)=\sum_{n_i} A_{n_1\cdots n_N} W[\alpha_1]^{n_1} W[\alpha_2]^{n_2}\cdots W[\alpha_N]^{n_N}.$$

The advantage of working with spin networks is that they form a convenient basis that diagonalizes the geometric operators, in particular, the area operator for simple intersections of the spin network and the surface S.

4. Discussion

In the previous sections we have shown that the ES-area operator as proposed by Alekseev, Polychronakos, and Smedback (APS) [10], and used in Refs. 11 and 12 is not a valid operator in LQG from the mathematical viewpoint, using arguments in both the GNS construction and in the projective families construction. This conclusion also applies to the length operator recently suggested in $2+1$ gravity in [16].

It has been noted that one might modify the operator (1) such that it *is* well defined, by changing its action when acting on a $j = 0$ edge. The choice that makes it well defined is to ask that the new operator $\hat{A}'(S)$ annihilates the state (*i.e.* it yields zero eigenvalue). This is not the action that was originally proposed by APS [10] (and analyzed later on by Polychronakos in Ref. 11), where the operator was motivated by a new regularization that included 'quantum corrections', with a resulting behavior similar to the zero point energy of a harmonic oscillator [10]. With this modification, the new and well defined operator $\hat{A}'(S)$ would cease to be Equally-Spaced (ES), since there would be a larger area gap from $j = 0$ to $j = 1/2$ edges of $8\pi l_P^2\gamma$, as opposed to $4\pi l_P^2\gamma$ that is the the area gap in the rest of the ES part of the spectrum. This would presumably make it less appealing for providing an explanation of the QNM frequencies.

There might be some further considerations on why an ES-area spectrum (without the nontrivial contribution from $j = 0$) is not the most desirable one, such as the so-called Bekenstein-Mukhanov effect [5, 7, 15], but we shall not go further into that discussion (see, for instance, the first Ref. 1 and 11 for some discussion).

The standard spectrum of Rovelli-Smolin has not only been obtained by different regularization procedures [2], but seems to be robust given its physical and mathematical properties. However, whether or not Loop Quantum Gravity (with the Rovelli-Smolin spectrum) should have anything to say about the asymptotic Quasi-Normal Modes frequencies remains, in our opinion, an open issue. The reason for this is that recent numerical and analytical explorations of charged and rotating Black Holes do not show the asymptotic behavior that one would expect if one assumes a Bohr correspondence principle, as originally conjectured by Hod [6] (for an incomplete list of recent references in QNM see Ref. 17).

Finally, let us note that a similar argument to that presented in Sec. 3. has already been given in Ref. 18, from a slightly different perspective.

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1. C. Rovelli, *Quantum Gravity* (Cambridge U. Press, 2004); T. Thiemann, *Introduction to modern canonical quantum general relativity* (Cambridge U. Press, 2004). arXiv:gr-qc/0110034; A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report," arXiv:gr-qc/0404018. A. Ashtekar and J. Lewandowski, *Class Quantum Grav.* **21** (2004) R53.
2. C. Rovelli and L.Smolin, *Nucl. Phys. B* **442** (1995) 593; A. Ashtekar and J. Lewandowski, *Class Quantum Grav.* **14** (1997) A55.
3. A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, *Phys. Rev. Lett.* **80** (1998) 904; A. Ashtekar, J. Baez and K. Krasnov, *Adv. Theor. Math. Phys.* **4** (2000) 1.
4. G. Immirzi, *Nucl. Phys. Proc. Suppl.* **57** (1997) 65.
5. O. Dreyer, *Phys. Rev. Lett.* **90** (2003) 081301.
6. S. Hod, *Phys. Rev. Lett.* **81** (1998) 4293; S. Hod, *Gen. Rel. Grav.* **31** (1999) 1639.
7. L. Motl, *Adv. Theor. Math. Phys.* **6** (2003) 1135; L. Motl and A. Neitzke, *Adv. Theor. Math. Phys.* **7** (2003) 307.
8. J.D. Bekenstein, *Lett. Nuovo Cimento* **11** (1974) 467; J.D. Bekenstein and V.F. Mukhanov, *Phys. Lett. B* **360** (1995) 7.
9. A. Corichi, *Phys. Rev. D* **67** (2003) 087502; J. Swain, *Int. J. Mod. Phys. D* **12** (2003) 1729; Y. Ling and H. b. Zhang, *Phys. Rev. D* **68** (2003) 101501.
10. A. Alekseev, A. P. Polychronakos and M. Smedback, *Phys. Lett. B* **574** (2003) 296.
11. A. P. Polychronakos, *Phys. Rev. D* **69** (2004) 044010.
12. G. Gour and V. Suneeta, *Class. Quantum Grav.* **21** (2004) 3405.
13. A. Ashtekar and C. J. Isham, *Class. Quant. Grav.* **9** (1992) 1433; A. Ashtekar and J. Lewandowski, arXiv:gr-qc/9311010.
14. A. Ashtekar and J. Lewandowski, *J. Math. Phys.* **36** (1995) 2170.
15. L. Smolin, "Macroscopic deviations from Hawking radiation?", in Matters of Gravity **7** Spring 1996.

16. L. Freidel, E. R. Livine and C. Rovelli, *Class. Quant. Grav.* **20** (2003) 1463.

17. E. Berti, V. Cardoso and S. Yoshida, *Phys. Rev. D* **69** (2004) 124018; E. Berti, V. Cardoso, K. D. Kokkotas and H. Onozawa, *Phys. Rev. D* **68** (2003) 124018. N. Andersson and C. J. Howls, *Class. Quant. Grav.* **21** (2004) 1623. E. Berti and K. D. Kokkotas, *Phys. Rev. D* **68** (2003) 044027.

18. J.A. Zapata, *Class. Quantum Grav.* **21** (2004) L115.