The general relativistic geometry of the Navarro-Frenk-White model

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We derive the space time geometry associated with the Navarro-Frenk-White dark matter galactic halo model. We discuss several properties of this geometry, paying particular attention to the corresponding Newtonian limit and stressing the qualitative and quantitative nature of the differences between the relativistic and Newtonian descriptions. We also discuss the characteristics of the possible stress energy tensors which could produce such a geometry, using Einstein's equations.

Keywords: Dark matter; relativity and gravitation; cosmology.

Obtenemos la geometría en el espacio-tiempo asociada con el modelo materia obscura par el halo galáctico propuesto por Navarro, Frenk y White. Analizamos algunas de las propiedades del espacio-tiempo obtenido, en especial las relacionadas con su correspondiente límite newtoniano, subrayando la naturaleza de las diferencias cuantitativas y cualitativas entre la descripción newtoniana y la descripción dentro de la teoría de la relatividad general que presentamos. También discutimos sobre las características de los posibles tensores de materia-energía que, via las ecuaciones de Einstein, podríamos dar lugar a la geometría presentada.

Descriptores: Materia obscura; relatividad y gravitación; cosmología

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The geometry generated by the galactic halo is generally thought to be "almost flat". This assumption is based, first of all, on the fact that the galactic dark halo has very low density, at most some orders of magnitude above the critical density of the Universe. Secondly, the velocities involved are small compared with the speed of light, and third, the dust treatment gives a description of the dynamics which is in good agreement with observation. These facts validate the Newtonian physics as an adequate treatment of the dark halo.

These very same arguments are used for studying the Solar system, where the geometry is also taken as "almost flat". Nevertheless, the general relativistic treatment of the Solar system has made it possible to give important corrections to the Newtonian one and, furthermore, in the general relativistic treatment, this "almost not flatness" is precisely what explains the motion of the planets!

We consider that using a general relativistic version of the galactic dark-matter halo allows one to make a more accurate analysis of the dynamics of the objects, including the study of gravitational lenses, to mention just one application.

In a nutshell: in the present work, we describe how the observations can be related to part of the geometry; we then use Newtonian limits to propose an expression for the complete geometry associated with the Navarro-Frenk-White, NFW, model. Next, we describe the properties of this type of geometry and explain why the Newtonian description works so well. Armed with the geometry, we discuss the type of matter-energy which generates the geometry, that is, the nature of dark matter, a point upon which the Newtonian analysis remains silent.

Specifically, given the fact that the dark halo in the galaxies seems to be spherical and at rest, at least on the average, we consider a general spherically symmetric static space-time, see Eq. (1) below, and were able to determine, on purely geometrical ground, an expression for the tangential velocity of objects moving in circular stable geodesics in terms of the metric coefficients, which turned out to be a very simple one, Eq. (9). We then take a sort of inverse point of view. Instead of considering such an equation as an expression for the velocity, we take it as an expression for the metric coefficient, given the fact that what is being observed is the velocity profile, thus being able to determine part of the geometry based only on observational data, Eq. (10). We determine the other metric coefficient using the fact that the matter-energy distribution is mostly due to the dark matter density, and then identify the M function in the metric coefficient with the mass given by the Newtonian model. In this way, we fix the geometry and can compute the Einstein tensor and the geometric scalars. Finally, by means of Einstein's equations, again with an inverse point of view, we can determine some properties of a given stress-energy tensor, that is, for the matter responsible of the geometry.

In a previous series of works, Matos et al. [1], Guzmán *et al.* [2], we discussed the possibility of determining the geometry of the space-time, and then limiting the type of matter-energy which generates this geometry, based on observational data. In particular, we addressed the problem of making those determinations based on the observed profile of the tangential velocities of objects orbiting galaxies.

In the present work, we present this type of a program applied to the NFW model, Navarro *et al.* [3], which has

proved to have a remarkable power of prediction and agrees very well with observations, particularly with those outside the central galactic region. In what follows, we reproduce the main steps in the reasoning leading to the conditions which the tangential velocity of circular orbits impose on the metric coefficients for the spherically symmetric static case.

We begin with the general line element for this geometry:

$$ds^2 = -\alpha^2(r)\, c^2\, dt^2 + \frac{dr^2}{\left(1 - \frac{2\,G\,M(r)}{c^2\, r}\right)} + r^2\, d\Omega^2, \quad (1)$$

where c is the speed of light and G the gravitational constant, and

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

is the solid angle element. From the corresponding Lagrangian for a test particle in this space,

$$2\mathcal{L} = \left(\frac{ds}{d\tau}\right)^2,$$

where τ stands for the proper time, we find that the energy,

$$E = \alpha^2(r)\, c^2\, \dot{t},$$

the φ-momentum

$$L_\varphi = r^2\, \sin^2\theta \dot{\varphi},$$

and the total angular momentum,

$$L^2 = {L_\theta}^2 + (\frac{L_\varphi}{\sin\theta})^2,$$

with $L_\theta = r^2\dot{\theta}$, where the dot stands for the derivative with respect to the proper time, are conserved quantities under this motion. Notice that we can write the total angular momentum in terms of the solid angle as: $L^2 = r^2\,\dot{\Omega}^2$.

With this information, the fact that the four-velocity, $u^\mu = dx^\mu/d\tau$, is normalized, $u_\nu\, u^\nu = -1$, translates into a radial motion equation:

$$\dot{r}^2 + V(r) = 0, \quad (2)$$

with the potential $V(r)$ given by

$$V(r) = -\left(1 - \frac{2\,G\,M(r)}{c^2\, r}\right)\left(\frac{E^2}{c^2\,\alpha^2(r)} - \frac{L^2}{r^2} - 1\right). \quad (3)$$

Restricting the radial motion to circular stable orbits implies imposing the conditions $\dot{r} = 0$, and $\partial V/\partial r = 0$, so that it describes an extremum of the motion, and $\partial^2 V/\partial r^2 > 0$, in order for the extremum to be a minimum. These three conditions guarantee that the motion will be circular and stable. They also imply the following expressions for the energy and total momentum of the particles in such orbits:

$$E^2 = \frac{c^2\,\alpha^3(r)}{\alpha(r) - r\,\alpha(r)_r}, \quad (4)$$

$$L^2 = \frac{r^3\,\alpha(r)_r}{\alpha(r) - r\,\alpha(r)_r}, \quad (5)$$

where the subindex r stands for the derivative with respect to r.

On the other hand, we can rewrite the line element for this geometry, Eq. (1), in terms of the modulus of the spatial velocity, normalized with the speed of light, measured by an inertial observer far from the source, as $ds^2 = -dt^2\left(1 - v^2\right)$, where

$$v^2 = \frac{1}{c^2\,\alpha^2(r)}\left(\frac{\left(\frac{dr}{dt}\right)^2}{1 - \frac{2\,G\,M(r)}{c^2\, r}} + r^2\left(\frac{d\Omega}{dt}\right)^2\right). \quad (6)$$

This last equation implies that the modulus of the angular velocity, which is the tangential velocity in the case of circular orbits, is defined as:

$${v_{\rm tg}}^2 = \frac{r^2}{c^2\,\alpha^2(r)}\left(\frac{d\Omega}{dt}\right)^2 = \frac{1}{c^2\,\alpha^2(r)}\left(\frac{d\tau}{dt}\right)^2 \dot{\Omega}^2, \quad (7)$$

thus, in terms of the conserved quantities, the angular velocity takes the form:

$${v_{\rm tg}}^2 = \frac{c^2\,\alpha^2(r)}{r^2}\,\frac{L^2}{E^2}. \quad (8)$$

Using the expression derived for these conserved quantities, Eq. (5), we find that the tangential velocity can be expressed in terms of the metric coefficient α as

$${v_{\rm tg}}^2 = \frac{r\,\alpha(r)_r}{\alpha(r)}. \quad (9)$$

This last equation allows us to determine the metric coefficient $\alpha(r)$ in terms of the observed velocity profile:

$$\alpha(r) = \exp\int \frac{v_{\rm tg}^2(r)}{r}\, dr. \quad (10)$$

This is the key equation of the reasoning: to use the observations, $v_{\rm tg}(r)$, in order to partially determine the geometry of the surrounding spacetime.

Now we combine these results with the Navarro-Frenk-White model. This model predicts the density profile [3]

$$\rho_{\rm NFW} = \frac{\rho_0}{\frac{r}{r_s}(1 + \frac{r}{r_s})^2}, \quad (11)$$

where $\rho_0 = \rho_{\rm crit}\,\delta_c$, r_s is a scale radius, δ_c is a characteristic (dimensionless) density, and

$$\rho_{\rm crit} = \frac{3H^2}{8\,\pi\,G}$$

is the critical density for closure. The mass function,

$$M_{\rm NFW}(r) = 4\,\pi\int r^2 \rho_{\rm NFW}\, dr,$$

with the integration constant chosen so that

$$M_{\rm NFW}(r=0)=0,$$

takes the form

$$M_{\rm NFW}(r)=4\,\pi\,r_s^3\,\rho_0\,\left(\ln\left(1+\frac{r}{r_s}\right)-\frac{\frac{r}{r_s}}{1+\frac{r}{r_s}}\right). \quad (12)$$

This implies, equating the gravitational force with the centrifugal one, the following profile for the tangential velocity:

$$v^2_{{\rm tg}_{NFW}}=v_0^2\left(\frac{\ln(1+\frac{r}{r_s})}{\frac{r}{r_s}}-\frac{1}{1+\frac{r}{r_s}}\right), \quad (13)$$

where $v_0^2=4\,\pi\,G\,\rho_0\,r_s^2$, which is a velocity proportional to the maximal velocity of the profile. The maximum is located at $r_{\rm max}=2.1626\,r_s$, and the velocity at this point is $v_0=2.15\,v_{{\rm tg}_{\rm max}}$. Thus, this parameter gives a measure of the maximal tangential velocity reached by the orbiting particles. It is proportional to the V_{200} value of the velocity at the virial radius given by NFW, [3].

These expressions are directly predicted by the NFW model and, except for the central parts of galaxies, it has been successfully compared with the actual observations [3].

Using the expression derived for the tangential velocity within the NFW model, Eq. (13), in the expression we obtained above, Eq. (9), we obtain a remarkably simple expression for the g_{tt} coefficient:

$$\alpha^2(r)=\left(1+\frac{r}{r_s}\right)^{-2\frac{v_0^2\,r_s}{c^2\,r}}, \quad (14)$$

where we have set the integration constant such that α goes to one for large radii, and we have normalized the NFW velocity with the speed of light. There are several noticeable features in this last expression. First, it is regular everywhere. The divergency problem that the NFW-density has in the central region, is not reflected in the metric coefficient:

$$\lim_{r\to 0}\alpha^2=e^{-\frac{v_0^2}{c^2}},\ \lim_{r\to\infty}\alpha^2=1. \quad (15)$$

Actually, the α-function goes to one for large radii, as can be seen in the Fig. 1, recovering and validating the Newtonian assumption in that region.

We can go on and work with the other metric coefficient,

$$g_{rr}=(1-\frac{2\,G\,M(r)}{c^2\,r})^{-1}.$$

The unknown function $M(r)$ can not be directly identified with the mass function. Essentially, this is due to the fact that, in General Relativity, the mass also includes the energy of the system, gravitational, kinetic, or any other form of energy. Thus, in GR, $M(r)$ is given by matter plus energy. Explicitly, we find that, in the present case, the matter-energy density is given by $\rho=\rho_{NFW}+\mathcal{O}(v_0/c)^4$, and the $M(r)$ function is given by the Einstein's equations as

$$M(r)=\int_0^r \rho x^2 dx. \quad (16)$$

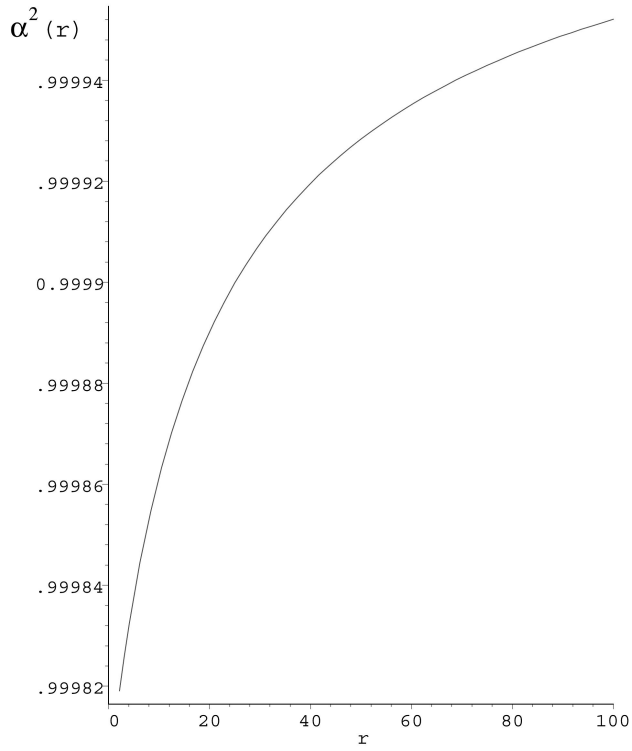

FIGURE 1. The g_{tt} metric coefficient, for values $\frac{v_0^2}{c^2}=10^{-4}$, $r_s=10kpc$.

Thus, it will coincide with the expression for mass given by the NFW model, Eq. (12), up to terms of the order $\mathcal{O}(v_0/c)^4$, which are negligible. This fact establishes a solid basis for the approximation for the mass function $M(r)$ given by the NFW expression.

Accordingly, the line element of a galaxy with the NFW density profile takes the form:

$$ds^2=-\left(1+\frac{r}{r_s}\right)^{-2\frac{v_0^2\,r_s}{c^2\,r}}c^2\,dt^2 + \frac{dr^2}{1-2\,\frac{v_0^2}{c^2}\left(\frac{\ln\left(1+\frac{r}{r_s}\right)}{\frac{r}{r_s}}-\frac{1}{1+\frac{r}{r_s}}\right)}+r^2\,d\Omega^2, \quad (17)$$

which is then the geometry associated with the Navarro-Frenk-White model.

The line element has two free parameters, namely v_0, the characteristic speed, and r_s, the characteristic radius.

With respect to the g_{rr} metric coefficient, it is also regular everywhere, we see its behavior in Fig. (2), and it has the following limits:

$$\lim_{r\to 0}g_{rr}=\lim_{r\to\infty}g_{rr}=1. \quad (18)$$

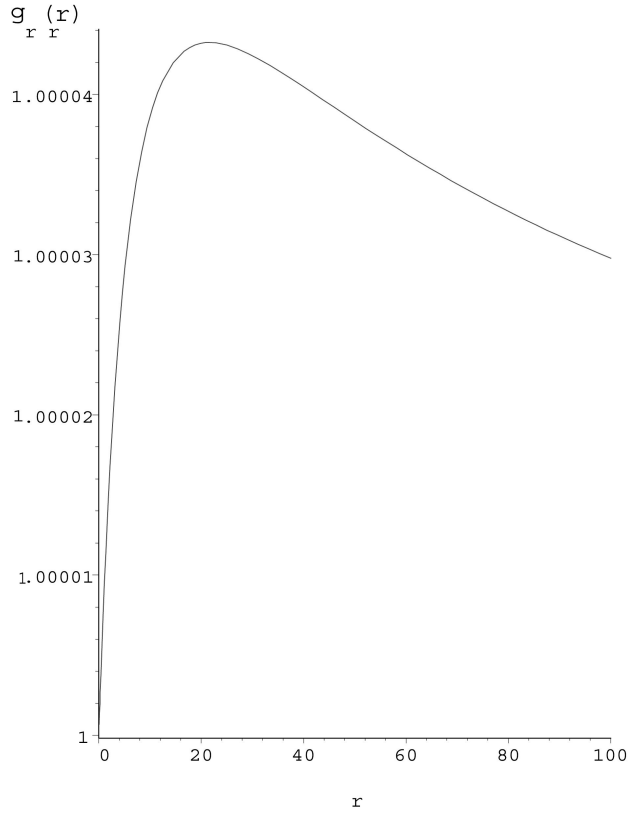

FIGURE 2. The g_{tt} metric coefficient, for values $v_0^2/c^2 = 10^{-4}$, $r_s = 10kpc$.

Thus, the line element given by Eq. (17) is regular at all points and, far from the source, takes the form of flat spacetime.

However, even though the line element does not show any divergence, we expect to have one, directly inherited from the divergence of the NFW density at the center. Actually, when one derives the scalar of curvature, R, it is seen that

$$R_{r\to 0} \sim 2\frac{v_0^2}{c^2\, r_s\, r} - 2\frac{v_0^2}{c^2\, r_s^2}\left(2 - \frac{v_0^2}{c^2}\right) + \mathcal{O}(r), \qquad (19)$$

showing a clear divergence at the center, as expected. This implies that we do not have a solution describing a spacetime; as long as there is a naked singularity, the divergence is not covered by any horizon. What we have is a geometry that implies a dynamic similar to the one described by the NFW model, so we claim that this geometry will be related to the exterior part of a complete spacetime with the observed dynamics.

The metric coefficients differ from the unity by very tiny amounts, validating the flat space approximation, that is, the Newtonian analysis. However, within the General relativistic theory, these tiny amounts of non-flat geometry are the ones responsible for the observed dynamics!

Finally, we can construct the Einstein tensor in order to check that our approximation, which produces the space time given in Eq. (17), is well defined and consistent. The Einstein tensor gives three independent, non-zero components:

$$G_t^t = -2\frac{v_0^2}{c^2\, r_s^2}\,\frac{1}{x\, u^2} \sim v_0^2,$$

$$G_r^r = -4\frac{v_0^4}{c^4\, r_s^2}\,\frac{1}{x^4\, u^2}\,\left[u\,\ln u\,(u\,\ln u - 2\,x) + x^2\right] \sim v_0^4,$$

$$G_\theta^\theta = -\frac{v_0^4}{c^4\, r_s^2}\,\frac{1}{x^5\, u^3}\,(u\,\ln u - x)$$
$$\times\left[2\,u\,\ln u\,\left(\frac{v_0^2}{c^2}\,u\,\ln u - 2\,x\,\left(u + \frac{v_0^2}{c^2}\right)\right)\right.$$
$$\left. + x^2\,\left(7\,x + 4 + 2\,\frac{v_0^2}{c^2}\right)\right] \sim v_0^4, \qquad (20)$$

where we have defined $x = r/r_s$, and $u = 1 + r/r_s$.

Being aware of the caveats on promoting this geometry to a spacetime, we can still say something about the type of matter which could produce this type of geometry, by means of the Einstein equations:

$$G_\nu^\mu = \frac{8\,\pi\, G}{c^4}\,T_\nu^\mu, \qquad (21)$$

where G stands for the gravitational constant, and T_ν^μ describes the tensor of distribution of the matter-energy in the spacetime.

It is common to identify the T_t^t component of the matter-energy tensor with the density of matter, and energy, present in the spacetime, ρ, that is $T_t^t = -c^2\,\rho$. Thus, using the corresponding expression for the Einstein tensor, Eq. (20), we obtain the same expression relating mass and density as that obtained in the Newtonian theory, Eq. (12). This fact supports the interpretation of the function $M(r)$ in the line element, Eq. (17), as the mass of the system, as we discussed above.

About the other components of the matter-energy tensor, by means of the Einstein equations, we can conclude that such matter-energy tensor can not be dust or even a perfect fluid. The reason for these conclusions are clear. Again, if we take the matter-energy tensor to be a perfect fluid one

$$T_\nu^\mu = \left(\rho\, c^2 + p\right)\, u^\mu\, u_\nu + p\,\delta_\nu^\mu, \qquad (22)$$

with u^μ the four velocity for the spherically symmetric case we are dealing with, the matter-energy tensor takes the form $T_\nu^\mu = \mathrm{diag}\left(-\rho\, c^2, p, p, p\right)$. But, from the Einstein tensor, we see that clearly G_r^r, and G_θ^θ are non zero and different from each other. Thus, the matter-energy curving the spacetime to form the NFW geometry can not be dust or a perfect fluid.

It is important to notice that, as the density, ρ, is very small, the pressures are even smaller, and tend very quickly to zero. These, again validate the Newtonian treatment taking the fluid as dust, in an exterior region. Nevertheless, we must recall that this analysis aims at determining the actual nature of dark matter, pointing out the physical properties it must have.

Thus, the geometry associated with the NFW model that we have derived, allows us to explain the validity of the Newtonian treatment, pointing out its limits, and to discuss on the nature of dark matter, and opens the road to conducting several dynamical studies within the general relativistic theory.

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