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The semiclassical theory of quantized fields in classical electromagnetic backgrounds

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We formulate mathematically the process of pair production in electromagnetic fields for spinless particles. We compute the probability that n pairs are created in the semiclassical approximation, and herein we prove that the pair creation phenomenon is a stochastic Poisson process. Finally, we prove rigorously and interpret suitably the Schwinger formula.

Keywords: Pair production; Schwinger's formula; semiclassical approach.

Damos la formulación matemática del proceso de creación de pares de partículas sin spin en campos electromagnéticos. Calculamos la probabilidad de que se creen n pares en la aproximación semiclásica, y probamos, en esta aproximación, que la creación de pares es un proceso estocástico de Poisson. Finalmente, damos una demostración rigurosa y una interpretación correcta de la fórmula de Schwinger.

Descriptores: Creación de pares; fórmula de Schwinger; aproximación semiclásica.

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1. Introduction

The phenomenon of pair production in the presence of a classical electromagnetic background has been studied by some authors. Many of them say that particle-antiparticle creation is due to a kind of relativistic tunneling effect. This is stated, generally, when the electromagnetic potential is time-independent and has the following form: $(V(z), 0, 0, 0)$, *i.e.*, the potential vector is zero, and therefore the field is purely electric.

These authors study for spinless (spin) particles the normal modes of the Klein-Gordon (Dirac) equation, which is, in this case, an ordinary differential equation in the variable z. The equation depends on the two first components of the momentum, namely $p_\perp := (p_1, p_2)$. When the electric field is spatially confined, these modes are asymptotically plane waves, and we can compute the transmission and reflection coefficients corresponding to the j-mode. Therefore, when the energy of the j-mode, namely E_j, verifies the inequality

$$E_j - eV(z) > \sqrt{c^2p_\perp^2 + m^2c^4}$$

in a region of $\mathbb{R}$, and

$$eV(z) - E_j > \sqrt{c^2p_\perp^2 + m^2c^4}$$

in another region, (*i.e.*, when the kinetic energy in a region is greater than $\sqrt{c^2p_\perp^2 + m^2c^4}$, and the kinetic energy is less than $-\sqrt{c^2p_\perp^2 + m^2c^4}$ in another region), we have Klein's paradox. It is also well-known that, when Klein's paradox exists, the modulus of the transmission coefficient corresponding to the j-mode is interpreted as the relative probability that a pair is created in the j-state [8,10,17,23]. This is the interpretation of the particle-antiparticle production using the relativistic tunneling effect.

However, there are authors who interpret the relativistic tunneling effect in another way. For example, in [26], the authors study particle production in purely time-independent magnetic fields. It is clear that, in this case, the energy of the system is only kinetic. Consequently, Klein's paradox is not present and no pairs are created. However, in [26], the authors always interpret the transmission coefficient as the relative probability that a pair is produced. They then conclude that, in the presence of a purely time-independent magnetic field, pairs are created from the vacuum state. On the other hand, using Schwinger's interpretation [10,18,19,25,26], the exponential of minus twice the imaginary part of the effective action gives the probability that the vacuum state remains unchanged. However, since purely magnetic fields do not have imaginary part [26], they reach the conclusion that no pairs are created in this situation. Consequently, the authors claim that a contradiction arises from the tunneling interpretation and Schwinger's interpretation. It is clear that, if we do not correctly interpret the relativistic tunneling effect, some type of contradiction appears.

In this paper, in order to avoid wrong interpretations, we prefer to approach the pair production phenomenon using Dirac's point of view, that is, using the Quantum Field Theory of the electron provided by Fock [27]. For this reason, it is appropriate to study first the case of particles with spin 1/2. Due to the Dirac Sea hypothesis and the Pauli Exclusion Principle, the vacuum state is the one in which all (Dirac would say nearly all [4]) the negative **kinetic** energy states are fulfilled. The state of a particle is that in which all the negative kinetic energy states are fulfilled and one positive kinetic energy state is fulfilled. In the same way, the state of an antiparticle is that in which the negative kinetic energy states are fulfilled except one, etc. [5,13].

From this interpretation we can deduce very interesting consequences. For example, when the electromagnetic background is time-independent two different situations arise:

A) The scalar potential is zero, *i.e.*, the field is purely magnetic. In this situation, the energy is kinetic only, and is constant over time. Consequently, the eigenfunctions of the energy operator are stationary states, and there is no particle creation.

B) The scalar potential is not zero. In this case, there is an electric field. The energy of the system is decomposed into kinetic and potential energy. Now, the states that describe a definite number of particles and antiparticles, *i.e.*, the eigenfunctions of the kinetic energy operator, are **not** eigenfunctions of the total energy of the system. Consequently, there is particle production. Physically, when a state with a definite number of particles and antiparticles evolves, a part of its potential energy becomes transformed into kinetic energy. This can become mass, and pair production appears.

Another important case appears when the potential does not depend on the spatial variables, *i.e.*, there is only an electric field. Using a gauge transformation, it is easy to see that the system is equivalent to another system that only has kinetic energy. In this case, the energy operator depends on time and so do the eigenfunctions. For this reason, the state with a definite number of particles and antiparticles is not stationary, thus there is particle creation. Physically, when the kinetic energy of the system changes, part of it becomes mass and pairs are produced.

For spinless particles we obtain similar results. Namely, the kinetic energy operator is composed by an infinite number of harmonic oscillators with frequencies that depend on time when the vector potential is time-dependent. For this reason, we will find situations similar to the case of particles with spin 1/2.

The way to compute any type of probability is clear through this interpretation. In fact, it can be demonstrated that, when the potential is spatially and temporally confined, the probability that the vacuum state remains unchanged is equal to the exponential of minus twice the imaginary part of the effective action. Proof of this result for particles with spin 1/2 is in [19]. In order to prove it, the authors use the Perturbation Theory in all the orders. Following this demonstration step, by step in the case of the Klein-Gordon field, we obtain that the result is also valid for spinless particles. We thus conclude that our interpretation coincides with Schwinger's interpretation of the exponential of minus twice the imaginary part of the effective action. Schwinger and some authors [10,19,24-26] also interpreted that twice the imaginary part of the effective action is the probability that a pair is created. In this paper, following [5,11,14,23] we can see that this interpretation is not correct at all. Precisely, we see that twice the imaginary part of the effective action is, within the semiclassical limit, the average number of produced pairs.

The paper is organized as follows: In Sec. 2, we study the production of spinless particles by uniform electric fields. We use the Schrödinger picture afterwards the Heisenberg picture. Using the Bogolubov coefficients, we show the way to compute probabilities in the two pictures. At the end of the section we prove that, in the semiclassical approximation, pair production is a stochastic Poisson process, and we compute the average number of pairs produced.

In Sec. 3, we study the spinless particle production by potentials that are spatially and temporally confined. We provide the method to obtain the kinetic energy decomposition in harmonic oscillators. Once we have obtained this decomposition, we introduce the creation and annihilation operators, that are time-dependent when the potential vector is so. From these operators, we can construct the function corresponding to the states with a definite number of particles and antiparticles.

Finally, in Sec. 4, we give rigorous proof and a suitable interpretation of the Schwinger formula.

2. Spinless particles in homogeneous fields

2.1. The Schrödinger Picture

In this section we consider the Klein-Gordon equation

$$-\hbar^2\partial_t\psi(\vec{x},t) = [i\hbar c\vec{\nabla} - e\vec{f}(t)]^2\psi(\vec{x},t) + m^2c^4\psi(\vec{x},t),$$

in a box of volume L^3, with a periodic boundary condition.

In the Schrödinger picture, the Klein-Gordon equation is equivalent to a Hamiltonian system, composed of an infinite number of harmonic oscillators with frequencies which depend on time [10,12].

The energy and the electric charge of the system are as follows:

$$E_S(t) = \frac{1}{2}\sum_{\vec{k}\in\mathbb{Z}^3}\left(P^2_{\vec{k}}(t) + \omega^2_{\vec{k}}(t)Q^2_{\vec{k}}(t)\right) + \left(\bar{P}^2_{\vec{k}}(t) + \omega^2_{\vec{k}}(t)\bar{Q}^2_{\vec{k}}(t)\right)$$

$$\rho_S(t) = \frac{1}{\hbar}\sum_{\vec{k}\in\mathbb{Z}^3}\left(\bar{Q}_{\vec{k}}(t)P_{\vec{k}}(t) - Q_{\vec{k}}(t)\bar{P}_{\vec{k}}(t)\right),$$

where

$$\omega_{\vec{k}}(t) := \frac{1}{\hbar}\epsilon_{\vec{k}}(t) = \frac{1}{\hbar}\sqrt{c^2\left|\frac{2\pi\hbar\vec{k}}{L} + \frac{e}{c}\vec{f}(t)\right|^2 + m^2c^4}$$

is the frequency.

In order to obtain the quantum theory we must quantize these oscillators, *i.e.* we make the replacement

$$Q_{\vec{k}}(t) \to \hat{Q}_{\vec{k}} := Q_{\vec{k}}; \qquad \bar{Q}_{\vec{k}}(t) \to \hat{\bar{Q}}_{\vec{k}} := \bar{Q}_{\vec{k}}$$
$$P_{\vec{k}}(t) \to \hat{P}_{\vec{k}} := -i\hbar\partial_{Q_{\vec{k}}}; \quad \bar{P}_{\vec{k}}(t) \to \hat{\bar{P}}_{\vec{k}} := -i\hbar\partial_{\bar{Q}_{\vec{k}}},$$

and the quantum equation, in the Schrödinger picture, becomes

$$i\hbar\partial_t|\Phi\rangle_S = \frac{1}{2}\sum_{\vec{k}\in\mathbb{Z}^3}\Big[\Big(-\hbar^2\partial^2_{Q_{\vec{k}}} + \omega^2_{\vec{k}}(t)Q^2_{\vec{k}}\Big) + \Big(-\hbar^2\partial^2_{\bar{Q}_{\vec{k}}} + \omega^2_{\vec{k}}(t)\bar{Q}^2_{\vec{k}}\Big)\Big]|\Phi\rangle_S - \sum_{\vec{k}\in\mathbb{Z}^3}\epsilon_{\vec{k}}(t)|\Phi\rangle_S.$$

The eigenfunctions of the energy and electric charge operators must now be found. First, we must introduce the creation and annihilation operators for particles and antiparticles in the Schrödinger picture, at time t [12].

$$\hat{a}_{S,\vec{k}}(t) = \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}}\Big[\Big(i\hat{P}_{\vec{k}} + \omega_{\vec{k}}(t)\hat{Q}_{\vec{k}}\Big) + i\Big(i\hat{\bar{P}}_{\vec{k}} + \omega_{\vec{k}}(t)\hat{\bar{Q}}_{\vec{k}}\Big)\Big]$$

$$\hat{b}^\dagger_{S,-\vec{k}}(t) = \frac{1}{2\sqrt{\epsilon_{\vec{k}}(t)}}\Big[\Big(-i\hat{P}_{\vec{k}} + \omega_{\vec{k}}(t)\hat{Q}_{\vec{k}}\Big) + i\Big(-i\hat{\bar{P}}_{\vec{k}} + \omega_{\vec{k}}(t)\hat{\bar{Q}}_{\vec{k}}\Big)\Big].$$

Then, using these operators, we obtain

$$\hat{E}_S(t) = \sum_{\vec{k}\in\mathbb{Z}^3}\epsilon_{\vec{k}}(t)\left(\hat{a}^\dagger_{\vec{k}}(t)\hat{a}_{\vec{k}}(t) + \hat{b}^\dagger_{-\vec{k}}(t)\hat{b}_{-\vec{k}}(t)\right);$$

$$\hat{\rho}_S(t) = \sum_{\vec{k}\in\mathbb{Z}^3}\left(\hat{a}^\dagger_{\vec{k}}(t)\hat{a}_{\vec{k}}(t) - \hat{b}^\dagger_{-\vec{k}}(t)\hat{b}_{-\vec{k}}(t)\right).$$

Now, we construct the vacuum state at time t. If we consider

$$|0_{\vec{k},t}\rangle_S = \sqrt{\frac{\omega_{\vec{k}}(t)}{\pi\hbar}}\exp\left(-\frac{\omega_{\vec{k}}(t)}{2\hbar}(Q^2_{\vec{k}} + \bar{Q}^2_{\vec{k}})\right),$$

then the vacuum state at time t, $|0_t\rangle$, is

$$|0_t\rangle_S = \prod_{\vec{k}\in\mathbb{Z}^3}|0_{\vec{k},t}\rangle_S,$$

since $\hat{E}_S(t)|0_t\rangle_S = 0$ and $\hat{\rho}_S(t)|0_t\rangle_S = 0$.

Therefore, starting at the vacuum state and using the creation operators we can construct the Fock space.

2.2. The Heisenberg Picture, "in" and "out" formalism, and Bogolubov coefficients

In order to obtain the Heisenberg picture, we must first define $\hat{E}_H(t) = T(0,t)\hat{E}_S(t)T(t,0)$, where $T(t,0)$ is the quantum evolution operator, *i.e.*, it verifies

$$\begin{cases} i\hbar\dot{T}(t,0) = \hat{E}_S(t)T(t,0) \\ T(0,0) = Id. \end{cases} \tag{1}$$

Let $|\psi_t\rangle_S$ be an eigenfunction of the operator $\hat{E}_S(t)$ with eigenvalue $\lambda(t)$, then $|\psi_t\rangle_H := T(0,t)|\psi_t\rangle_S$ is an eigenfunction of the operator $\hat{E}_H(t)$ with eigenvalue $\lambda(t)$. That is, $T(0,t)$ maps the eigenfuctions of the energy operator in the Schrödinger picture to the eigenfuctions of the energy operator in the Heisenberg picture.

In this picture, the creation and annihilation operators are [27]:

$$\hat{a}_{H,\vec{k}}(t) = T(0,t)\hat{a}_{S,\vec{k}}(t)T(t,0), \ldots, \hat{b}^\dagger_{H,-\vec{k}}(t) = T(0,t)\hat{b}^\dagger_{S,-\vec{k}}(t)T(t,0).$$

Then, the "in" and "out" creation and annihilation operators are

$$\hat{a}_{\binom{\text{in}}{\text{out}},\vec{k}} := \lim_{t\to\mp\infty}\hat{a}_{H,\vec{k}}(t), \ldots, \hat{b}^\dagger_{\binom{\text{in}}{\text{out}},\vec{k}} := \lim_{t\to\mp\infty}\hat{b}^\dagger_{H,\vec{k}}(t),$$

and the "in" and "out" vacuum state is

$$|0_{\vec{k},\binom{\text{in}}{\text{out}}}\rangle := \lim_{t\to\mp\infty}|0_{\vec{k},t}\rangle_H,$$

$$|0_{\binom{\text{in}}{\text{out}}}\rangle := \prod_{\vec{k}}|0_{\vec{k},\binom{\text{in}}{\text{out}}}\rangle = \lim_{t\to\mp\infty}|0_t\rangle_H.$$

In order to obtain the relationship between the creation and annihilation operators in the Heisenberg picture at different times, we define the Bogolubov coefficients in the following form [11]:

$$\begin{pmatrix}\hat{a}_{H,\vec{k}}(t_2)\\ \hat{b}^\dagger_{H,\vec{k}}(t_2)\end{pmatrix} = \begin{pmatrix}\alpha^*_{\vec{k}}(t_2,t_1) & \beta_{\vec{k}}(t_2,t_1)\\ \beta^*_{\vec{k}}(t_2,t_1) & \alpha_{\vec{k}}(t_2,t_1)\end{pmatrix}\begin{pmatrix}\hat{a}_{H,\vec{k}}(t_1)\\ \hat{b}^\dagger_{H,\vec{k}}(t_1)\end{pmatrix}$$

Remark 2.1. *The Bogolubov coefficients verify* $|\alpha_{\vec{k}}(t_2,t_1)|^2 - |\beta_{\vec{k}}(t_2,t_1)|^2 = 1.$

It is easy to check that [11,14]

$$|0_{\vec{k},t_2}\rangle_H = c_{\vec{k}}\sum_{n=0}^{\infty}\left(-\frac{\beta_{\vec{k}}(t_2,t_1)}{\alpha^*_{\vec{k}}(t_2,t_1)}\right)^n|n_{\vec{k},t_1}\rangle_H;$$

$$|0_{\vec{k},t_1}\rangle_H = \bar{c}_{\vec{k}}\sum_{n=0}^{\infty}\left(\frac{\beta_{\vec{k}}(t_2,t_1)}{\alpha_{\vec{k}}(t_2,t_1)}\right)^n|n_{\vec{k},t_1}\rangle_H,$$

where $|c_{\vec{k}}|^2 = |\bar{c}_{\vec{k}}|^2 = |\alpha_{\vec{k}}(t_2,t_1)|^{-2}$, and $|n_{\vec{k},t}\rangle_H$ is the vector that, at time t, contains n particles in the $\vec{k}$-state and n antiparticles in the $-\vec{k}$-state, i.e.

$$|n_{\vec{k},t}\rangle_H = \frac{(\hat{a}^\dagger_{H,\vec{k}}(t))^n(\hat{b}^\dagger_{H,\vec{k}}(t))^n}{n!}|0_{\vec{k},t}\rangle_H.$$

2.3. Probability Formulae

Let $P_{n,\vec{k}}(t_2,t_1)$ be the probability that n pairs are created in the $\vec{k}$-state, after the evolution of the vacuum state from t_1 to t_2. Then, in the Schrödinger and Heisenberg picture, the formulae that give this probability are:

$$P_{n,\vec{k}}(t_2,t_1) = |{}_S\langle n_{\vec{k},t_2}|T(t_2,t_1)|0_{t_1}\rangle_S|^2 = |{}_H\langle n_{\vec{k},t_2}|0_{t_1}\rangle_H|^2, \tag{2}$$

and for the average number of produced pairs in the $\vec{k}$-state at time t_2 created from the vacuum state at time t_1, namely $N_{\vec{k}}(t_2, t_1)$, the formulae are

$$\begin{aligned} & N_{\vec{k}}(t_2, t_1) \\ &= {}_S\langle 0_{t_1}|T(t_1, t_2)\hat{a}^\dagger_{S,\vec{k}}(t_2)\hat{a}_{S,\vec{k}}(t_2)T(t_2, t_1)|0_{t_1}\rangle_S \\ &= {}_H\langle 0_{t_1}|\hat{a}^\dagger_{H,\vec{k}}(t_2)\hat{a}_{H,\vec{k}}(t_2)|0_{t_1}\rangle_H. \end{aligned} \tag{3}$$

Now, using the Bogolubov coefficients we have [23]

$$P_{n,\vec{k}}(t_2, t_1) = |\alpha_{\vec{k}}(t_2, t_1)|^{-2}\left|\frac{\beta_{\vec{k}}(t_2, t_1)}{\alpha_{\vec{k}}(t_2, t_1)}\right|^{2n} = P_{0,\vec{k}}(t_2, t_1)\left(\frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}(t_2, t_1)}\right)^n$$

$$N_{\vec{k}}(t_2, t_1) = |\beta_{\vec{k}}(t_2, t_1)|^2 = \frac{P_{1,\vec{k}}(t_2, t_1)}{P^2_{0,\vec{k}}(t_2, t_1)},$$

and, using the "in" and "out" formalism, we obtain [9]

$$P_{0,\vec{k}}(+\infty, -\infty) = |\langle 0_{out,\vec{k}}|0_{in,\vec{k}}\rangle|^2;$$

$$N_{\vec{k}}(+\infty, -\infty) = \langle 0_{in}|\hat{a}^\dagger_{out,\vec{k}}\hat{a}_{out,\vec{k}}|0_{in}\rangle.$$

Therefore, the formula that gives the probability that the vacuum state remains unchanged between times t_1 and t_2 is [9,11,23]:

$$\begin{aligned} P_0(t_2, t_1) &= \prod_{\vec{k}\in\mathbb{Z}^3} P_{0,\vec{k}}(t_2, t_1) = \prod_{\vec{k}\in\mathbb{Z}^3} |\alpha_{\vec{k}\in\mathbb{Z}^3}(t_2, t_1)|^{-2} \\ &= \prod_{\vec{k}\in\mathbb{Z}^3} \frac{1}{1 + N_{\vec{k}}(t_2, t_1)} \\ &= \exp\left\{-\sum_{\vec{k}\in\mathbb{Z}^3} \log[1 + N_{\vec{k}}(t_2, t_1)]\right\}. \end{aligned} \tag{4}$$

In general, if we define (for details see Refs. 14 and 23)

$$g(x) = \prod_{\vec{k}\in\mathbb{Z}^3}\left(1 - x\frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}(t_2, t_1)}\right)^{-1},$$

then, the probability that n pairs are created at time t_2, namely $P_n(t_2, t_1)$, is

$$P_n(t_2, t_1) = \frac{1}{n!}\frac{D^n g(0)}{g(1)}.$$

Finally, in accordance with Feynman [7], the relative probability that a pair is produced, namely $P_{R,1}(t_2, t_1)$, is

$$P_{R,1}(t_2, t_1) := \frac{P_1(t_2, t_1)}{P_0(t_2, t_1)} = \sum_{\vec{k}\in\mathbb{Z}^3}\frac{P_{1,\vec{k}}(t_2, t_1)}{P_{0,\vec{k}}(t_2, t_1)} = \sum_{\vec{k}\in\mathbb{Z}^3}\frac{N_{\vec{k}}(t_2, t_1)}{1 + N_{\vec{k}}(t_2, t_1)}.$$

2.4. Semiclassical Results

We shall now show the results obtained using the W.K.B. method [3,6,8,12,16,21].

Theorem 2.1. *In the semiclassical approach, if we assume that $\vec{f} \in C_0^\infty(-\infty, \infty)$, the probability that n pairs are produced at time t is [12]*

$$P_n(t, -\infty) = \frac{1}{n!}\left(\frac{\alpha}{64mc^2}\mathcal{E}(t)\right)^n \exp\left(-\frac{\alpha}{64mc^2}\mathcal{E}(t)\right),$$

where, α is the fine structure constant and

$$\mathcal{E}(t) := \frac{L^3}{8\pi c^2}\left|\dot{\vec{f}}(t)\right|^2$$

is the energy of the electric field at time t. Moreover, the average number of produced pairs at time t is

$$N(t, -\infty) := \sum_{\vec{k}\in\mathbb{Z}^3} N_{\vec{k}}(t, -\infty) = \sum_{n=0}^{\infty} n P_n(t, -\infty) = \frac{\alpha}{64mc^2}\mathcal{E}(t).$$

Consequently, in the semiclassical approximation, pair production is an stochastic Poisson process with expected value $(\alpha/64mc^2)\mathcal{E}(t)$.

Remark 2.2. *For particles with spin $\frac{1}{2}$ we have [13]*

$$P_n(t, -\infty) = \frac{1}{n!}\left(\frac{3\alpha}{32mc^2}\mathcal{E}(t)\right)^n \exp\left(-\frac{3\alpha}{32mc^2}\mathcal{E}(t)\right).$$

Using the results in [3], we can prove the following [16]:

Theorem 2.2. *Assuming that the electric field is $\mathcal{C}^N(\mathbb{R}\backslash\{T\})$ and $\mathcal{C}^{N-1}$ in T; and that the field is switched on and off, the average number of produced pairs after the field is switched off, in the semiclassical approximation, is*

$$N(+\infty, -\infty) \sim \frac{\hbar^{2N}\alpha L^3||D^{N+1}\vec{f}||^2_\infty}{(mc^2)^{2N+1}c^2}. \tag{5}$$

For $N = 0$ this average number is

$$\frac{\alpha}{64mc^2}\frac{1}{8\pi}|\vec{E}(T^+) - \vec{E}(T^-)|^2,$$

where $\vec{E}(T) := (1/c)\dot{\vec{f}}(T)$ is the electric field at time T. In particular, when $N = 0$, if we assume $\vec{E}(T^+)=\vec{0}$ or $\vec{E}(T^-)=\vec{0}$, we have

$$N(+\infty, -\infty) = \frac{\alpha}{64mc^2}\mathcal{E}(T)$$

in the semiclassical approximation.

Remark 2.3. *Using the results obtained in ([6], [21]), and assuming that the electric field is analytic in $\mathbb{R}$ with gentle properties, then $N(+\infty, -\infty)$ is exponentially small in $\hbar$ (see for example Refs. 20 and 24).*

3. Spinless particles in non-homogeneous fields

In this case, the Klein-Gordon equation is ([10], [15])

$$(i\hbar\partial_t - eV(\vec{x},t))^2\psi(\vec{x},t) = (i\hbar c\vec{\nabla} - e\vec{A}(\vec{x},t))^2\psi(\vec{x},t) + m^2c^4\psi(\vec{x},t).$$

The Lagrangian density at time t is:

$$\mathcal{L}(\vec{x},t) = |i\hbar\partial_t\psi(\vec{x},t) - eV(\vec{x},t)\psi(\vec{x},t)|^2 - |i\hbar c\vec{\nabla}\psi(\vec{x},t) - e\vec{A}(\vec{x},t)\psi(\vec{x},t)|^2 - m^2c^4|\psi(\vec{x},t)|^2.$$

Let $\phi(\vec{x},t) = -i\hbar(i\hbar\partial_t - eV(\vec{x},t))\psi(\vec{x},t)$ be the momentum. Then, the energy density is

$$\mathcal{E}(\vec{x},t) = \phi^*(\vec{x},t)\dot{\psi}(\vec{x},t) + \dot{\psi}^*(\vec{x},t)\phi(\vec{x},t) - \mathcal{L}(\vec{x},t) = \mathcal{E}_c(\vec{x},t) + \mathcal{E}_p(\vec{x},t),$$

where

$$\mathcal{E}_c(\vec{x},t) = \frac{1}{\hbar^2}|\phi(\vec{x},t)|^2 + |i\hbar c\vec{\nabla}\psi(\vec{x},t) - e\vec{A}(\vec{x},t)\psi(\vec{x},t)|^2 + m^2c^4|\psi(\vec{x},t)|^2$$

is the kinetic energy density, and $\mathcal{E}_p(\vec{x},t){=}V(\vec{x},t)\rho(\vec{x},t)$ is the potential energy density. We have introduced the electric charge density

$$\rho(\vec{x},t) = -\frac{i}{\hbar}e\left[\phi(\vec{x},t)\psi^*(\vec{x},t) - \psi(\vec{x},t)\phi^*(\vec{x},t)\right].$$

Please note that if we perform the change $\bar{\psi} = (i/\hbar)\phi$, then the Klein-Gordon equation is [10]

$$i\hbar\partial_t\begin{pmatrix}\psi(\vec{x},t)\\ \bar{\psi}(\vec{x},t)\end{pmatrix} = \begin{pmatrix} eV(\vec{x},t) & 1\\ (i\hbar c\vec{\nabla} - e\vec{A}(\vec{x},t))^2 + m^2c^4 & eV(\vec{x},t)\end{pmatrix}\begin{pmatrix}\psi(\vec{x},t)\\ \bar{\psi}(\vec{x},t)\end{pmatrix}.$$

In order to simplify, let us assume that the operator $(i\hbar c\vec{\nabla} - e\vec{A}(\vec{x},t))^2 + m^2c^4$ has a discrete spectrum, and let $\xi_j(\vec{x},t)$ be the eigenfuction with eigenvalue $\lambda(t)$. We write

$$\psi(\vec{x},t) = \sum_j A_j(t)\xi_j(\vec{x},t)$$

and

$$\phi(\vec{x},t) = \sum_j B_j(t)\xi_j(\vec{x},t).$$

Then, we have

$$E_c(t) := \int_{\mathbb{R}^3}\mathcal{E}_c(\vec{x},t)d\vec{x} = \sum_j \frac{|B_j(t)|^2}{\hbar^2} + \lambda^2(t)|A_j(t)|^2.$$

Now, if we make the canonical change:

$$B_j(t) = \frac{\hbar}{\sqrt{2}}\left(P_j(t) + i\bar{P}_j(t)\right);$$

$$A_j(t) = \frac{1}{\hbar\sqrt{2}}\left(Q_j(t) + i\bar{Q}_j(t)\right),$$

we obtain

$$E_c(t) = \frac{1}{2}\sum_j\left(P_j^2(t) + \omega_j^2(t)Q_j^2(t)\right) + \left(\bar{P}_j^2(t) + \omega_j^2(t)\bar{Q}_j^2(t)\right),$$

where $\omega_j(t) := \lambda_j(t)/\hbar$. This is the kinetic energy decomposition in oscillators. We can now quantize these oscillators.

Therefore, the lowest kinetic energy state, *i.e.*, the vacuum state is $|0_t\rangle = \prod_j|0_{j,t}\rangle$, where

$$|0_{j,t}\rangle = \sqrt{\frac{\omega_j(t)}{\pi\hbar}}\exp\left[-\frac{\omega_j(t)}{2\hbar}\left(Q_j^2 + \bar{Q}_j^2\right)\right].$$

In this case, the creation and annihilation operators are

$$\hat{a}_j(t) = \frac{1}{2\sqrt{\lambda_j(t)}}\left[\left(i\hat{P}_j + \omega_j(t)\hat{Q}_j\right) + i\left(i\hat{\bar{P}}_j + \omega_j(t)\hat{\bar{Q}}_j\right)\right]$$

$$\hat{b}^\dagger_{-j}(t) = \frac{1}{2\sqrt{\lambda_j(t)}}\left[\left(-i\hat{P}_j + \omega_j(t)\hat{Q}_{\vec{k}}\right) + i\left(-i\hat{\bar{P}}_j + \omega_j(t)\hat{\bar{Q}}_j\right)\right].$$

Using these operators, we have

$$\hat{E}_c(t) = \sum_j\lambda_j(t)\left(a_j^\dagger(t)a_j(t) + b^\dagger_{-j}(t)b_{-j}(t)\right);$$

$$\hat{\rho}(t) = \sum_j\left(a_j^\dagger(t)a_j(t) - b^\dagger_{-j}(t)b_{-j}(t)\right).$$

$$\hat{\psi}(\vec{x},t) = \sum_j\frac{1}{\sqrt{2\lambda_j(t)}}\left(\hat{a}_j(t) + \hat{b}^\dagger_{-j}(t)\right)\xi_j(\vec{x},t),$$

$$\hat{\bar{\psi}}(\vec{x},t) = \sum_j\sqrt{\frac{\lambda_j(t)}{2}}\left(\hat{a}_j(t) - \hat{b}^\dagger_j(t)\right)\xi_j(\vec{x},t).$$

Now, the different states that contain a definite number of particles and antiparticles are eigenfunctions of the kinetic energy operator, and have the following form:

$$\prod_j\frac{(\hat{a}_j^\dagger(t))^{n_j}}{\sqrt{n_j!}}\frac{(\hat{b}^\dagger_{-j}(t))^{m_j}}{\sqrt{m_j!}}|0_t\rangle, \quad \text{with} \quad n_j, m_j\in\mathbb{N}.$$

We also have

$$\hat{E}_p(t) := \int_{\mathbb{R}^3} \mathcal{E}_p(\vec{x},t)d\vec{x} = \int_{\mathbb{R}^3} V(\vec{x},t) : \hat{\rho}(\vec{x},t) : d\vec{x},$$

where $\hat{\rho}(\vec{x},t) = e(\hat{\psi}(\vec{x},t)\hat{\bar{\psi}}^*(\vec{x},t) + \hat{\psi}^*(\vec{x},t)\hat{\bar{\psi}}(\vec{x},t))$, and :: is the normal ordering operator.

Finally, the Schrödinger equation is

$$i\hbar\partial_t|\Psi\rangle = (\hat{E}_c(t) + \hat{E}_p(t))|\Psi\rangle.$$

3.1. Example

A very interesting example is the case where

$$A_\mu(\vec{x},t)=(V(\vec{x},t),0,0,0).$$

Let us suppose that the potential is switched on and off, and assume that it is confined in a box of volume L^3. In this case the operators $\hat{a}_{\vec{k}}, \ldots \hat{b}^\dagger_{\vec{k}}$, are time independent. Moreover, we have

$$\hat{E}_c = \sum_{\vec{k}\in\mathbb{Z}^3} \lambda_{\vec{k}}(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{b}^\dagger_{-\vec{k}}\hat{b}_{-\vec{k}}),$$

where

$$\lambda_{\vec{k}} = \sqrt{c^2\left|\frac{2\pi\hbar\vec{k}}{L}\right|^2 + m^2c^4}.$$

Furthermore

$$\hat{\psi}(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{\sqrt{2\lambda_{\vec{k}}}}\left(\hat{a}_{\vec{k}} + \hat{b}^\dagger_{-\vec{k}}\right)\xi_{\vec{k}}(\vec{x});$$

$$\hat{\bar{\psi}}(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3} \sqrt{\frac{\lambda_{\vec{k}}}{2}}\left(\hat{a}_{\vec{k}} - \hat{b}^\dagger_{-\vec{k}}\right)\xi_{\vec{k}}(\vec{x}),$$

where

$$\xi_{\vec{k}}(\vec{x}) = \exp\left(i\frac{2\pi}{L}\vec{k}.\vec{x}\right)/L^{\frac{3}{2}}.$$

The Schrödinger equation is $i\hbar\partial_t|\Psi\rangle = (\hat{E}_c+\hat{E}_p(t))|\Psi\rangle$. In interaction picture, the equation behaves as follows: $i\hbar\partial_t|\Psi\rangle_I = \hat{E}_{p,I}(t)|\Psi\rangle_I$, where $\hat{E}_{p,I}(t)$ is the potential energy in the interaction picture, which in this case is

$$\hat{E}_{p,I}(t) = \int V(\vec{x},t) : \hat{\rho}_I(\vec{x},t) : d\vec{x}.$$

In this picture, the electric charge density operator is $\hat{\rho}_I(\vec{x},t) = e(\hat{\psi}_I(\vec{x},t)\hat{\bar{\psi}}^*_I(\vec{x},t) + \hat{\psi}^*_I(\vec{x},t)\hat{\bar{\psi}}_I(\vec{x},t))$, with

$$\hat{\psi}_I(\vec{x},t) = \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{\sqrt{2\lambda_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\xi_{\vec{k}}(\vec{x},t) + \hat{b}^\dagger_{\vec{k}}\xi^*_{\vec{k}}(\vec{x},t)\right);$$

$$\hat{\bar{\psi}}_I(\vec{x},t) = \sum_{\vec{k}\in\mathbb{Z}^3} \sqrt{\frac{\lambda_{\vec{k}}}{2}}\left(\hat{a}_{\vec{k}}\xi_{\vec{k}}(\vec{x},t) - \hat{b}^\dagger_{\vec{k}}\xi^*_{\vec{k}}(\vec{x},t)\right),$$

where

$$\xi_{\vec{k}}(\vec{x},t) = \exp\left[i\left(\frac{2\pi}{L}\vec{k}.\vec{x} - \frac{\lambda_{\vec{k}}}{\hbar}t\right)\right]/L^{\frac{3}{2}}.$$

In the semiclassical limit, we have (see [15]):

$$P_n(t,-\infty) = \frac{1}{n!}\left(\frac{\alpha}{64mc^2}\mathcal{E}(t)\right)^n \exp\left(-\frac{\alpha}{64mc^2}\mathcal{E}(t)\right),$$

where the energy of the electric field at time t is now

$$\mathcal{E}(t) := \frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla V(\vec{x},t)|^2 d\vec{x}.$$

4. Schwinger's formula for spinless particles

In this section, we deduce and interpret the Schwinger's formula for spinless particles. In order to deduct this formula, we consider the potential $\vec{f}(t) = (0,0,\chi(t))$, where

$$\chi(t) = \begin{cases} -cET & \text{if} \quad t < -T \\ cEt & \text{if} \quad -T \le t \le T \\ cET & \text{if} \quad t > T, \end{cases}$$

we have assumed that $eE > 0$ [2,9,14]. A formal deduction of this formula is obtained if we take [12,14]

$$N_{\vec{k}} = \begin{cases} \exp\left[-\frac{\pi\left(c^2p_\perp^2 + m^2c^4\right)}{\hbar ceE}\right] & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| \le eET \\ 0 & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| > eET, \end{cases} \tag{6}$$

where $p_\perp := (2\pi\hbar/L)(k_1,k_2)$ and $N_{\vec{k}} := N_{\vec{k}}(+\infty,-\infty)$. Then, using formula (4), we have

$$|\langle 0_{out}|0_{in}\rangle|^2 = \exp\left[-\sum_{\vec{k}\in\mathbb{Z}^3}\log\left(1+N_{\vec{k}}\right)\right] = \exp\left[-\sum_{\vec{k}\in\mathbb{Z}^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}N^n_{\vec{k}}\right]$$
$$= \exp\left[-\frac{2TL^3E^2\alpha}{8\pi^3\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-n\frac{\pi m^2c^4}{\hbar ceE}\right)\right].$$

This agrees with Schwinger's results [18,19,24,25].

In this case, the generating function $g(x)$ has the following form (see for details Ref. 14):

$$g(x) = \exp\left\{ \frac{2TL^3E^2\alpha}{8\pi^3\hbar} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[(-1)^{n+1} + (x-1)^n\right] \exp\left(-n\frac{\pi m^2c^4}{\hbar ceE}\right) \right\}.$$

Therefore, using this generating function, we obtain that the average number of pairs produced per unit of volume and per unit of time, is [11,14,23]

$$\frac{E^2\alpha}{8\pi^3\hbar} \exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right), \tag{7}$$

and that the relative probability that a pair is created per unit of volume and per unit of time, is [14]

$$\frac{E^2\alpha}{8\pi^3\hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp\left(-n\frac{\pi m^2c^4}{\hbar ceE}\right). \tag{8}$$

This is in contrast with the interpretation of some authors [10,19,24,25], who interpreted that

$$\frac{E^2\alpha}{8\pi^3\hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp\left(-n\frac{\pi m^2c^4}{\hbar ceE}\right) \tag{9}$$

is the probability that a pair is created per unit of volume and per unit of time.

Now we show the way to obtain a rigorous demonstration. Firstly, using semiclassical methods [12,16] it is easy to prove that

$$\sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar k_3}{L}\in[eEt_1,eEt_2]}} N_{\vec{k}} \leq 40\pi^3 \frac{L^3}{(2\pi\hbar)^3} \frac{(\hbar ceE)^2}{(mc^2)^2c^3} ceE(t_2-t_1). \tag{10}$$

$$\sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ \frac{2\pi\hbar|k_3|}{L}\geq eE\left(T+\sqrt{\frac{mcT}{eE}}\right)}} N_{\vec{k}} \leq \frac{\pi^2(\hbar ceE)^2L^3}{8(2\pi\hbar)^3c^3mc^2} + \frac{32\pi^2\hbar^2(eEc)^3L^3}{c^3mc^2(2\pi\hbar)^3} \frac{T}{(mc^3eET)^{\frac{1}{4}}(mc^2)^{\frac{1}{2}}}. \tag{11}$$

Remark 4.1. *Similar bounds are obtained in [9] using another method.*

After obtaining these bounds we must study the problem [2,9,16]

$$\ddot{u}_{\vec{k}} + \frac{1}{\hbar^2}\left[c^2p_\perp^2 + c^2\left(\frac{2\pi\hbar k_3}{L} + eEt\right)^2 + m^2c^4\right] u_{\vec{k}} = 0; \quad t \in (-T,T). \tag{12}$$

If we make the following change

$$y = \sqrt{\frac{2c}{\hbar eE}}(p_3 + eEt),$$

the differential equation

$$u''_{\vec{k}} + \left(\frac{1}{4}y^2 - A\right) u_{\vec{k}} = 0, \tag{13}$$

with $A = (-1/2eEc\hbar)(c^2p_\perp^2 + m^2c^4)$, is obtained.

It is a well-known fact that the Kummer function allows the construction of an independent set of solutions which verifies, when $y<0$ [1,16,22]:

$$\varphi^+_{\vec{k}}(y) = \bar{A}B \exp\left(\frac{\pi A}{4}\right) \exp\left(-\frac{i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A} \times \exp\left(\frac{i}{4}y^2\right) \left[1 + R_1(A,y^2)\right] \tag{14}$$

$$\varphi^-_{\vec{k}}(y) = -\bar{A}B \exp\left(\frac{\pi A}{4}\right) \exp\left(\frac{i\pi}{8}\right) \left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A} \times \exp\left(-\frac{i}{4}y^2\right) \left[1 + R_2(A,y^2)\right], \tag{15}$$

with

$$\bar{A} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{i}{2}A\right)\Gamma\left(\frac{3}{4}+\frac{i}{2}A\right)};$$

$$B = \frac{\Gamma\left(\frac{1}{4}+\frac{i}{2}A\right)}{\Gamma\left(\frac{1}{4}-\frac{i}{2}A\right)} + i\frac{\Gamma\left(\frac{3}{4}+\frac{i}{2}A\right)}{\Gamma\left(\frac{3}{4}-\frac{i}{2}A\right)}.$$

Also, when $y>0$ verifies:

$$\varphi^+_{\vec{k}}(y) = \exp\left(\frac{\pi A}{4}\right)\left\{ 2\bar{A}\exp\left(\frac{i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A} \exp\left(\frac{-iy^2}{4}\right)\left[1+R_3(A,y^2)\right] + \bar{A}C\exp\left(\frac{-i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A} \times \exp\left(\frac{iy^2}{4}\right)\left[1+R_4(A,y^2)\right] \right\} \tag{16}$$

$$\varphi^{-}_{\vec{k}}(y) = \exp\left(\frac{\pi A}{4}\right)\left\{2i\bar{A}^{*}\exp\left(\frac{-i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}-\frac{i}{2}A}\exp\left(\frac{iy^2}{4}\right)[1+R_5(A,y^2)] + \bar{A}C\exp\left(\frac{i\pi}{8}\right)\left(\frac{y^2}{2}\right)^{-\frac{1}{4}+\frac{i}{2}A}\right.$$
$$\left.\times\exp\left(\frac{-iy^2}{4}\right)\left[1+R_6(A,y^2)\right]\right\}, \quad (17)$$

with

$$C = \frac{\Gamma\left(\frac{1}{4}+\frac{i}{2}A\right)}{\Gamma\left(\frac{1}{4}-\frac{i}{2}A\right)} - i\frac{\Gamma\left(\frac{3}{4}+\frac{i}{2}A\right)}{\Gamma\left(\frac{3}{4}-\frac{i}{2}A\right)}.$$

In order to prove the Schwinger formula, the key is the following bound:

$$|R_j(A,y^2)| \le -K\frac{A}{y^2}\exp\left(-\frac{A\pi}{2}\right) \quad j=1,..6, \quad (18)$$

where K is a positive, dimensionless constant that is independent of A and y. We used [22] to obtain this bound.

Remark 4.2. *For the derivative we obtain expressions similar to (14),. . . ,(17).*

Thus, when

$$\frac{2\pi\hbar|k_3|}{eEL} \le T - \sqrt{\frac{Tmc}{eE}},$$

we have $y(-T) < 0$ and $y(T) > 0$. Therefore, for

$$\frac{2\pi\hbar|k_3|}{eEL} \le T - \sqrt{\frac{Tmc}{eE}},$$

using the bound (18) and formulae (14). . . (17) we obtain,

$$N_{\vec{k}} = \exp\left[-\frac{\pi}{eEc\hbar}\left(c^2p_\perp^2 + m^2c^4\right)\right] + F(\vec{p},T), \quad (19)$$

with

$$|F(\vec{p},T)| \le \widetilde{K}\frac{c^2p_\perp^2+m^2c^4}{mc^3TeE} \times \exp\left[-\frac{3\pi}{4eEc\hbar}\left(c^2p_\perp^2+m^2c^4\right)\right],$$

where $\widetilde{K}$ is a dimensionless constant that is independent of T, $p_\perp$ and $\hbar$ (for details see Ref. 16).

With formulae (10), (11) and (19) we can calculate the average number of pairs produced per unit of volume and unit of time when $T\to\infty$. We thus obtain

$$\lim_{\substack{T\to\infty\\ L\to\infty}} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{N_{\vec{k}}}{2TL^3} = \frac{E^2\alpha}{8\pi^3\hbar}\exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right). \quad (20)$$

Remark 4.3. *In Refs. 18 and 23 the authors calculate the quantity*

$$\lim_{L\to\infty}\frac{1}{2TL^3}\sum_{\vec{k}\in\mathbb{Z}^3}\lim_{T\to\infty}|N_{\vec{k}}|^2 = \frac{1}{2T(2\pi\hbar)^3} \times\int_{\mathbb{R}^3}\exp\left(-\frac{\pi(c^2p_\perp^2+m^2c^4)}{\hbar ceE}\right)d\vec{p},$$

and perform the replacement $\int_{\mathbb{R}} dp_3 \to 2eET$ in order to obtain the formula (20). Clearly, this argument is meaningless.

On the same basis, we can prove that, when $T\to\infty$, the relative probability that a pair is produced per unit of volume and per unit of time, is [16]

$$\lim_{\substack{T\to\infty\\ L\to\infty}} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{2TL^3}\frac{|\beta_{\vec{k}}|^2}{|\alpha_{\vec{k}}|^2} = \frac{E^2\alpha}{8\pi^3\hbar} \times\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left(-\frac{n\pi m^2c^4}{\hbar ceE}\right). \quad (21)$$

5. Conclusions

It is important to realize that, in order to understand the pair production phenomenon in the presence of electromagnetic backgrounds, we must interpret the eigenfunctions of the kinetic energy operator as the states that represent a definite number of pairs. Therefore, from the Schrödinger equation of the quantized Klein-Gordon field (and, in the case of spin particles, the quantized Dirac field) we can compute the probability that pairs are created. ¿From this interpretation, it is easy to verify that, when the field is spatially and temporally confined, the probability that the vacuum state remains unchanged, is the exponential of minus twice the imaginary part of the effective action defined by Schwinger. We have also seen that, in the semiclassical approximation, twice the imaginary part of the effective action is the average number of produced pairs. Finally, we have shown how to prove and interpret the Schwinger formula.

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