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Nambu-Goto action and classical rebits in any signature and in higher dimensions

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We perform an extension of the relation between the Nambu-Goto action and classical rebits. Of course, the Cayley hyperdeterminant is the key mathematical tool in such generalization. Using the Wick rotation, we find that in four dimensions such a relation can be established no only with the signature (2+2) but also with any signature. We generalize our result to a curved space-time of (2^{2n}+2^{2n})-dimensions and (2^{2n+1}+2^{2n+1})-dimensions.

Keywords: Nambu-Goto action; rebit theory; general relativity

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1. Introduction

Some years ago, Duff [1] discovers hidden new symmetries in the Nambu-Goto action [2-3]. It turns out that the key mathematical tool in such a discovery is the Cayley hyperdeterminant [4]. In this pioneer work, however, the target space-time turns out to have an associated $(2+2)$-signature, corresponding to two time and two space dimensions. It was proved in Ref. 5 and 6 that the Duff's formalism can also be generalized to $(4+4)$-dimensions and $(8+8)$-dimensions. Here, we shall prove that if one introduces a Wick rotations for various coordinates then one can actually extend the Duff's procedure to any signature in 4-dimensions. Moreover, we also prove that our method can be extended to curved space-time in $(2^{2n}+2^{2n})$-dimensions and $(2^{2n+1}+2^{2n+1})$-dimensions.

There are a number of physical reasons to be interested on these developments, but perhaps the most important is that eventually our work may be useful on a possible generalization of the remarkable correspondence between black-holes and quantum information theory (see Refs. 7 to 10] and references therein).

2. Mathematical development

Let us start recalling the Duff's approach on the relation between the Nambu-Goto action and the $(2{+}2)$-signature. Consider the Nambu-Goto action [2,3],

$$S=\int d\xi^2\sqrt{\epsilon\det(\partial_a x^\mu\partial_b x^\nu\eta_{\mu\nu})}. \tag{1}$$

Here, the space-time coordinates x^μ are real function of two parameters $(\tau,\sigma)=\xi^a$ and $\eta_{\mu\nu}$ is a flat metric, determining the signature of the target space-time. Moreover, the parameter ϵ takes the values $+1$ or -1, depending whether the signature of $\eta_{\mu\nu}$ is Euclidean or Lorenziana, respectively.

It turns out that by introducing the world-sheet metric g^{ab} one can prove that (1) is equivalent to the action [11] (see also Ref. 12 and references therein)

$$S=\int d\xi^2\sqrt{-\epsilon\det g}g^{ab}\partial_a x^\mu\partial_b x^\nu\eta_{\mu\nu}, \tag{2}$$

which is, of course, the Polyakov action (see Ref. 12 and references therein). In fact, from the expression

$$\partial_a x^\mu\partial_b x^\nu\eta_{\mu\nu}-\frac{1}{2}g_{ab}g^{cd}\partial_c x^\mu\partial_d x^\nu\eta_{\mu\nu}=0, \tag{3}$$

obtained by varying the action (2) with respect to g^{ab}, it is straightforward to show that from (2) one obtains (1) and *vise versa*. Hence, the actions (1) and (2) are equivalents.

It is convenient to define the induced world-sheet metric

$$h_{ab}\equiv\partial_a x^\mu\partial_b x^\nu\eta_{\mu\nu}. \tag{4}$$

Using this definition, the Nambu-Goto action (1) becomes

$$S=\int d\xi^2\sqrt{\epsilon\det(h_{ab})}. \tag{5}$$

It is not difficult to see that in $(2+2)$-dimensions the expression (4) can be written as

$$h_{ab}=\partial_a x^{ij}\partial_b x^{kl}\varepsilon_{ik}\varepsilon_{jl}, \tag{6}$$

where x^{ij} denotes a the 2×2- matrix

$$x^{ij}=\begin{pmatrix} x^1+x^3 & x^2+x^4\\ -x^2+x^4 & x^1-x^3\end{pmatrix}. \tag{7}$$

It is important to observe that (7) corresponds to the set $M(2,R)$ of any 2×2-matrix. In fact, by introducing the fundamental base matrices

$$\delta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon^{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\eta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \lambda^{ij} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

one observes that (7) can be rewritten as the linear combination

$$x^{ij} = x^1\delta^{ij} + x^2\varepsilon^{ij} + x^3\eta^{ij} + x^4\lambda^{ij}. \tag{9}$$

Let us now introduce the expression

$$h = \frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd}h_{ac}h_{bd}. \tag{10}$$

If one uses (4) one gets

$$h = \det(h_{ab}). \tag{11}$$

However, if one considers (6) one obtains

$$h = \mathcal{D}et(h_{ab}), \tag{12}$$

where $\mathcal{D}et(h_{ab})$ denotes the Cayley hyperdeterminant of h_{ab}, namely

$$\mathcal{D}et(h_{ab}) = \frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd} \times \varepsilon_{ik}\varepsilon_{jl}\varepsilon_{mr}\varepsilon_{ns}\partial_a x^{ij}\partial_c x^{kl}\partial_b x^{mn}\partial_d x^{rs}. \tag{13}$$

Of course, (11) and (12) imply that

$$\det(h_{ab}) = \mathcal{D}et(h_{ab}). \tag{14}$$

In turn, (14) means that in $(2+2)$-dimensions the Nambu-Goto action (5) can also be written as

$$S = \int d\xi^2\sqrt{\mathcal{D}et(h_{ab})}. \tag{15}$$

Note that, since in this case one is considering the $(2+2)$-signature one must set $\epsilon = +1$ in (5).

In $(4+4)$-dimensions the key formula (6) can be generalized as

$$h_{ab} = \partial_a x^{ijm}\partial_b x^{kls}\varepsilon_{ik}\varepsilon_{jl}\eta_{ms}. \tag{16}$$

While in $(8+8)$-dimensions one has

$$h_{ab} = \partial_a x^{ijmn}\partial_b x^{klsr}\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{ms}\varepsilon_{nr}. \tag{17}$$

(see Refs. 5 and 6 for details). So by considering the real variables $x^{i_1\cdots i_n}$ and properly considering the matrices ε_{ij} and η_{ij} the previous formalism can be generalized to higher dimensions. Of course, in such cases the Cayley hyperdeterminant $\mathcal{D}et(h_{ab})$ must be modified accordingly.

Observing (7) one wonders whether one can consider in (6) other signatures in 4-dimensions besides the $(2+2)$-signature. It is not difficult to see that using the Wick rotation in any of the coordinates x^1, x^2, x^3 or x^4 one can modify the signature. For instance, one can achieve the $(1+3)$-signature if one uses the prescription $x^2 \to ix^2$ in (6). This method lead us inevitable to generalize our method to a complex structure. One simple introduce the complex matrix

$$z^{ij} = z^1\delta^{ij} + z^2\varepsilon^{ij} + z^3\eta^{ij} + z^4\lambda^{ij}, \tag{18}$$

where the variables z^1, z^2, z^3 and z^4 are complex numbers. The expression (6) is generalized accordingly as [13]

$$h_{ab} = \partial_a z^{ij}\partial_b z^{kl}\varepsilon_{ik}\varepsilon_{jl}. \tag{19}$$

Thus, in this case, the Cayley hyperdeterminant becomes

$$\mathcal{D}et(h_{ab}) = \frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd} \times \varepsilon_{ik}\varepsilon_{jl}\varepsilon_{mr}\varepsilon_{ns}\partial_a z^{ij}\partial_b z^{kl}\partial_a z^{mn}\partial_b z^{rs} \tag{20}$$

and consequently the Nambu-Goto action must be written using (20). Of course, the Nambu-Goto action, or the Polyakov action, must be real and therefore one must choose any of the coordinates z^1, z^2, z^3 and z^4 in (20) either as pure real or pure imaginary.

Similarly, the generalization to a complex structure can be made by introducing the complex variables $z^{i_1\cdots i_n}$ and writing

$$\mathcal{D}et(h_{ab}) = \frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{i_1j_1}\ldots\varepsilon_{i_{n-1}j_{n-1}}\eta_{i_nj_n}\varepsilon_{k_1l_1\ldots} \times \varepsilon_{k_{n-1}l_{n-1}}n_{k_nl_n}\cdot\partial_a z^{i_1\ldots i_n}\partial_c z^{j_1\ldots j_n} \times \partial_b z^{k_1\ldots k_n}\partial_d z^{l_1\ldots l_n} \tag{21}$$

or

$$\mathcal{D}et(h_{ab}) = \frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{i_1j_1}\ldots\varepsilon_{i_nj_n}\varepsilon_{k_1l_1}\ldots\varepsilon_{k_nl_n} \cdot\partial_a z^{i_1\ldots i_n}\partial_c z^{j_1\ldots j_n}\partial_b z^{k_1\ldots k_n}\partial_d z^{l_1\ldots l_n}, \tag{22}$$

depending whether the signature is $(2^{2n}+2^{2n})$ or $(2^{2n+1}+2^{2n+1})$, respectively.

One can further generalize our procedure by considering a target curved space-time. For this purpose let us introduce the curved space-time metric

$$g_{\mu\nu} = e^A_\mu e^B_\nu\eta_{AB}. \tag{23}$$

Here, e^A_μ denotes a vielbein field and η_{AB} is a flat metric. The Polyakov action in a curved target space-time becomes

$$S = \int d\xi^2\sqrt{-\epsilon\det g}g^{ab}\partial_a x^\mu\partial_b x^\nu g_{\mu\nu}. \tag{24}$$

Using (23), one sees that this action can be written as

$$S = \int d\xi^2\sqrt{-\epsilon\det g}g^{ab}(\partial_a x^\mu e^A_\mu)(\partial_b x^\nu e^B_\nu)\eta_{AB}. \tag{25}$$

So, by defining the quantity

$$E_a^A \equiv \partial_a x^\mu e_\mu^A, \tag{26}$$

the action in (25) reads as

$$S = \int d\xi^2 \sqrt{-\epsilon \det g} g^{ab} E_a^A E_b^B \eta_{AB}. \tag{27}$$

Hence, in a target space-time of $(2+2)$-dimensions one can write (27) in the form

$$S = \int d\xi^2 \sqrt{-\epsilon \det g} g^{ab} E_a^{ij} E_b^{kl} \varepsilon_{ik} \varepsilon_{jl}, \tag{28}$$

where

$$E_a^{ij} \equiv \partial_a x^\mu e_\mu^{ij}. \tag{29}$$

Here, we considered the fact that one can always write

$$e_\mu^{ij} = e_\mu^1 \delta^{ij} + e_\mu^2 \varepsilon^{ij} + e_\mu^3 \eta^{ij} + e_\mu^4 \lambda^{ij}. \tag{30}$$

Observe that in this development one can consider a generalization of (4) namely

$$h_{ab} = E_a^A E_b^B \eta_{AB} \tag{31}$$

and therefore in $(2+2)$-dimensions this expression becomes

$$h_{ab} = E_a^{ij} E_b^{kl} \varepsilon_{ik} \varepsilon_{jl}, \tag{32}$$

while in $(4+4)$-dimensions and $(8+8)$-dimensions one obtains

$$h_{ab} = E_a^{ijm} E_b^{klr} \varepsilon_{ik} \varepsilon_{jl} \eta_{mr} \tag{33}$$

and

$$h_{ab} = E_a^{ijmn} E_b^{klrs} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns}, \tag{34}$$

respectively.

At this stage, it is evident that if one wants to generalize the procedure to any signature in a curved space-time one simply substitute in the action (27) either

$$h_{ab} = \mathcal{E}_a^{i_1 \ldots i_n} \mathcal{E}_b^{j_1 \ldots j_n} \varepsilon_{ik} \ldots \varepsilon_{i_{n-1} j_{n-1}} \eta_{i_n j_n} \tag{35}$$

or

$$h_{ab} = \mathcal{E}_a^{i_1 \ldots i_n} \mathcal{E}_b^{j_1 \ldots j_n} \varepsilon_{ik} \ldots \varepsilon_{i_{n-1} j_{n-1}} \varepsilon_{i_n j_n}, \tag{36}$$

depending whether the signature is $(2^{2n} + 2^{2n})$ or $(2^{2n+1} + 2^{2n+1})$, respectively. Here, we used the prescription $E_a^{i_1 \ldots i_n} \to \mathcal{E}_a^{i_1 \ldots i_n}$, with $\mathcal{E}_a^{i_1 \ldots i_n}$ a complex function.

In order to include p-branes in our formalism, one notes that the expression (35) and (36) can still be used. In such a case, one allows the indice a in (35) and (36) to run from 0 to p. Braking such kind of indices as $a = (\hat{a}_1, \hat{a}_2)$ for a 3-brane, as $a = (\hat{a}_1, \hat{a}_2, \hat{a}_3)$, for a 5-brane and so on one observes that (35) and (36) can be written as

$$h_{\hat{a}_1 \ldots \hat{a}_2 \hat{b}_1 \ldots \hat{b}_2} = \mathcal{E}_{\hat{a}_1 \ldots \hat{a}_2}^{i_1 \ldots i_p} \mathcal{E}_{\hat{b}_1 \ldots \hat{b}_2}^{j_1 \ldots j_p} \varepsilon_{ik} \ldots \varepsilon_{i_{p-1} j_{p-1}} \eta_{i_p j_p} \tag{37}$$

or

$$h_{\hat{a}_1 \ldots \hat{a}_2 \hat{b}_1 \ldots \hat{b}_2} = \mathcal{E}_{\hat{a}_1 \ldots \hat{a}_2}^{i_1 \ldots i_p} \mathcal{E}_{\hat{b}_1 \ldots \hat{b}_2}^{j_1 \ldots j_p} \varepsilon_{ik} \ldots \varepsilon_{i_{p-1} j_{p-1}} \varepsilon_{i_p j_p}, \tag{38}$$

respectively. The analogue of Cayley hyperdeterminant in this case will be

$$\begin{aligned} &\hat{\mathcal{D}}et(h_{\hat{a}_1 \ldots \hat{a}_2 \hat{b}_1 \ldots \hat{b}_2}) \\ &= \varepsilon^{\hat{a}_1 \hat{b}_1} \ldots \varepsilon^{\hat{a}_p \hat{b}_p} \mathcal{E}_{\hat{a}_1 \ldots \hat{a}_2}^{i_1 \ldots i_p} \mathcal{E}_{\hat{b}_1 \ldots \hat{b}_2}^{j_1 \ldots j_p} \varepsilon_{ik} \ldots \varepsilon_{i_{p-1} j_{p-1}} \varepsilon_{i_p j_p} \end{aligned} \tag{39}$$

and therefore the corresponding Nambu-Goto action becomes

$$S = \int d\xi^{p+1} \sqrt{\epsilon \hat{\mathcal{D}}et(h_{\hat{a}_1 \ldots \hat{a}_2 \hat{b}_1 \ldots \hat{b}_2})}. \tag{40}$$

3. Conclusions and comments

We have generalized the Duff's procedure concerning the combination of the Nambu-Goto action and the Cayley hyperdeterminant in target space-time of $(2+2)$-dimensions. Such a generalization first corresponds to a curved worlds with $(2^{2n} + 2^{2n})$-signature or $(2^{2n+1} + 2^{2n+1})$-signature. Using complex structure we may be able to extend the procedure to any signature. Further, we generalize the method to p-branes.

It turns out that these generalization may be useful in a number of physical scenario beyond string theory and p-branes. In fact, since the quantity $z^{j_1 \cdots j_n}$ can be identified with a n-complex rebit one may be interested in the route leading to oriented matroid theory [14] (see also Ref. 15 and 16). In this direction, using the phirotope concept (see Ref. 17 and references therein), which is a complex generalization of the concept of chirotope in oriented matroid theory, a link between super p-branes and qubits (in this context) has already been established [17]. Thus, it may be interesting for further developments to explore the connection between the results of the present work and supersymmetry *via* the Grassmann-Plücker relations (see Refs. 8 and 9 and references therein). It is worth mentioning that such relations are natural mathematical notions in information theory linked to n-qubit entanglement. Indeed, in such a case, the Hilbert space can be broken in the form $C^{2n} = C^L \otimes C^l$ with $L = 2n-1$ and $l = 2$. This allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of 2-planes in C^{2n} *via* the Plücker embedding. In this context, the Plücker coordinates of Grassmannians $Gr(L,l)$ are natural invariants of the theory (see Ref. 9 for details). However, it has been mentioned in Ref. 18, and proved in Refs. 19 and 20, that for normalized qubits the complex 1-qubit, 2-qubit and the 3-qubit are deeply related to division algebras via the Hopf maps, $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$, respectively. In order to clarify the possible application of these observations in the context of our formalism let us consider the general complex state $\mid \psi\rangle \in C^{2n}$,

$$\mid \psi\rangle = \sum_{i_1 i_2 \ldots i_n = 0}^{1} C^{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle, \tag{41}$$

where $|i_1 i_2 ... i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes ... \otimes |i_n\rangle$ correspond to a standard basis of the n-qubit, and $C^{i_1 \cdots i_n}$ is a complex quantity which real and imaginary parts can be identified in terms of two rebits ($a^{i_1 \cdots i_n}$ and $b^{i_1 \cdots i_n}$) in the form $C^{i_1 \cdots i_n} = a^{i_1 \cdots i_n} + ib^{i_1 \cdots i_n}$. It is interesting to make the following observations. First, one finds that a 3-rebit and 4-rebit have 8 and 16 real degrees of freedom, respectively. Thus, one learns that the 4-rebit can be associated with the 16 degrees of freedom of a 3-qubit. It turns out that this is the kind of embedding discussed in Ref. 9. Second, one may expect that the quantum development of the Nambu-Goto action in n-dimensions leads to consider the 16-dimensions of target space-time as the maximum dimension required by division algebras via the Hopf map $S^{15} \xrightarrow{S^7} S^8$. Finally, the question arises whether in our generalized formalism one may also find hidden symmetries of the Nambu-Goto action in the sense of Ref. 1. In (2+2)-dimensions the hyperdeterminant turns out to be invariant under

$$[SL(2,R) \times SL(2,R) \times SL(2,R)] \times S_3. \qquad (42)$$

Here, the first $SL(2,R)$ is a global subgroup of the worldsheet diffeomorphism. The second two factors are spacetime Lorentz in $(2+2)$-dimensions, namely $Spin(2,2) \cong SL(2,R) \times SL(2,R)$. By complexifying the x^μ one may take different real forms, $Spin(2,2) \cong SL(2,R) \times SL(2,R)$, $Spin(1,3) \cong SL(2,C)$, $Spin(4) \cong SU(2) \times SU(2)$ to obtain various signatures. However, only in $(2+2)$-dimensions one has the three factors $SL(2,R)$ in the same footing and hence additional S_3. In the case of $(4+4)$-dimensions one may consider the chain of maximal embeddings and branches,

$$\begin{aligned} so(4,4) &\supset s(2,R) \oplus so(2,3) \\ &\supset so(1,1) \oplus sl(2,R) \oplus sl(2,2). \end{aligned} \qquad (43)$$

However, these subgroups are not full symmetry groups and therefore it is difficult to reveal hidden discrete symmetries of the Nambu-Goto action in this case. In other cases the analysis seems even more difficult, but this motivate us to explore in more detail these developments.

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