Self-similarity in a Kantowski-Sachs universe with a string cloud

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We study a cosmological model with very simple solutions characterized by string clouds. Assuming self-similar symmetry for a Kantowski-Sachs spacetime we study the gravitational effects of the cosmic strings. It is also assumed that the primitive universe enters in a false vacuum-dominated era, accelerating the expansion in a time period of the order of 10^{-35} sec (a phase transition), satisfying all the energy conditions. Finally, we examine the possibility that in the last stages of this evolution, the geometry of the universe could be flat.

Keywords: Self-similar; strings; inflation; universe.

Estudiamos un modelo cosmológico con soluciones muy simples caracterizadas por nubes de cuerdas. Exigiendo simetría auto-similar para el espacio-tiempo de Kantowski-Sachs estudiamos los efectos gravitacionales de las cuerdas cósmicas. Se supone además que el universo primitivo entra en una era dominada por el vacío, acelerando la expansión en un periodo de tiempo del orden de 10^{-35} sec (transición de fase), y satisfaciendo todas las condiciones de energía. Finalmente, estudiamos la posibilidad que en los últimos estadios de su evolución el universo puede ser plano.

Descriptores: Autosimilar; cuerdas; inflación; universo.

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1. Introduction

The cosmological theory has studied cosmic strings as the source of the inhomogeneities in the plasma, which then gave rise to the formation of large-scale structures in the universe [1]. In the primitive universe (string-dominated era), the strings could have produced fluctuations in the particle density and one may speculate that since strings were formed before the inflationary era, they were dispersed by the enlargement of the radius of the universe to a such small density that they would be virtually undetectable. When the strings disappeared and particles became important, the fluctuations grew in such a way that galaxies were finally formed and the anisotropy of spacetime introduced by them also disappeared.

The evolution of the universe is usually studied assuming that the only forces present are string tension and gravity [2]. Cosmic strings could have been produced during a 10^{-35} sec period (GUTs's time [3]) after the Big-Bang, but some mechanism should have prevented their existence during the subsequent evolution. The disappearanceof strings may be explained at an earlier time in the universe when a critical temperature T_c was reached and a phase transition made the vacuum energy density (ρ_V) the dominant form of energy density of the universe. A phase transition could have occurred these circumstances, as the universe cooled below T_c, in which a multiplet of scalar fields (Higgs fields) developed a vacuum expectation value, $\langle\phi\rangle = \eta$. This type of phase transition can result in the development of different kinds of vacuum structures, depending on the structure and topology of the gauge group. One possibility is that these vaccum structures gave origin to strings in spacetime. The possibility could also be considered that the early universe, by means of a phase transition, changed over from a state with a finite cosmological constant Λ to a state with zero Λ, that is, from similarity solutions of the second type to similarity solutions of the first type [4]. According to this, one could justify the inflationary scenario of the cosmic strings universe in a self-similar spacetime. The appearence of strings in the primitive universe has been a crucial topic in cosmological and relativistic models. They have been characterized in magnetic fields [5], equations of state [6] and in diffusive transport fluids [7]. Recently, curvature inheritance symmetry in a Riemannian space with applications to string cloud and string fluids has been considered by some authors [8, 9].

The purpose of this paper is to extend the work of Ponce de León (*J. Math. Phys.* **31**(1990) 371) to the cosmic strings fluid case. We study a model of cosmic strings characterized by a Kantowski-Sachs (K-S) self-similar spacetime coupled with a cosmic strings energy-momentum tensor which is equivalent to the one used by Letelier [10]. We study the self-similar solutions, the kinematic and dynamical variables, and the evolution of the self-similar universe with cosmic strings to a inflationary scenario. The kinematic and dynamical conditions of the large-scale universe are also considered.

2. Einstein's field equations

It is assumed that the spacetime metric is described by a K-S type line element

$$ds^2 = dt^2 - e^{\Omega} dr^2 - R^2(d\theta^2 + \text{sen}^2\theta d\phi^2) \qquad (1)$$

where Ω and R are the metric functions which depend on the temporal coordinate t.

We study the gravitational effects of a cosmic strings cloud. The energy-momentum tensor is taken as [2, 10]

$$T_{\mu\nu} = \rho u_\mu u_\nu - \lambda X_\mu X_\nu, \qquad (2)$$

where ρ is the rest energy density for a string cloud with particles attached to it and λ is the string cloud tension density. If the particle density of the configuration is given by ρ_p, then we have

$$\rho = \rho_p + \lambda. \qquad (3)$$

In a comobile frame $u^\mu = \delta_0^\mu$ and $X_\mu = e^{\Omega/2}\delta_\mu^1$. u_μ is the four-velocity of particles and X_μ is a spacelike unit vector (in the radial direction to the strings) orthogonal to u_μ. The X^μ four-vector satisfies the following identities

$$u_\mu u^\mu = 1 = -X_\mu X^\mu, \;\; u_\mu X^\mu = 0. \qquad (4)$$

The Einstein's field equations corresponding to (1) and (2) are as follows ($G = c = 1$):

$$8\pi\rho = (\Omega_{,t} R_{,t} + R_{,t}^2 + 1)R^{-2}, \qquad (5)$$

$$8\pi\lambda = (2R_{,tt}R + R_{,t}^2 + 1)R^{-2} \qquad (6)$$

and

$$2\Omega_{,t}R_{,t} + 2\Omega_{,tt}R + \Omega_{,t}^2 R + 4R_{,tt} = 0. \qquad (7)$$

The commas denote partial derivatives with respect to the coordinate indicated. There are four unknowns (R, Ω, ρ, λ) and only three equations. Therefore, in order to obtain analytic solutions, we impose an additional geometric condition to the spacetime.

3. Self-similar solutions

We shall assume that the spherical distribution admits a one-parametric group of homothetic motions. A homothetic vector field on the manifold is one that satisfies $\pounds_\xi g_{\mu\nu} = 2n g_{\mu\nu}$ on a local chart, where n is a constant on the manifold and $\pounds$ denotes the Lie derivative operator. If $n \neq 0$, we have a proper homothetic vector field and it can always be scaled so as to have $n = 1$; if $n = 0$ ξ, is a Killing vector on the manifold [11–13]. Then, after a constant rescaling, ξ satisfies $\pounds_\xi g_{\mu\nu} = 2g_{\mu\nu}$ and has the form $\xi = \Upsilon(r,t)\partial_t + \Gamma(r,t)\partial_r$. It is easy to check that the homothetic equations reduce to $\xi(X) = 0$, $\xi(Y) = 0$, $\Upsilon = t$ and $\Gamma(r)$, where $X = R/t$ and $Y = \Gamma \exp(\Omega/2)/t$. Therefore, $X = X(\zeta)$ and $Y = Y(\zeta)$ are solutions if the self-similar variable is defined as $\zeta \equiv texp(-\int dr/\Gamma)$. Here we assume $X = C_1\zeta^k$ and $Y = C_2\zeta^l$, where C_1, C_2, k and l are real constants. This kind of solution has been previously applied to different relativistic scenarios [14, 15]. Substituting the equations of the symmetry in Eq. (7), we obtain the relation $k^2+k+l+1 = 0$, where the only possible solution is $k = 0$ y $l = -1$. Therefore

$$\Gamma(r) = r, \qquad (8)$$

$$e^{\Omega} = \text{const} \equiv \beta, \qquad (9)$$

$$R = t/\alpha \qquad (10)$$

and

$$ds^2 = dt^2 - \beta dr^2 - t/\alpha(d\theta^2 + \text{sen}^2\theta d\phi^2) \qquad (11)$$

where β and α are integration constants. In a model with such simple solutions, the homothetic parameters Υ y Γ are completely determined.

4. Kinematic variables

The kinematic variables (shear, expansion and deceleration) we wish to study in this model are

$$\sigma = \pm\frac{1}{\sqrt{3}t}, \qquad (12)$$

$$\Theta = \frac{2}{t} \qquad (13)$$

and

$$q = \frac{1}{2}. \qquad (14)$$

Our primitive solutions provide a simple cosmological model for a primitive self-similar universe which expands in $\Theta{=}2\sqrt{3}\sigma$. Some other authors have reported similar solutions in this context [10, 16]. These solutions also give another important result: an expanding universe accelerating quickly to an inflationary era ($\Theta{>}\,0$, $q{=}1/2$, $t{>}0$). The value of $q{=}1/2$ was reported by Arbab through of the study a viscous model with variable gravitational and cosmological constant [17].

5. Dynamical variables

From the field Eqs. (5-6), we obtain the physical variables (energy and tension density)

$$\rho = \frac{(1+\alpha^2)}{8\pi t^2} \tag{15}$$

and

$$\lambda = \frac{(1+\alpha^2)}{8\pi t^2}. \tag{16}$$

The α parameter represents a physically meaningful quantity: it measures the energy and tensional density of the universe at a given time t.

The model begins with an initial singularity $t = 0$ and subsequently undergoes an expansion with a relation between ρ and λ which satisfies the equation of state of the cosmic strings, since $\lambda = \rho$ ($\rho_p = 0$).

6. Energy conditions

The standard energy conditions, corresponding to a K-S spacetime with cosmic strings [2], can be stated as

- Weak and strong energy conditions.

$$\rho \geq \lambda \text{ with } \lambda \geq 0,\ \rho \geq 0 \text{ with } \lambda < 0. \tag{17}$$

- Dominant energy conditions, which implies

$$\rho \geq 0 \text{ and } \rho^2 \geq \lambda^2. \tag{18}$$

These conditions do not restrict the sign of λ and do not impose any restriction upon α. They are satisfied for all physical variables of the energy-momentum tensor. This solution is physically reasonable because it satisfies the weak, strong and dominant energy conditions. It represents a universe which emerges at $t = 0$ from a singularity with infinite energy density and tension satisfying the cosmic strings equation of state.

7. Equations in the false vacuum phase

The inflationary theory of the universe states that, a few instants after the Big-Bang, the cosmos would have been in a vacuum excited state or false vacuum, during which the universe would have expanded extremely fast. This theory of the universe also defines a very short phase with ultra-fast expansion only 10^{-35} sec after the Big-Bang which expanded the universe size in a huge proportion, becoming geometrically flat [18]. This led the universe to its current, more stable, real vacuum condition.

We shall focus our attention on a cosmic strings universe which can enter an inflationary era, supported by the conditions mentioned above and fulfilling to the conditions $R > 0$, $\alpha > 0$ and $t > 0$. We shall assume a scenario in which the universe enters a false vacuum phase with constant positive energy and negative tensional density ($\lambda_V = -\rho_V$, $\rho_V = \Lambda/8\pi$, where λ_V is the tension density and ρ_V the energy density of the false vacuum). When $t \to t_p$, (t_p is the transition phase time) the strings phase of the universe disappears because λ becomes negative, *i.e.*, a universe dominated by the cosmological constant. This false vacuum with energy density ρ_V will have an associated strings energy-momentum tensor like

$$T_{\mu\nu} = \rho_V (U_\mu U_\nu + X_\mu X_\nu). \tag{19}$$

This can be interpreted as the energy-momentum tensor of a string fluid in the false vacuum. The solutions of the field equations (with the cosmological constant Λ [19]) corresponding to Eqs.(1) and (19) are given by

$$e^{\Omega} = A^2 R_{,t}^2 \tag{20}$$

and

$$R_{,t}^2 = \frac{1}{3}\Lambda R^2 + \frac{F}{R} - 1, \tag{21}$$

where A and F are arbitrary integration constants.

We can define a finite time t_p in which the energy of the universe is given by $\rho_V = \Lambda/8\pi$. We assume that this transition occurs everywhere at the same time. That is, with the field equations one must match the geometry of the spacetime before t_p with the geometry of spacetime after t_p.

8. Matching equations

Following Ponce de León [20] we couple the geometry of space time before and after the time t_p (defined in the previous section) by considering a matching hypersurface $(t - t_p)$. We shall study the matching equations (using the Darmois-Lichnerowicz junction conditions) corresponding to Eqs. (9), (10), (20), and (21).

The continuity of $R_{,t}$ in t_p leads to

$$\Lambda t_p^3 - 3(\alpha^2+1)t_p + 3F\alpha^3 = 0. \tag{22}$$

and the continuity of $\Omega_{,t}$ in t_p to

$$2\Lambda t_p^3 - 3\alpha^3 F = 0. \tag{23}$$

Combining (22) and (23), we obtain

$$t_p = \frac{1}{\sqrt{\Lambda}}(\alpha^2+1)^{1/2}. \tag{24}$$

We see that the parameter α determines the time of the phase transition t_p. From the continuity of the metric functions at t_p, we obtain

$$F = \frac{2}{3\sqrt{\Lambda}\alpha^3}(\alpha^2+1)^{3/2}. \tag{25}$$

and

$$A/\beta = \alpha. \tag{26}$$

The above equations show the assumed phase transitions. Thus the only condition required for the correct matching of

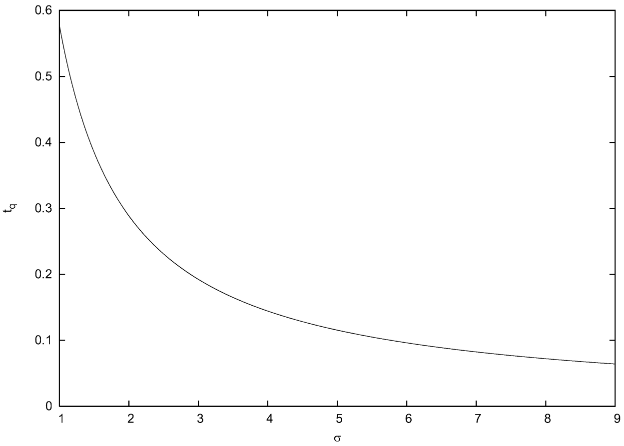

FIGURE 1. Behaviour of t_p (multiplied by 10^{-35} sec) and σ (multiplied by 10^{34} $\sec^{-1}$) as functions of parameter α with $\Lambda = 10^{70}$ $\sec^{-2}$.

both solutions is that $F > 0$ (this condition provides the occurrence of inflation). The behaviour of the shear σ and the expansion Θ at the time t_p of the phase transition depends on the choice of α. Using Eq. (24), we developed t_p in power series (for $\alpha \ll 1$) and we found the values for α that made the inflation possible

$$t_p = \frac{1}{\sqrt{\Lambda}}(\alpha^2+1)^{1/2} = \frac{1}{\sqrt{\Lambda}}(1+\frac{\alpha^2}{2}-\frac{\alpha^4}{8}+\frac{\alpha^6}{48}+\ldots). \tag{27}$$

However, for $\alpha \gg 1$, we have

$$t_p = \frac{1}{\sqrt{\Lambda}}(\alpha^2+1)^{1/2} = \frac{1}{\sqrt{\Lambda}}\alpha. \tag{28}$$

Equations (27) and (28) show that as α increases σ, decreases and the transition time t_p increases. There is a better chance for a phase transition associated with loss of shear. Therefore the inflation is valid for small values of α. This behaviour can be seen in Fig. 1.

The time required (in Guth's inflationary scenario) for a very low exponential growth is in the order of 10^{-35} sec, for an estimated cosmological constant $\Lambda \sim 10^{70}$ $\sec^{-2}$. Figure 1 justifies the compatibility of the possible values of Λ and t_p outlined by the inflationary theory. Figure 1 states that for a phase transition time to occur, this must be accompanied by a lost of the shear and of an exponential accelerated growth of the universe. This entire this process required an estimated cosmological constant of the order of $\Lambda \sim 10^{70}$ $\sec^{-2}$ to justify a transition phase time compatible with the time of Guth's inflationary scenario of the order of 10^{-35} sec.

9. Asymptotic behaviour and plane universe

We are interested in studying solutions which describe an asymptotically de Sitter-like behaviour for a later period. Under some particular conditions (see Ref. 20), we can express the expansion, shear and deceleration in the following way:

$$\Theta = \sqrt{3\Lambda}\left(1-\frac{2}{x^2}+\frac{\epsilon}{2x^3}\right)\left(1-\frac{3}{x^2}+\frac{\epsilon}{x^3}\right)^{-1/2}, \tag{29}$$

$$\sigma = \frac{\sqrt{\Lambda}}{x^2}\left(1-\frac{\epsilon}{2x}\right)\left(1-\frac{3}{x^2}+\frac{\epsilon}{x^3}\right)^{-1/2} \tag{30}$$

and

$$q = \frac{\left(1-\frac{\epsilon}{2x^3}\right)-3\left(1-\frac{2}{x^2}+\frac{\epsilon}{2x^3}\right)^2}{3\left(1+\frac{3}{x^2}+\frac{\epsilon}{2x^3}\right)^2} - \frac{\left(1+\frac{3}{x^2}-\frac{2\epsilon}{x^3}\right)\left(1-\frac{3}{x^2}+\frac{\epsilon}{x^3}\right)}{3\left(1+\frac{3}{x^2}+\frac{\epsilon}{2x^3}\right)^2} \tag{31}$$

where

$$\epsilon \equiv \frac{(1+\alpha)^{3/2}}{\alpha^3} > 1$$

and $x \equiv \sqrt{\Lambda}R$. These equations ($\epsilon > 1$) clearly show that the universe, after the transition to a vacuum-dominated phase, will rapidly evolve to a purely cosmological state with an expansion rate $\Theta \approx \sqrt{3\Lambda}$, shear $\sigma \approx 0$ and deceleration $q \approx -1$. The symmetry that follows the large-scale universe is homogeneous, isotropic, acelerated and with a flat scalar curvature ($\mathcal{R} \approx 0$).

The line-element, takes the form

$$ds^2 \simeq dt^2 - e^{2\sqrt{\Lambda/3}t}(dr^2 + r_o^2(d\Omega)) \tag{32}$$

From Eq. (21), we can see that, as the universe expands, the term $(F/R-1)$ becomes small compared with $\Lambda R^2/3$, and the line-element (with $A^2 = 3/\Lambda r_o^2$, where r_o is an integration constant) shows an asymptotic behaviour towards a de Sitter universe.

10. Conclusions

The homothetic symmetry of the K-S spacetime outlines a cosmic strings universe with a simple structure (at the level of the kinematic and dynamical variables). This symmetry clearly defines the evolution relations $\Theta = 2\sqrt{3}\sigma$ and $q = 1/2$ for a curvature $\mathcal{R} = [2(\alpha^2+1)]/t^2$ during the earliest geomeric conditions of the universe. The value of $q = 1/2$ is predicted by the modern cosmologic theory for a plane spacetime without the cosmological constant.

It was shown that a homothetic spacetime is compatible with Guth's inflationary theory. The phase transition is described as a change in the spacetime geometry. The universe undergoes a transition from the cosmic string-dominated (pre-inflationary) state described by Eqs. (9) and (10) to the vacuum-dominated (inflationary) state described by Eqs. (20) and (21). After this we have a universe that evolves to a particle-dominated era (post-inflationary), described by Eqs. (29) and 30. By virtue of this solutions the initial physical conditions, like the preservation of the energy conditions and the original energy-gravity balance, are compatible with a large scale asymptotically flat, homogeneous and isotropic universe.

Guth's theory, under the refinements of Ponce de León [20], warranties that the cosmic strings took a purely cosmologic effect on the primitive universe expanding exponentially. The universe evolved with very weak shear effects (see Fig. 1) and the false vacuum dominated era and the cosmological constant arised ($\Lambda \sim 10^{70}$ sec^{-2}). In the GUT's epoch the quantum effects would had generated an effective cosmological constant of about 10^{70} sec^{-2}. Observations show that the cosmological constant should be less than 10^{-35} sec^{-2}. Such a discrepancy is one of the greatest problems of theoretical physics.

The cosmological effect of λ and the lost of the effects of the shear σ can imply the existence of a repulsive force that initially boosted the inflation and could direct the current acceleration, making a flat universe unavoidable. Our model justifies this statement with a value of $q = 1/2$, initially, that would change on the large–scale to $q \approx -1$, defined by a curvature of the space–time $\mathcal{R} \approx 0$. The contributions concerning the value $q \approx -1$ and other possible values near this [21] have been a route of exploration and of cosmological interest in affirming the accelation of the present universe. Finally, the negative tension density would help to accelerate the universe and reconcile its expansion age with the ages of stars in globular clusters.

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