Image restoration of blurring due to rectilinear motion:
constant velocity and constant acceleration

J.S. Pérez Huerta and G. Rodriguez Zurita

Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla,
Apartado Postal 1152, Puebla Pue., México,
e-mail: est091@fcfm.buap.mx, gzurita@fcfm.buap.mx

If a relative motion exists between a given object's image and a recording medium (such as a photographic film) during exposure, the recorded image will be blurred. This effect can be expressed by an impulse response in irradiance function and the respective image restoration is made possible by a post-recording process. General one-dimension motion is considered here. Two cases are discussed: constant velocity motion and constant acceleration motion. Image simulations representing those degradation mechanisms are shown, as well as the corresponding restored images after using the method which is described in this work. This method is based on Dirac delta function propreties. We also show real digital images affected by those motions. It can be experimentally obtained by using an air-rail. The blurred images are processed digitally by spatial filters given by the proposed method. Numerical and experimental results are shown.

Keywords: Image forming and processing; Fourier optics; modulation and optical transfer functions

Si durante el registro de una imagen (una exposición fotográfica, por ejemplo) existiera un movimiento relativo entre ésta y el medio de registro, la imagen grabada resultará deteriorada (emborronada). Este efecto se expresa por medio de la función de respuesta impulso en irradiancia, y a partir de ésta es posible restaurar la imagen mediante procesos posteriores al registro. Aquí se considera movimiento rectilíneo general y se analizan los casos de movimiento uniformemente lineal y movimiento uniformemente acelerado. Se presentan simulaciones de imágenes afectadas por estos movimientos así como su restauración al aplicar el procedimiento sugerido por el análisis descrito, que se basa en propiedades de la función delta de Dirac. También se muestran imágenes reales digitalizadas afectadas por los movimientos mencionados con la ayuda de un riel sin fricción. Estas imágenes se procesan digitalmente implementando filtros espaciales sugeridos por el método propuesto. Se muestran los resultados correspondientes.

Descriptores: Formación y procesamiento de imágenes; óptica de Fourier; funciones de modulación y transferencia óptica

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1. Introduction

Certain factors that can degrade the recorded image in a photographic or holographic film have been studied for a long time. Relevant works include studies on transverse vibrations of one or several components (object, lens, or image plane) perpendicular to the optical axis, effects studied by A. W. Lohmann [1]. Uniformly linear motion between the original image and the film was treated by Edward L. O'Neill [2], the shutter operation was analyzed by Ronald V. Shack [3], longitudinal vibrations were worked on by A.W. Lohmann and D.P. Paris [4], out-of-focus degradation was studied by J.W. Goodman [9], the penumbra effect in radiograph images due to the nonzero dimension of the source was dealt with by John B. Minkoff, S.K. Hilal, W.F. Konig, M. Arm, and L.B. Lambert [10]. Nonuniform motion was worked on by Som [13], multiply exposed images were studied by D.P. Jablonowski and S.H. Lee [14]. A feature common to these works is the characterization of the image degradation by a point spread function (PSF), so that the degraded image can be expressed by the convolution of the ideal or original image with the respective PSF.

Works about restoration and evaluation images by a post-recording process have been demonstrated. In 1953, A. Maréchal and P. Croce showed the classical image restoration technique by inverse spatial filtering, while P.F. Mueller and G.O. Reynolds restored images affected by turbulence in 1966. J.L. Harris showed a formulation different from inverse filtering and experimental results are shown in Ref. 5. C.W. Helstrom showed image restoration by the method of least squares, which considers spatial noise in the image [7], A.W. Lohmann and D.P. Paris showed computer generated binary spatial filters which can represent complex inverse filters [8]. J.L. Horner showed restoration of photographic images when noise is considered [11]. George W. Stroke showed holographic image deblurring [12]. D.P. Jablonowski and S.H. Lee proposed a method of synthesizing the appropriate composite gratings, which represents a double convolution, with a set of the Dirac delta function, and experimental results are shown in Ref. 14, L. Celaya and S. Mallick deblurred images degraded by a linear smear by using a Wollaston prism and two linear polarizers [16].

From the above mentioned deblurring methods, many of them are optical analog in nature for historical reasons. A new consideration of these methods within today's scope, however, would undoubtedly bring benefits to digital image processing after proper updates or adaptations. It is under these considerations that the present work was done. First, a general description of blurring due to linear motion is proposed taking advantage of Dirac delta function proper-

ties [13] which have not been used in connection with this problem, to our knowledge. To test this formulation, two cases are numerically inspected: uniform velocity and uniform acceleration. By using a given form of velocity (PSF), the corresponding blurring of digital images is performed. Then, an inverse spatial filter is applied to the smeared image in order to obtain a deblurred one. The Spatial filtering that we propose was implemented with programs we wrote in MathCad. Experimental smeared images were achieved with slides on an air-rail. Inverse filtering used a sample of linear motion as the experimental PSF. In this case, no analytical knowledge about the movement was really employed to achieve restored images, but a further link between the experimental PSF and the theoretical one can be made to obtain typical parameters of movement.

In order to achieve image restoration and evaluation, one must know the nature of the degradation as well as its effect in order to operate the image to compensate the image for degradation. In this study, the images are recorded in order to know the nature of object within the field of view, although possible motion analysis technique is suggested. When image processing makes an intelligible image from an unintelligible image, it is because the information in the image has been displayed in such a way that the human visual system is able to extract it [5]. A qualitative measure of the gain achieved by processing is a comparison of the information extraction by the human visual system both before and after processing. This will be the criterion for evaluating the restoration. Linear smear restoration is a well-known example whereby a PSF (a rectangular or window function for the case of uniform velocity) is exemplified. It also shows an application of the convolution operation. Moreover, it suggests a restoration method which can be carried out directly [2]. Hence, it can be worthwhile to consider if the techniques now widely available can offer advantages to the practical solution of this otherwise classical problem, besides of being a case with the illustrative features already mentioned. The case of uniformly accelerated motion, on the other hand, is more general than the linear one [13], but has been less studied in the literature (even the effects of oscillatory motion has been more thoroughly considered.) So, in spite of the already venerable tradition of the restoration problem, we propose an analysis which focuses on linear with movement constant acceleration, but that includes also with constant velocity as a special case. This last case can be used to verify some aspects of the analysis. Only the case of constant acceleration and that of constant velocity are considered in detail but, in principle, other linear motion could be included using the general approach. Other "dynamic" effects such as shutter operation or time-varying aberrations are neglected. There are both numerically calculated images as well as experimental images shown to illustrate the restoration procedure.

Experimental results and simulations show that inverse filtering works in a satisfactory way for digital images processed by computer. Moreover, it would also be possible to extract information about the parameters of the movement which causes smear by means of the true parameters which can restore the image (motion analysis).

2. Analysis

A linear smearing process in recording images is assumed. For example, by employing a photographic material, it is supposed that it absorbs incident radiant energy over the linear range of the H and D curve. PSF of the recording material are considered as ideal points, thus using Dirac-delta functions instead of extended functions. For instance, the point spread function of the photographic emulsion resulting from the diffusion of light in the emulsion is not considered, since it is independent of other effects, for example, from the influence of image motion [13] or from the shutter operation [3].

Let $I(x)$ be the irradiance distribution in the original or ideal image, *i.e.*, that which would be obtained in the absence of any degradation (See appendix). So, the blurring effect due to linear motion in the degraded image that is recorded $I_b(x)$ can be described as a superposition on the photographic film of successive positions of the original image continuously way, that is

$$I_b(x) = I(x) \circledast IRI(x), \tag{1}$$

where $\circledast$ denotes a convolution, $IRI(x)$ is the *impulse response in irradiance* (or point spread function due to motion) given by

$$IRI(x) = \frac{1}{T} \int_0^T dt \, \delta(x - \Delta x[t])$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} dt \delta(x - \Delta x[t]) \mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right). \tag{2}$$

Here T is the register time, $\Delta x[t]$ is the displacement between the original image and the film at time t and $\delta(u)$ is the *Dirac delta function*.

If we substitute Eq. (2) in Eq. (1), we obtain

$$I_b(x) = I(x) \circledast IRI(x)$$

$$= \frac{1}{T} \int_0^T dt \, I(x) \circledast \delta(x - \Delta x[t])$$

$$= \frac{1}{T} \int_0^T dt \, I(x - \Delta x[t]).$$

This expression is similar to the equation given by Som [13]. However, the detector response (photographic emulsion) is not taken into account. It is the PSF of the recording medium which is considered a Dirac-delta function in the position. In other words, without other degradation mechanisms, the recorded image has a one-to-one correspondence with the original image.

We can express the smeared image $(I_b(x))$ as a convolution given in Eq. (1) whenever the function $IRI(x)$ is invariant, at least over a limited region of the image plane (isoplanatic patch). In Eq. (2), there is a factor T^{-1} in order to normalize function $IRI(x)$ over the space

$$\int\limits_{-\infty}^{\infty} dx\, IRI(x) = 1.$$

On the other hand, in order to change the argument of the delta function from the spatial domain to the time domain, we use the following property [17]:

Given the function $r(t)$, and if the equation $r(t) = 0$ has a finite or countably infinite numbers of zeros, that is, $r(t_n) = 0$, then

$$\delta(r(t)) = \sum_n \frac{\delta(t - t_n[x])}{\left|\frac{dr}{dt}\right|_{t_n[x]}},$$

whenever $r(t)$ has a continuous derivative

$$\left|\frac{dr}{dt}\right|_{t_n[x]} \neq 0.$$

Therefore, if $r(t) = x - \Delta x[t]$, then Eq. (2) can be written as

$$IRI(x) = \frac{1}{T}\int\limits_{-\infty}^{\infty} dt \sum_n \frac{\delta(t - t_n[x])}{\left|\frac{dr}{dt}\right|_{t_n[x]}}\mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

$$= \frac{1}{T}\sum_n |v(t)|_{t_n[x]}^{-1}\,\mathrm{rect}\left(\frac{t_n[x] - T/2}{T}\right). \quad (3)$$

The function velocity $(v(t))$ has been identified and it characterizes the movement. The absolute value of the velocity consequently gives positive values of the function $IRI(x)$.

The equation $r(t_n)=x-\Delta x[t_n]=0$ implies $x=\Delta x[t_n]$, which is a motion equation and hence we can attempt to solve this equation for t_n, which has a usual sense of time $(0 \leq t_n \leq T)$. Thus, the summation in Eq. (3) only has one term. When it is possible to find t, then, the velocity is expressed as function of position x, and thus $v(t) = v'(x)$.

So, bearing all this in mind, Eq. (3) can be rewritten as

$$IRI(x)=\begin{cases} |Tv'(x)|^{-1} & \text{if } x=\Delta x[t], \text{ and: } 0\leq t\leq T, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Now, taking the Fourier transform of Eq.(1) and using Eq.(2), we have

$$\tilde{I}_b(\mu) = \tilde{I}(\mu)\widetilde{IRI}(\mu)$$

$$= \tilde{I}(\mu)\frac{1}{T}\int\limits_0^T dt\, e^{-\imath 2\pi\mu\Delta x[t]}. \quad (5)$$

Making a shifted integration variable t by

$$\Delta x_r[t] = \Delta x[t] - x_o$$

which are related by

$$v'(\Delta x_r) = \frac{d\Delta x[t]}{dt} = \frac{d\Delta x_r[t]}{dt},$$

we obtain

$$\tilde{I}_b(\mu) = \tilde{I}(\mu)\frac{e^{-\imath 2\pi\mu x_o}}{T}\int\limits_{\Delta x_r(0)}^{\Delta x_r(T)} d(\Delta x_r)\frac{e^{-\imath 2\pi\mu\Delta x_r}}{v'(\Delta x_r)}. \quad (6)$$

Therefore, the *transfer function* is given by:

$$\widetilde{IRI}(\mu) = \int\limits_{\Delta x_r(0)}^{\Delta x_r(T)} d(\Delta x_r)\frac{e^{-\imath 2\pi\mu\Delta x_r}}{v'(\Delta x_r)\,T}e^{-\imath 2\pi\mu x_o}$$

$$= \int\limits_{-\infty}^{\infty} d(\Delta x_r)\, e^{-\imath 2\pi\mu\Delta x_r}$$

$$\left\{\frac{e^{-\imath 2\pi\mu x_o}}{v'(\Delta x_r)\,T}\mathrm{rect}\left(\frac{t(\Delta x_r) - T/2}{T}\right)\right\}. \quad (7)$$

This result is in agreement with Eq. (4). Here, the upper and the lower limits in the integral, $\Delta x_r(T)$ and $\Delta x_r(0)$, correspond to zero and maximum displacements respectively. This latter limit is called *length of trace* or *trace of motion*. Beyond these values, the integrand becomes zero, which is expressed by means of a rect(u) function.

Eq. (7) is similar to the equations derived by Shack and Som for the effect due to transverse motion of the image [3, 13]. Shack assumes a negative time coordinate and Som considers only relative displacement between the ideal image and the photographic film, whereas here a positive time coordinate is only considered and one initial displacement is proposed (x_o). This in some cases simplifies the transfer function to real values (displacement theorem).

In order to restore images, some methods have been proposed [5–8, 11, 12, 14]. We use the inverse filter method. It is performed by taking the Fourier transform of Eq. (1). It is written

$$\tilde{I}_b(\mu) = \tilde{I}(\mu)\widetilde{IRI}(\mu).$$

Solving for the transform of the original image gives

$$\tilde{I}(\mu) = \frac{\tilde{I}_b(\mu)}{\widetilde{IRI}(\mu)} = \tilde{I}_b(\mu)H_R(\mu),$$

where the inverse filter is defined by

$$H_R(\mu) = \begin{cases} \frac{1}{\mathscr{F}\{IRI(x)\}} & \text{if } \mathscr{F}\{IRI(x)\} \neq 0, \\ k & \text{otherwise.} \end{cases},$$

k is a constant value.

Although this is one of the most direct methods, there are problems related to points where the transfer function is zero. These points are replaced by the constant k. $k = 0$ was chosen and then the restored spectrum is multiplied by an absorption grating whose lines coincide with the zeros of $\mathscr{F}\{IRI(x)\}$. This becomes a convolution in the image space between the restored image and the Fourier transform of the resulting grating, thus giving rise to replicating images (ghost images.) In any case, image degradation always reduces the information content in the image [5].

3. Typical cases of motion

Now, special cases of smear due to motion are studied: uniform velocity and uniform acceleration.

3.1. Smear due to uniformly linear motion

This case is characterized by a constant velocity v_o and satisfies: $\Delta x[t] = v_o t + x_o$, $r(t) = x - x_o - v_o t$, $dr/dt = -v_o$,

using Eq. (2) and Eq. (3), we get

$$
\begin{aligned}
IRI_l(x) &= \frac{1}{T} \int\limits_{-\infty}^{\infty} dt\, \delta\left(x - x_o - v_o t\right) \mathrm{rect}\left(\frac{t - T/2}{T}\right) \\
&= \frac{1}{L}\mathrm{rect}\left(\frac{\Delta x_o}{L} - \frac{1}{2}\right),
\end{aligned}
\tag{8}
$$

where $\Delta x_o \equiv x - x_o$ and $L \equiv v_o T$ is the length of trace in this case. Putting Eq. (8) in to Eq. (1) , the degraded image in this case will be:

$$
\begin{aligned}
I_{lb}(x) &= I(x) \circledast \frac{1}{L}\mathrm{rect}\left(\frac{\Delta x_o}{L} - \frac{1}{2}\right) \\
&= I(x) \circledast \frac{1}{L}\mathrm{rect}\left(\frac{\Delta x_o - L/2}{L}\right).
\end{aligned}
\tag{9}
$$

Taking the Fourier transform of Eq. (9) we obtain

$$
\tilde{I}_{lb}(\mu) = \tilde{I}(\mu)\mathrm{sinc}(L\mu)\exp(-\imath 2\pi\mu(x_o + L/2)).
\tag{10}
$$

Using the inverse filter method,

$$
\tilde{I}'_{lb}(\mu) = \begin{cases} \tilde{I}_{lb}(\mu)(\mathrm{sinc}(L\mu))^{-1}e^{\imath 2\pi\mu(x_o + L/2)}, & \text{if } L\mu \neq n; \\ 0, & \text{otherwise.} \end{cases}
\tag{11}
$$

where n is an integer.

Thus, the restored image is reached by taking the inverse Fourier transform $I'(x) = \mathscr{F}^{-1}\left\{\tilde{I}'_{lb}(\mu)\right\}$.

If there is a symmetric displacement about the origin coordinate, *i.e.*, $x_o = -L/2$, the exponential factor will be unity. Here, we can see the advantage of using the initial position.

3.2. Smear due to uniformly accelerated motion

This case is characterized by one initial velocity v_o, and one uniform acceleration a_c and then:

$$
\Delta x[t] = x_o + v_o t + \frac{1}{2}a_c t^2,
$$

$$
r(t) = x - x_o - v_o t - \frac{1}{2}a_c t^2,
$$

$$
\frac{dr}{dt} = -v_o - a_c t.
$$

We have to find the roots of

$$
r(t_n) = 0 \text{ or } t^2 + 2\frac{v_o}{a_c}t - 2\frac{\Delta x_o}{a_c} = 0.
$$

Two solutions are found:

$$
t_{1,2} = -\frac{v_o}{a_c}\left(1 \mp \left[1 + \frac{2a_c\Delta x_o}{v_o^2}\right]^{1/2}\right);
$$

$$
t_1 = \frac{v_o}{a_c}\left(\left[1 + \frac{2a_c\Delta x_o}{v_o^2}\right]^{1/2} - 1\right) > 0,
\tag{12}
$$

which are valid for an initial velocity different from zero ($v_o \neq 0$). Only t_1 has a physical meaning.

If the initial velocity is zero ($v_o = 0$), the positive root will be:

$$
t_1 = \sqrt{\frac{2\Delta x_o}{a_c}}.
\tag{13}
$$

Substituting t_1 of Eq. (12) (when $v_o \neq 0$) in the absolute value of the derivative, then the velocity is expressed as a function of displacement.

$$
\begin{aligned}
\left|\frac{dr}{dt}\right|_{t_1} &= v'_{ac}(\Delta x_o) \\
&= v_o\left[1 + \frac{2a_c\Delta x_o}{v_o^2}\right]^{1/2}.
\end{aligned}
\tag{14}
$$

Similarly for the case of an initial velocity zero (when $v_o=0$).

$$
\left|\frac{dr}{dt}\right|_{t_1} = v'_{ao}(\Delta x_o) = \sqrt{2a_c\Delta x_o}.
\tag{15}
$$
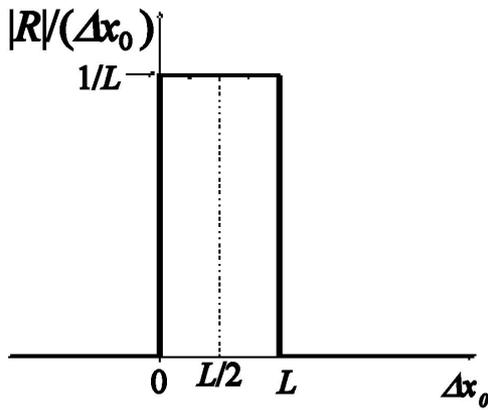
FIGURE 1. Impulse response in irradiance for uniform velocity.
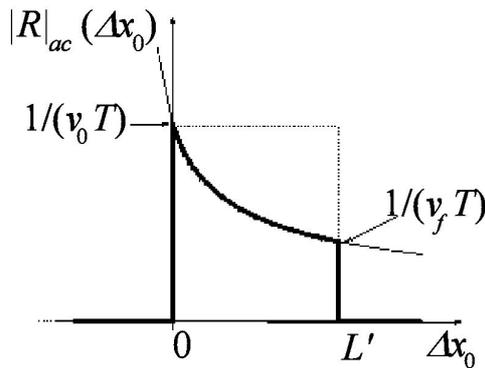
FIGURE 2. Impulse response in irradiance for uniform acceleration and non-zero initial velocity ($v_o \neq 0$).

3.2.1. Non-zero initial velocity ($v_o \neq 0$)

Here, the function is obtained by substituting Eq. (14) in Eq. (3)

$$IRI_{ac}(x) = \frac{1}{T} \int_{-\infty}^{\infty} dt \, \frac{\delta \left(t + \frac{v_o}{a_c} \left(1 - \left[1 + \frac{2a_c \Delta x_o}{v_o^2} \right]^{\frac{1}{2}} \right) \right)}{v_o \left[1 + \frac{2a_c \Delta x_o}{v_o^2} \right]^{\frac{1}{2}}}$$

$$\times \text{rect} \left(\frac{t - \frac{T}{2}}{T} \right)$$

$$= \frac{1}{v'_{ac}(\Delta x_0)T} \text{rect} \left(\frac{v'_{ac}(\Delta x_0) - v_o}{v_f - v_o} - \frac{1}{2} \right), \quad (16)$$

where v_f is the final velocity given by $v_f = v_o + a_c T$. In this case, the length of trace is

$$L' \equiv x(T) - x_o = v_o T + \frac{1}{2} a_c T^2.$$

Evaluating the zero and maximum displacement in to Eq. (16) we have

$$IRI_a(\Delta x_o = 0) = \frac{1}{v_o T} \text{rect} \left(-\frac{1}{2} \right)$$

and

$$IRI_a(\Delta x_o = L') = \frac{1}{v_f T} \text{rect} \left(\frac{1}{2} \right).$$

The respective transfer function for (16) by using Eq.(7) is given by

$$\widetilde{IRI}_{ac}(\mu) = \int_0^{L'} d(\Delta x_o) e^{-\imath 2\pi \mu \Delta x_o} \left[\frac{e^{-\imath 2\pi \mu x_o}}{[v_o^2 + 2a_c \Delta x_o]^{1/2} \, T} \right]$$

$$= \frac{e^{-\imath 2\pi \mu x_o}}{T} \int_0^{L'} d(\Delta x_o) e^{-\imath 2\pi \mu \Delta x_o}$$

$$\times \frac{1}{[v_o^2 + 2a_c \Delta x_o]^{1/2}}$$

$$= \frac{e^{-\imath 2\pi \mu x_o}}{T} e^{\imath \mu K_1}$$

$$\times \left\{ \left(\frac{K_2}{\pi a_c} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} [j_{2n}(\mu K_2) - \imath j_{2n+1}(\mu K_2)] \right.$$

$$\left. - \left(\frac{K_1}{\pi a_c} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} [j_{2n}(\mu K_1) - \imath j_{2n+1}(\mu K_1)] \right\},$$

where

$$K_1 = \frac{\pi v_o^2}{a_c}, \quad K_2 = \frac{\pi v_o^2}{a_c} + 2\pi L',$$

and $j_{2n}(y)$ and $j_{2n+1}(y)$ are spherical Bessel functions of the first kind [13].

3.2.2. Initial velocity of zero value ($v_o = 0$)

Here $v_o = 0$ and the $IRI(x)$ function is obtained from Eq. (15) and Eq. (3)

$$IRI_{ao}(x) = \frac{1}{v'_{ao}(\Delta x_0)T} \text{rect} \left(\frac{\frac{v'_{ao}(\Delta x_o)}{a_c} - T/2}{T} \right).$$
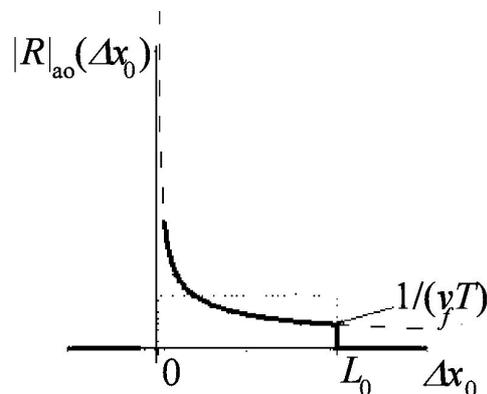
FIGURE 3. Impulse response in irradiance for uniform acceleration and initial velocity of zero value ($v_o = 0$).

Smear simulation and restorin processes

Original Image

Program written in MathCad

Smear process

Original image $\circledast$ *IRI*   =   Smeared image

Furier transforms

Transfer funcion   Degrades spectum

Inverse filter method

The degraded spectrum is multiplied by H_R, where

$$H_R(\mu) = \begin{cases} \frac{1}{\mathscr{F}\{IRI(x)\}} & \text{if } \mathscr{F}\{IRI(x)\} \neq 0, \\ k & \text{otherwise.} \end{cases}$$

We take the inverse transform of this product

Retrieved image
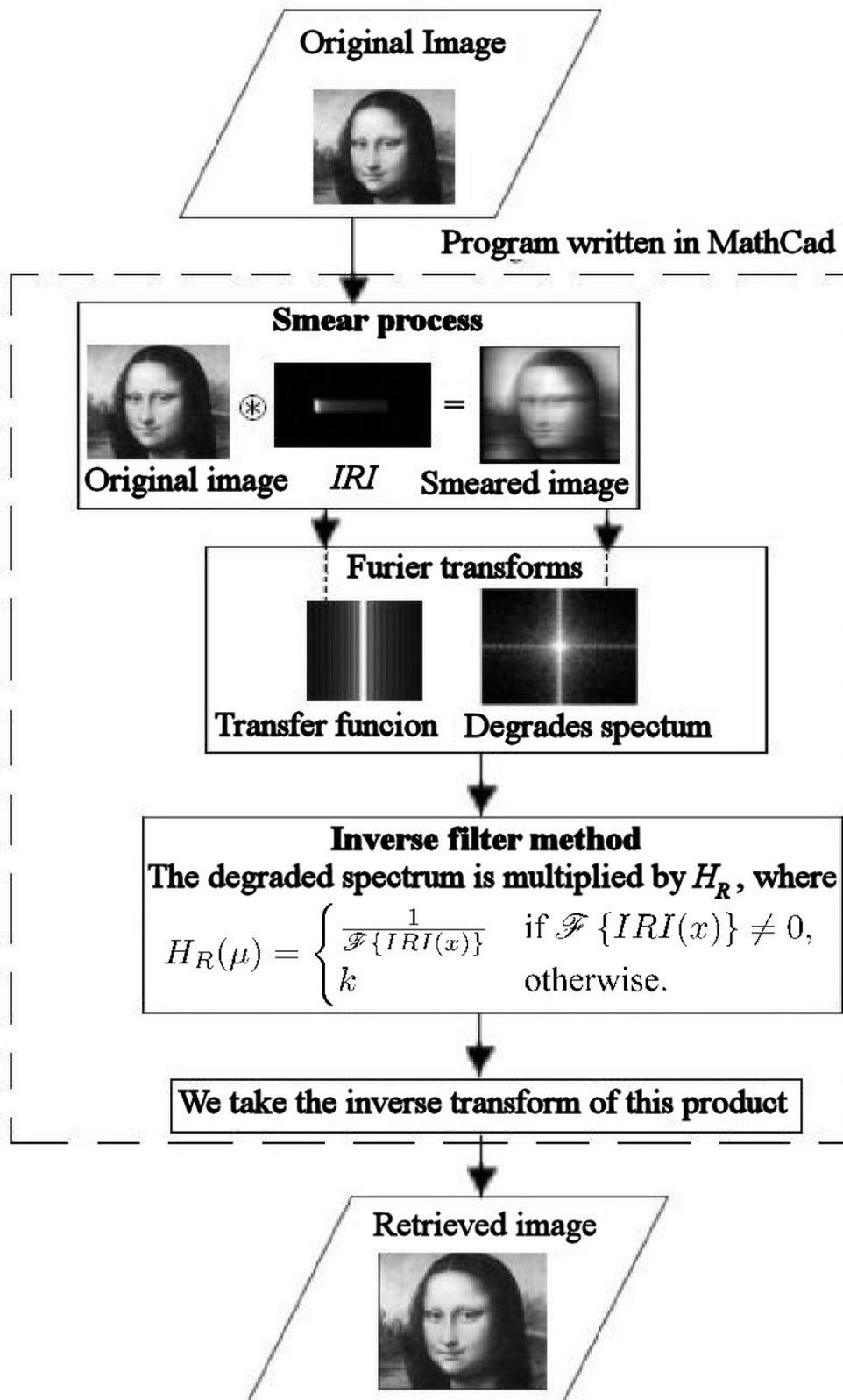
FIGURE 4. Flux diagram showing the smear simulation and restoring process. In this case, the restoration of blurring due to uniform acceleration is shown.
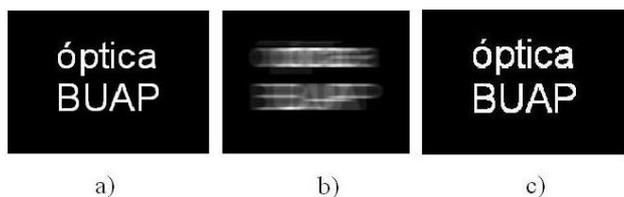
FIGURE 5. Restoration of smeared simulation letters. Original image in a) was convolved with a blurring representing by uniform velocity in b); and corresponding restoration in c).

In this case, the length of trace is $L_o = (1/2)a_c T^2$, so the $IRI_{ao}(x)$ function can rewritten as

$$IRI_{ao}(x) = \frac{1}{[2a_c \Delta x_o]^{1/2} T} \text{rect} \left(\sqrt{\frac{\Delta x_o}{L_o}} - \frac{1}{2} \right). \quad (17)$$

4. Numerical results

In order to visualize how the present method of image restoration performs, a gray scale image was deliberately blurred by a convolution process with the $IRI(x)$ function for different cases of movement. Then, the inverse filter method was applied by means of a program written in Mathcad, and the restored image was obtained. More images suffering from degradation by uniform acceleration than by uni-
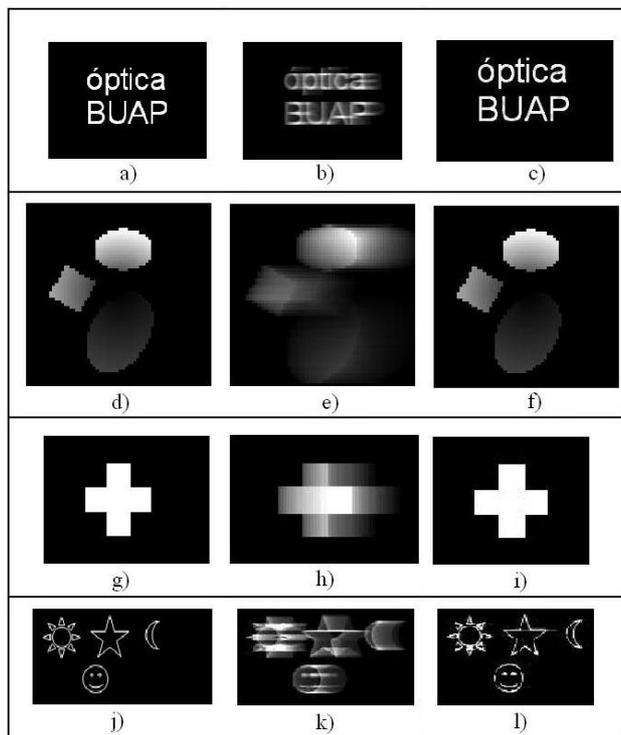
FIGURE 6. a), d), g), j) are original images; b), e), h), k) smear images by convolution representing blurring by uniform acceleration; in c), f), i), l), the corresponding restoration.
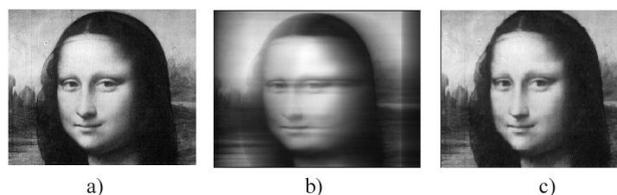
FIGURE 7. a) Original photographic image; b) smeared simulation image by convolution; c) restored image.

form velocity are shown because the case of non-zero acceleration is less known that the uniform velocity case (Figs. 6 and 7).

In binary images, like Figs. 5, 6a, 6g, 6j, 10, and 11, this restoration method has a practical feature. In particular, the restoration of uniform velocity insomuch as some extra be information has been put in (absorption grating) and it can seen included in the restored image. This extraneous information can be identified and eliminated more easily in the practical binary case. However, the numerical procedure works satisfactory for both gray scale images and binary images even in the case of constant velocity.

The image restorations show here is satisfactory according to visual evaluation criteria; in other words, the original structure of the image can be distinguishable. In simulation cases, we can say there is a perfect restoration because the function IRI is completely known. So numerical procedure shows that if adequate IRI is used, will have excellent results in the restored image.

In the simulated smear images, it is possible to distinguish which images were smear by uniform linea and which by uniform acceleration. This difference is due to the form of the function IRI. For the first case this is symmetric, as long as the second is non-symmetric. This fact has significant consequences in the processing insomuch as the transfer function due to uniform acceleration case does not vanish, but the transfer function due to uniform velocity does. According to S.C. Som [13], "As far as the magnitude of the transfer function is concerned, a smear due to uniform acceleration is preferable to one of the same extent due to uniform velocity". The transfer function zeros consequently a loss of have as a consequence, resolution lost.

5. Experimental results

Actual photographic images of letters painted on a carrier which went moving over an air-rail were obtained (Fig. 9). Each motion was captured by means of a given rail inclination. Actual degraded images are shown in Figs. 10 and 11 for constant velocity and constant acceleration respectively. Also, the restored images of each letter are shown after being subjected the method described. One small white point was put on the carrier and was taken as the trace of the movement. The trace was used as the function $IRI(x)$ in the program we wrote in Mathcad, in order to process the images.

Experimental image restoration
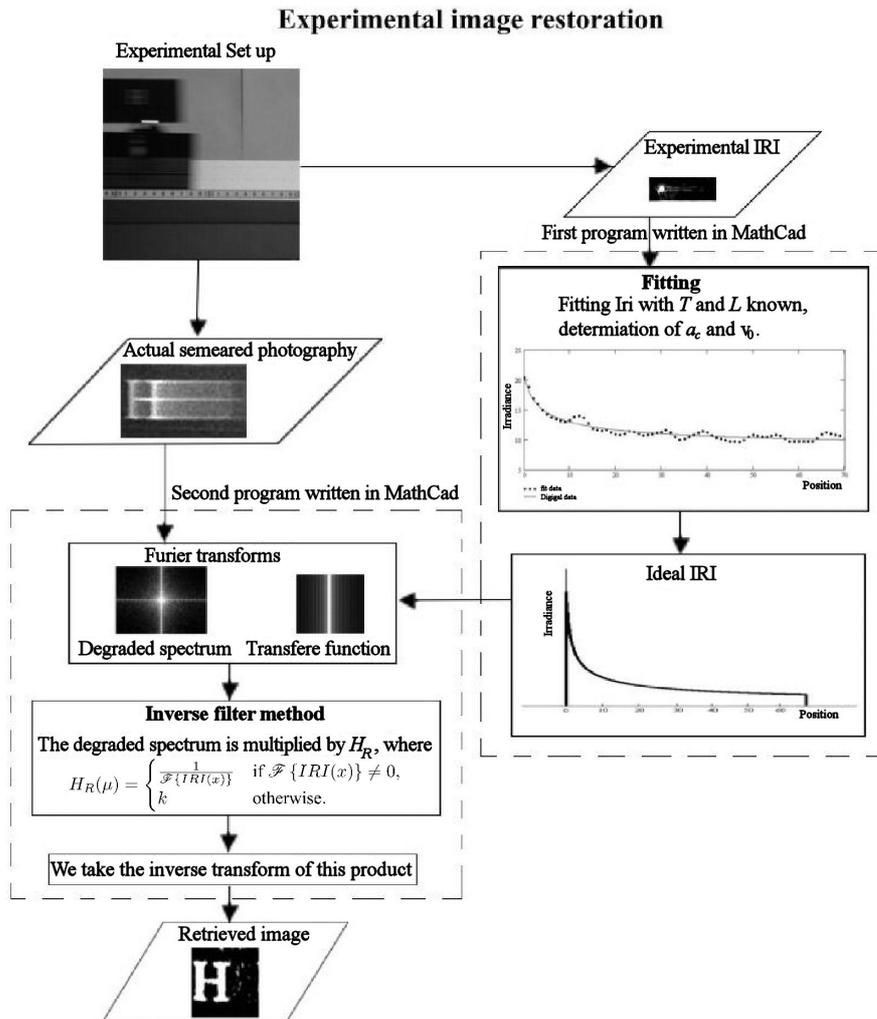
FIGURE 8. Flux diagram showing the experimental image restoration. In this case, the restoration of blurring due to uniform acceleration is shown.
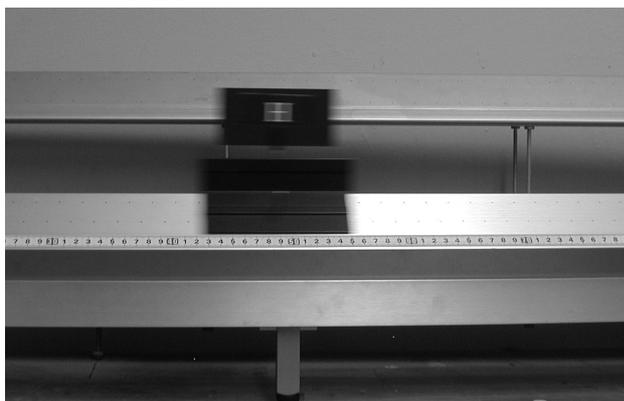
FIGURE 9. Experimental setup: air-rail and carrier employed to get experimentally smeared photographs due to relative motion of a carrier with respect to the photographic camera.

6. Final Comments

The present analysis describes image blurring based on Dirac-delta functions as an alternative to previous ones. An-
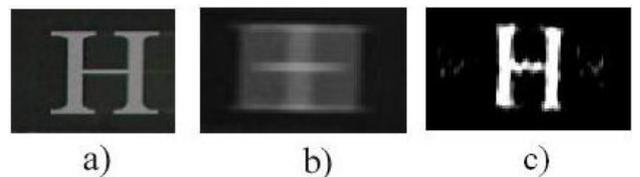
FIGURE 10. a) Photograph taken of the letter "H" which is in static position; b) Actual photograph taken of the letter "H" which is mounted in a carrier with linear motion over a level air-rail at constant velocity; c) restored letter.
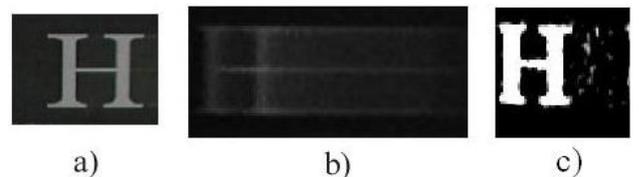
FIGURE 11. a) Photograph taken of the letter "H" which is in static position; b) Actual photograph taken of the letter "H" which is mounted in a carrier with linear motion over a tilted air-rail at constant acceleration; b) restored letter.

alytical results for some types of movement were presented. Examples of restoration of both numerically generated images and experimental images which have been blurred due to uniform linear motion or constant acceleration motion have been presented. Restorations are carried out on the digitalized corresponding images by using the proposed analysis.

The Dirac-delta function formalism was used to define a PSF function. Purposely blurred images were calculated by convolving a given image with the corresponding PSF. Two types of linear movements were considered: constant velocity and constant acceleration. Because analytical knowledge of IRI was given beforehand, implementation of the inverse filter avoiding zeros was carried out directly.

As for the experimental data, there was no need for such knowledge as long as it was possible to isolate the blurring of a point. This blur was information identified as the PSF function and, from that, the inverse filter was implemented. However, when the analytical form of the blur was known, motion parameters could be determined as well, leading to motion analysis.

Analysis of every function PSF is simplified when it is expressed in terms of Dirac delta functions. The restoration program for experimental data does not need the analytic form of the transfer function. Numerical and experimental results obtained by implementation of our analysis show that the inverse filter method can work adequately.

Although state-of-the-art grabbing resources seem to lessen the applicability of restoration techniques to blurred images with degradation due to motion (high-speed CMOS cameras with 60,000 fps capability, or short laser flashes of duration in the range of ps, for example), movement inspection can be still be useful as a technique for movement analysis. For this purpose, stroboscopic illumination techniques constitute a well-known particular case of composite imaging with non-overlapping superposition of discrete, not blurred images. The more general case of composite multiple, overlapping images could be treated with the basic elements described in this work.

Acknowledgments

A Appendix

As a rule, one image can be described by a two-dimensional function $I(x,y)$ when there is linear motion in one direction at an angle θ with respect to the x-axis. The impulse response in irradiance is given by

$$IRI_{2D}(x,y) = IRI(x')\delta(y'),$$

where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For the sake of simplicity, we consider motion across the x-axis. Then

$$IRI_{2D}(x,y) = IRI(x)\delta(y),$$

and therefore

$$I_b(x,y) = I(x,y) \circledast \circledast (IRI(x)\delta(y))$$
$$= I(x,y) \circledast IRI(x).$$

Only one dimension has to be considered.

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