Two stream approximation to radiative transfer equation: An alternative method of solution

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An alternative analytical method of solution to radiative transfer equation in the two-stream approximation is studied. The method is formulated in terms of the *diffusion-type equation for radiative transfer* associated with the fluxes (*irradiances*) $F_d = F^+ - F^-$ and $F_s = F^+ + F^-$, where F^+ and F^- are defined as the upward and downward fluxes respectively. The diffusion-type equations are independent and therefore the method of solution is algebraically easier and faster than that used to solve the two coupled differential equations associated with F^+ and F^-.

Keywords: F^+ upward flux; F^- downward flux.

Se presenta un método alternativo para la solución análitica de las ecuaciones que describen la transmisión de radiación en la atmósfera, mediante la aproximación de dos flujos. El método se basa en una formulación de *ecuaciones de tipo difusivo* asociadas a los flujos (*irradiancias*) $F_d = F^+ - F^-$ y $F_s = F^+ + F^-$, donde F^+ y F^- se definen como flujos ascendente y descendente respectivamente. Estas ecuaciones resultan ser independientes y en consecuencia se obtiene un método de solución directo.

Descriptores: F^+ flujo ascendente; F^- flujo descendente.

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1. Introduction

The multiple scattering process in the atmosphere is the physical process by means of which the radiant energy is transferred through the atmosphere, especially when aerosols and clouds are involved. The fundamentals of the multiple scattering process are based on the radiative transfer equation which is a integro-differential equation [1]. The exact solution of the radiative transfer equation in a scattering and absorbing media is difficult to obtain even for plane-parallel atmospheres; under these circumstances approximate methods are necessary. In the last three decades, special attention has been paid to find simple and effective methods for solving the radiative transfer equation. The simplest method to determine the radiative flux is the two-stream approximation, which has been widely used in radiative flux calculations in climate models, as described in several review papers like Meador and Weaver (1980), Shettle and Weinman (1970), Zdunkowski *et al.* (1980), and King and Harshvardhan (1986), Li and Ramaswamiy , (1995), etc. In these works, the Eddington approximation, quadrature discrete ordinate method, and hemispheric constant method have been incorporated into a standard solution form with appropriate choice of the parameters. The two-stream approximation allows the formulation of a coupled pair of differential equation for the upward F^+ and downward F^- fluxes; which represents integrals of the intensity over hemispheres. The solution for those two coupled differential equations can be given in a matrix scheme by calculating the eigenvalues and their corresponding eigenvectors. Our purpose in this work is to give an alternative method of solution for those two coupled differential equations, based on the *diffusion-type equation for radiative transfer*, Liou, (2002). The method consists in transforming the coupled differential equations into a pair of independent second order differential equations associated with the fluxes $F_d = F^+ - F^-$ and $F_s = F^+ + F^-$, which are easer to solve. The solution for F^+ and F^- obtained by this alternative method can easily be transformed as that used by Toon *et al.* (1989) in the study of heating rates and photodissociation rates in homogeneous multiple scattering atmospheres. The method can also be applied to solve the two-stream approximation method proposed by Shettle and Weinman (1970) and used later by Ruíz-Suárez *et al.* (1993) to study the Photolytic rates for NO_2, O_3 and $HCHO$ in the atmosphere of México City.

In this work, we first establish the radiative transfer equation for plane-parallel atmospheres in the manner of Chandrasekhar, (1960) and modify a little bit the Meador and Weaver proposal to establish, in the Eddington approximation, the two coupled differential equations for the upward and downward fluxes. We propose the alternative method of solution based on the diffusion-type equation, and finally give our comments.

2. The radiative transfer equation

The general equation of the radiative transfer equation in a plane parallel scattering atmosphere can be written (Chandrasekhar, 1960) for the diffuse intensity as

$$\mu \frac{dI(\tau, \mu, \varphi)}{d\tau} = I(\tau, \mu, \varphi) - J(\tau, \mu, \varphi) - J_0(\tau, \mu_0, \phi_0), \quad (1)$$

where $I(\tau, \mu, \varphi)$ is the diffuse intensity, τ is the optical depth measured along the zenith direction beginning at the top of the atmosphere, $\mu = \cos\theta$, with θ is the local zenith angle, and φ the local azimuth angle. The function $J(\tau, \mu, \varphi)$ is known as the *internal source function* due to multiple scattering, and defined as

$$J(\tau, \mu, \varphi) = \frac{\tilde{\omega}}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \wp(\mu, \varphi; \mu', \varphi')$$

$$\times I(\tau, \mu', \varphi') d\varphi' d\mu', \quad (2)$$

where $\tilde{\omega}$ is called as the single-scattering albedo, that is, the ratio of the scattering coefficient to the sum of the scattering and absorption coefficients. The single scattering albedo is, in general, a function of the optical depth τ. The function $\wp(\mu, \varphi; \mu', \varphi')$ is the phase function or single-particle scattering law for radiation scattered from the direction (μ', φ') into the direction (μ, φ).

The function J_0 is known as the *external source function* due to single scattering of the direct solar beam, it is defined as

$$J_0(\tau, \mu_0, \varphi_0) = \frac{\tilde{\omega}}{4\pi} \wp(\mu, \varphi; -\mu_0, \varphi_0) F_\odot \, e^{-\tau/\mu_0}, \quad (3)$$

such that $F_\odot$ is the incident solar flux on the top of the atmosphere at angle θ_0, such that $\mu_0 = -\cos\theta_0$ (see Fig. 1).

We can express the phase function in terms of Legendre polynomials P_l to solve Eq. (1), in the following way

$$\wp(\cos\Theta) = \sum_{l=0}^{N} \omega_l P_l(\cos\Theta), \quad (4)$$

where

$$\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\varphi - \varphi'), \quad (5)$$

and Θ being the angle between incident and scattered radiation, ω_l can be determined from the orthogonal properties of Legendre Polynomials, such that

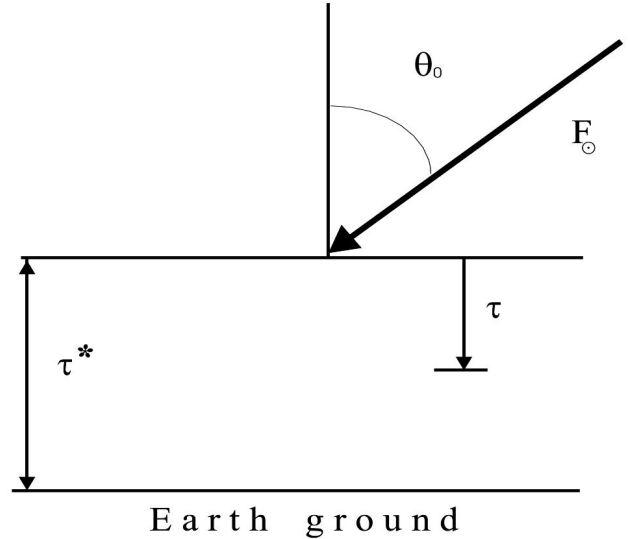$$\omega_l = \frac{2l+1}{2} \int_{-1}^{1} \wp(\cos\Theta) P_l(\cos\Theta) \, d\cos\Theta. \quad (6)$$

FIGURE 1. Illustration of the incident solar flux $F_\odot$ on the top of a plane-parallel atmosphere, at an angle θ_0. The parameter τ^* is the total optical depth and τ represents any point inside the layer.

When $l = 0$, $\omega_0 = 1$ which represents the normalization of the phase function. When $l = 1$, we have

$$g = \frac{\omega_1}{3} = \frac{1}{2} \int_{-1}^{1} \wp(\cos\Theta) \cos\Theta \, d\cos\Theta, \quad (7)$$

which is referred to as *asymmetry factor*, and is the first moment of the phase function. It is an important parameter in radiative transfer because it characterizes the scattering pattern of a particle. Substituting Eq. (5) into Eq. (4), we have for the phase function

$$\wp(\mu, \varphi; \mu', \varphi')$$
$$= \sum_{l=0}^{N} \omega_l P_l[\mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\varphi - \varphi')]. \quad (8)$$

Expanding this function for its argument and using the addition theorem of spherical harmonics [1], it can be shown that

$$\wp(\mu, \varphi; \mu', \varphi') = \sum_{l=0}^{N} \omega_l \left\{ P_l(\mu) P_l(\mu') \right.$$

$$\left. +2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)1} P_l^m(\mu) P_l^m(\mu') \cos m(\varphi - \varphi') \right\}. \quad (9)$$

In the azimuth-independent case, the phase function, according to Eq.(9), reduces to

$$\wp(\mu, \mu') = \frac{1}{2\pi} \int_{0}^{2\pi} \wp((\mu, \varphi; \mu', \varphi') \, d\varphi$$

$$= \sum_{l=0}^{N} \omega_l P_l(\mu) P_l(\mu'), \quad (10)$$

and the intensity $I(\tau, \mu)$ is defined as [1]

$$I(\tau, \mu) = \frac{1}{2\pi} \int\limits_0^{2\pi} I(\tau, \mu, \varphi)\, d\varphi\,. \tag{11}$$

According to Eqs. (10) and (11), the azimuthal integration of the radiative transfer equation leads to

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}}{2} \int\limits_{-1}^{1} I(\tau, \mu')\wp(\mu, \mu')\, d\mu'$$
$$- \frac{\tilde{\omega}}{4\pi} \wp(\mu, -\mu_0) F_\odot e^{-\tau/\mu_0}\,. \tag{12}$$

2.1. Two-stream method and the Eddington approximation

Eq. (12) can be written in terms of the upward F^+ and downward F^- fluxes defined as

$$F^+(\tau) = 2\pi \int\limits_0^1 I(\tau, \mu)\, \mu\, d\mu\,, \tag{13}$$

$$F^-(\tau) = 2\pi \int\limits_0^1 I(\tau, -\mu)\, \mu\, d\mu\,. \tag{14}$$

In this case, the radiative transfer equation (12) can be decomposed in two differential equations, one for the upward flux and the other one for the downward flux; that is

$$\frac{dF^+}{d\tau} = 2\pi \int\limits_0^1 I(\tau, \mu)\, d\mu$$
$$- \pi\tilde{\omega} \int\limits_0^1 \int\limits_{-1}^1 I(\tau, \mu')\wp(\mu, \mu')d\mu\, d\mu'$$
$$+ \frac{\tilde{\omega}}{2}\gamma_3\, F_\odot\, e^{-\tau/\mu_0}\,, \tag{15}$$

$$\frac{dF^-}{d\tau} = -2\pi \int\limits_0^1 I(\tau, -\mu)\, d\mu$$
$$+ \pi\tilde{\omega} \int\limits_0^1 \int\limits_{-1}^1 I(\tau, \mu')\wp(-\mu, \mu')d\mu\, d\mu'$$
$$- \frac{\tilde{\omega}}{2}\gamma_4 F_\odot\, e^{-\tau/\mu_0}\,, \tag{16}$$

where

$$\gamma_3 = \frac{1}{2}\int\limits_0^1 \wp(\mu, -\mu_0)\, d\mu, \quad \gamma_4 = \frac{1}{2}\int\limits_0^1 \wp(-\mu, -\mu_0)\, d\mu\,. \tag{17}$$

To write the right hand side of Eqs. (15) and (16) in terms of F^+ and F^-, we must evaluate the integrals of such expres-

sions using some approximations. Firstly, from the normalization of the phase function given by

$$\frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_{-1}^1 \wp(\mu, \varphi; \mu', \varphi')\, d\mu\, d\varphi = \frac{1}{2}\int\limits_{-1}^1 \wp(\mu, \mu')\, d\mu = 1, \tag{18}$$

it can be shown that $\gamma_3 + \gamma_4 = 1$, where we have assumed that $\mu' = \mu_0$.

On the other hand, in a similar way as the phase function, the diffuse intensity may also be expanded in terms of Legendre polynomials such that

$$I(\tau, \mu) = \sum_{l=0}^N I_l(\tau)P_l(\mu)\,. \tag{19}$$

The *Eddington approximation* is obtained for $N = 1$ and therefore Eqs. (4) and (19) are approximated by

$$I(\tau, \mu) = I_0(\tau) + \mu\, I_1(\tau),$$
$$\wp(\mu, \mu') = 1 + 3g\mu\mu'\,. \tag{20}$$

In this approximation it can be shown that

$$2\pi I(\tau, \pm\mu) = \frac{1}{2}[(2 \pm 3\mu)F^+ + (2 \mp 3\mu)F^-]\,, \tag{21}$$

and therefore

$$2\pi \int\limits_0^1 I(\tau, \pm\mu)\, d\mu = \frac{1}{4}[(4 \pm 3)F^+ + (4 \mp 3)F^-]\,. \tag{22}$$

From Eq. (20), we can check that $\wp(-\mu, \mu') = \wp(\mu, -\mu')$, and therefore the integrals of Eqs. (15) and (16) reduce to

$$\pi\tilde{\omega} \int\limits_0^1 \int\limits_{-1}^1 I(\tau, \mu')\wp(\mu, \pm\mu')\, d\mu\, d\mu'$$
$$= \frac{\tilde{\omega}}{4}[(4 \pm 3g)F^+ + (4 \mp 3g)F^-]\,. \tag{23}$$

The coefficients γ_3 and γ_4 are approximated by

$$\gamma_3 = \frac{1}{4}(2 - 3g\mu_0), \qquad \gamma_4 = \frac{1}{4}(2 + 3g\mu_0)\,, \tag{24}$$

where the asymmetry factor reads explicitly as

$$g = \frac{1}{2}\int\limits_{-1}^1 \mu\, \wp(\mu, 1)\, d\mu\,. \tag{25}$$

Substituting Eqs. (22) and (23) into their respective expressions in Eqs. (15) and (16), we finally get the *two-stream approximation* for the upward and downward irradiances; which can be written as [8], [9]

$$\frac{dF^+}{d\tau} = \gamma_1 F^+(\tau) - \gamma_2 F^-(\tau) - \gamma_3\tilde{\omega}\, F_\odot e^{-\tau/\mu_0}\,, \tag{26}$$

$$\frac{dF^-}{d\tau} = \gamma_2 F^+(\tau) - \gamma_1 F^-(\tau) + \gamma_4\tilde{\omega}\, F_\odot e^{-\tau/\mu_0}\,, \tag{27}$$

where

$$\gamma_1 = \frac{1}{4}[7 - \tilde{\omega}(4 + 3g)], \qquad \gamma_2 = -\frac{1}{4}[1 - \tilde{\omega}(4 - 3g)]\,. \tag{28}$$

2.2. Diffusion-type method of solution

The solution of Eqs. (26) and (27) can be formulated in a matrix scheme associated with the eigenvalues and their corresponding eigenvectors. The alternative method reduces the system (26) and (27) in two independent second order differential equations: one for the difference $F_d = F^+ - F^-$, and the other for the sum $F_s = F^+ + F^-$. So, for these two quantities we have after an immediate algebra

$$\frac{dF_d}{d\tau} = (\gamma_1 - \gamma_2)F_s - \tilde{\omega}\, F_\odot e^{-\tau/\mu_0}\,, \tag{29}$$

$$\frac{dF_s}{d\tau} = (\gamma_1 + \gamma_2)F_d - (\gamma_3 - \gamma_4)\tilde{\omega}\, F_\odot e^{-\tau/\mu_0}\,. \tag{30}$$

For these equations we get rapidly

$$\frac{d^2 F_d}{d\tau^2} = k^2 F_d - \xi\, e^{-\tau/\mu_0}\,, \tag{31}$$

$$\frac{d^2 F_s}{d\tau^2} = k^2 F_s - \eta\, e^{-\tau/\mu_0}\,, \tag{32}$$

where $k^2 = \gamma_1^2 - \gamma_2^2$ and the parameters

$$\xi = \tilde{\omega}F_\odot[(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4) - 1/\mu_0]\,, \tag{33}$$

$$\eta = \tilde{\omega}F_\odot[\gamma_1 + \gamma_2 - (\gamma_3 - \gamma_4)/\mu_0]\,. \tag{34}$$

Clearly Eqs. (31) and (32) are the two independent second order differential equations, called the *diffusion-type equations for radiative transfer.*

It is easy to see that the solutions to Eqs. (31) and (32) are given by the sum of the homogeneous part plus a particular solution; thus

$$F_d(\tau) = C_1 e^{k\tau} + C_2 e^{-k\tau} + \alpha e^{-\tau/\mu_0}\,, \tag{35}$$

$$F_s(\tau) = C_3 e^{k\tau} + C_4 e^{-k\tau} + \beta e^{-\tau/\mu_0}\,. \tag{36}$$

By substituting the particular solution we obtain

$$\alpha = \frac{\mu_0^2\, \tilde{\omega} F_\odot}{(k\mu_0)^2 - 1}\left[(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4) - 1/\mu_0\right]\,, \tag{37}$$

$$\eta = \frac{\mu_0^2\, \tilde{\omega} F_\odot}{(k\mu_0)^2 - 1}\left[\gamma_1 + \gamma_2 - (\gamma_3 - \gamma_4)/\mu_0\right]\,. \tag{38}$$

The constants C_1 and C_3, C_2 and C_4 are not independent, in fact it can be shown from Eqs. (29) and (30) that $C_1 = [(\gamma_1 - \gamma_2)/k]C_3$ and $C_2 = -[(\gamma_1 - \gamma_2)/k]C_4$. According to the definitions of F_d and F_s, we get immediately the following

$$F^+(\tau) = uC_3 e^{k\tau} + vC_4 e^{-k\tau} + G^+ e^{-\tau/\mu_0}\,, \tag{39}$$

$$F^-(\tau) = vC_3 e^{k\tau} + uC_4 e^{-k\tau} + G^- e^{-\tau/\mu_0}\,, \tag{40}$$

where

$$u = \frac{1}{2}\left[1 + \frac{(\gamma_1 - \gamma_2)}{k}\right]\,, \qquad v = \frac{1}{2}\left[1 - \frac{(\gamma_1 - \gamma_2)}{k}\right]\,, \tag{41}$$

and

$$G^+ = \frac{\mu_0^2\, \tilde{\omega} F_\odot}{(k\mu_0)^2 - 1}\left[(\gamma_1 - 1/\mu_0)\gamma_3 + \gamma_2\gamma_4\right]\,, \tag{42}$$

$$G^- = \frac{\mu_0^2\, \tilde{\omega} F_\odot}{(k\mu_0)^2 - 1}\left[(\gamma_1 + 1/\mu_0)\gamma_4 + \gamma_2\gamma_3\right]\,. \tag{43}$$

These solutions for F^+ and F^- can be transformed as that used by Toon *et al.* (1989), if we define the constant

$$\Gamma = \frac{\gamma_1 - k}{\gamma_2} = \frac{\gamma_2}{\gamma_1 + k}\,, \tag{44}$$

which satisfies because $k^2 = \gamma_1^2 - \gamma_2^2$. In this case u and v are related by $v = \Gamma u$. If we redefine the constants $uC_3 = K_1$ and $uC_4 = K_2$, we finally get from Eqs. (39) and (40) the following solutions

$$F^+(\tau) = K_1 e^{k\tau} + \Gamma K_2 e^{-k\tau} + G^+ e^{-\tau/\mu_0}\,, \tag{45}$$

$$F^-(\tau) = \Gamma K_1 e^{k\tau} + K_2 e^{-k\tau} + G^- e^{-\tau/\mu_0}\,, \tag{46}$$

which are the same as those used by Toon *et al.* (1989). The constants K_1 and K_2 can be calculated by applying the boundary conditions for the upward and downward fluxes at the top and at the bottom of the atmosphere. It is assumed, in general, that there is no incident flux at the top of the atmosphere, in this case $F^-(0) = 0$. At the surface, it can be assumed that the upward flux is R_s times the downward flux, where R_s is known as the reflectivity of the bottom, that is $F^+(\tau*) = R_s F^-(\tau*)$, where τ^* is the total optical depth of the layer as shown in Fig.1. By applying these boundary conditions we get

$$K_1 = \frac{(R_s G^- - G^+)e^{-\tau^*/\mu_0} + G^-(1 - \Gamma R_s)e^{-k\tau^*}}{(1 - \Gamma R_s)[e^{k\tau^*} - \Gamma e^{-k\tau^*}]} \tag{47}$$

$$K_2 = -\frac{[\Gamma(R_s G^- - G^+)e^{-\tau^*/\mu_0} + G^-(1 - \Gamma R_s)e^{k\tau^*}]}{(1 - \Gamma R_s)[e^{k\tau^*} - \Gamma e^{-k\tau^*}]} \tag{48}$$

3. Conclusions

We conclude saying that the method of solution proposed in this work is clearly very easy and fast and no matrix method is required. It can immediately be extended to solve others coupled differential equations, as for instance that obtained by Li and Ramaswamy, (1995) in the four-stream approximation method. The radiative fluxes $F^+(\tau)$ and $F^-(\tau)$ as functions of the optical depth can be calculated using the expressions of the constants K_1 and K_2 given in Eqs. (47) and (48), respectively, we comment that those radiative fluxes can directly be applied to atmospheric photochemistry [2], [5], [10], [13].

Acknowledgments

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