Transient effects in two channel interactions and an application to the behavior of a time dependent shutter

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The subject of transient effects in quantum mechanics has been of interest to one of the authors (MM) since long before it became possible to study them experimentally. In particular, the problem of opening a shutter instantaneously led him to the concept of diffraction in time [1]. Physically, it is only possible to open a shutter as a function of time and this complicates the problem greatly, as it is then not invariant under time translations and thus, among other difficulties, the energy of the system is not a constant of the motion. Kleber and Scheitler [2] analyzed the problem describing the shutter as a $\delta-$function potential at the origin $x = 0$, but whose strength was an inverse function of time. In this paper, we follow another procedure by adding to our initial two particle channel another one, and make them interact through appropriate time independent boundary conditions at the point of coincidence. The full problem conserves the total energy, but this does not happen if we restrict ourselves to a description in the first channel only. Thus, we have a rough but analytically solvable problem using a Laplace transform, which mimics some aspects of a time-dependent shutter, and compare our results with some of those derived by the different procedure of reference [2].

Keywords: Transient effects; time dependent shutter.

El tema de efectos transitorios en mecácanica cuántica ha sido de interés para uno de los autores (MM) desde hace mucho tiempo, cuando aún no había técnicas experimentales para observarlos. En particular, el problem ade abrir instantáneamente un obturador lo llevo al concepto de difracción en el tiempo [1]. Desde el punto de vista físico sólo se puede abrir un obturador como función del tiempo, y esto complica grandemente el problema ya que entonces no es invariante ante translaciones en el tiempo y por ello, entre otras dificultades, la energía del sistema no es una constante de movimiento. Kleber y Scheitler [2] analizaron el problema describiendo el obturador como un potencial δ en el origen $x = 0$, pero cuya intensidad es una función inversa del tiempo. En este trabajo seguimos otro procedimiento agregando a nuestro canal inicial de dos partículas otro nuevo, y los hacemos interactuar a través de apropiadas condiciones estacionarias a la frontera en el punto de coincidencia de los dos canales. El problema completo conserva la energía total, pero esto no sucede si nos restringimos sólo a la descripción en el primer canal. Por ello tenemos un modelo, (soluble analíticamente con ayuda de una transfomrada de Laplace) de algunos aspectos de un obturador que se abre como función del tiempo y lo comparamos con el análisis de la referencia [2].

Descriptores: Efectos transitorios; obturadores dependiente del tiempo.

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1. Introduction

One of the authors (MM) was interested in transient effects in quantum mechanics since long ago [1] when there was no possibility of observing them experimentally. This situation has changed dramatically in the last decades as can be seen, for example in Ref. [2] in the review article by Kleber.

One of the articles mentioned above [1] dealt with a one dimensional problem $-\infty \leq x \leq \infty$ in which, initially, there was a particle characterized by a plane wave coming from $-\infty$ and satisfying a time dependent Schroedinger equation and interrupted at $x = 0$ by a shutter.

If the shutter was opened fully at $t = 0$ the question was to determine the transient probability or current density in the interval $0 \leq x \leq \infty$. The solution was elementary but, at that time, was not well known and, as it can be expressed in terms of Fresnel integrals, it gave time dependent oscillation effects at any point $x > 0$ which were denoted as diffraction in time [1].

The problem mentioned in the previous paragraph can also be considered as a two channel problem in which both of them are restricted to the interval $0 \leq x \leq \infty$, while in one of them there is a matter wave satisfying a free particle Schroedinger equation for $t < 0$, the other is empty. Connecting the two channels appropriately at $x = 0$ we can get again a diffraction in time effect in the second channel.

In the problems discussed in the last paragraphs, the shutter is removed instantaneously and obviously can not be performed experimentally. If the shutter is removed in accordance with some time dependence the problem is much more complicated mathematically as discussed in many papers mentioned in Ref. [2], as well as in another publication [3] by the present authors. The presence of an obstacle to matter waves that is removed following a certain time dependence, or the effect of a potential function for both coordinates and time, clearly invalidates the energy as an integral of motion, as the problem is not then invariant under time translation, which may be one of the causes of the complexity. On the other hand if we have two channel interactions we can appropriately couple them through time independent boundary conditions [4] at $x = 0$ so that energy remains an integral of motion for the complete system, but not if we restrict ourselves to the analysis of only the first channel. This is what we plan to do in the next section in which we actually think of motion of each particle in three dimensions, but we shall restrict ourselves to the frame of reference where the total momentum is zero and the function for the relative coordinate is restricted to S waves.

2. The equations of motion of the problems

Following the observations at the of the last section, we shall discuss a state in three stages

1) First, we have a system of two free particles of masses m_1', m_1'' interacting only at their point of coincidence.
2) The two particles form a compound of mass M through an appropriate boundary condition at their point of coincidence [4].
3) The compound particle disintegrates either into the two original particles or into a new channel where we have them with masses m_3', m_3''.

Thus our state will be represented in a Fock space as

$$\Psi = \begin{bmatrix} \psi_1(\mathbf{r}_1', \mathbf{r}_1'', t) \\ \psi_2(\mathbf{r}_2', t) \\ \psi_3(\mathbf{r}_3', \mathbf{r}_3'', t) \end{bmatrix}. \tag{1}$$

The total momenta should be the same for all three stages so we should have

$$\mathbf{p}_1' + \mathbf{p}_1'' = \mathbf{p}_2' = \mathbf{p}_3' + \mathbf{p}_3'', \tag{2}$$

and taking as reference frame the one of the observer, in which we assume they vanish, we get

$$\mathbf{p}_1'' = -\mathbf{p}_1' \equiv \mathbf{p}_1 \quad \mathbf{p}_3' = -\mathbf{p}_3'' \equiv \mathbf{p}_3 \quad \mathbf{p}_2' = 0. \tag{3}$$

The first component of Ψ becomes only $\psi_1(\mathbf{r}_1, t)$, the second $\psi_2(t)$, and the last one $\psi_3(\mathbf{r}_3, t)$. As we are only considering the interaction in the S-wave (*i.e.* orbital angular momentum equal to zero) ψ_1, ψ_3 depend on the magnitude of $\mathbf{r}_1, \mathbf{r}_3$ so for s-waves they can be written as

$$\psi_1(\mathbf{r}_1, t) = \frac{u_1(r,t)}{r}, \qquad \psi_2 = u_2(t),$$
$$\psi_3(\mathbf{r}_3, t) = \frac{u_3(r,t)}{r}, \tag{4}$$

where we use the same radial variable r for r_1, r_3 as the channel is already indicated by the indices of u_1 and u_3.

We shall use units in which

$$\hbar = c = 1, \qquad m_1' + m_1'' = m_3' + m_3'' \equiv m = 1. \tag{5}$$

We assume that the total mass m of particles in channels 1 or 3 is the same, to simplify the mathematics, as then the wave numbers in channels 1 or 3 are proportional, and will be shown later.

In the units units of Eq. (5) all the variables will be dimensionless and, in particular, the reduced masses in channel 1 and 3 will be

$$\mu_1 \equiv m_1'(1 - m_1'), \quad \mu_3 \equiv m_3'(1 - m_3') \tag{6}$$

Taking in to account the derivative of μ_i with respect to $m_i', (i = 1, 3)$ as zero, we will see that m_i' is restricted to the interval $0 \leq m_i' \leq (1/2)$

As our interaction will be limited to the point $r = 0$ we have that, for $r \neq 0$, $u_1(r,t)$ and $u_3(r,t)$ satisfy the time dependent radial Schroedinger equations which, in our units, are

$$\frac{1}{i}\frac{\partial u_1}{\partial t} = \frac{1}{2\mu_1}\frac{\partial^2 u_1}{\partial r^2}, \quad \frac{1}{i}\frac{\partial u_3}{\partial t} = \frac{1}{2\mu_3}\frac{\partial^2 u_3}{\partial r^2}, \quad r \neq 0. \tag{7}$$

The total probability of the system is given by

$$P = \int_0^\infty u_1^* u_1 dr + u_2^* u_2 + \int_0^\infty u_3^* u_3 dr \tag{8}$$

where $*$ indicates a complex conjugation and P should not change with time. Thus

$$\begin{aligned} \frac{1}{i}\frac{dP}{dt} &= \int_0^\infty \left[u_1^*\left(\frac{1}{i}\frac{\partial u_1}{\partial t}\right) - \left(\frac{1}{i}\frac{\partial u_1}{\partial t}\right)^* u_1\right] dr \\ &+ \left[u_2^*\left(\frac{1}{i}\frac{\partial u_2}{\partial t}\right) - \left(\frac{1}{i}\frac{\partial u_2}{\partial t}\right)^* u_2\right] \\ &+ \int_0^\infty \left[u_3^*\left(\frac{1}{i}\frac{\partial u_3}{\partial t}\right) - \left(\frac{1}{i}\frac{\partial u_3}{\partial t}\right)^* u_3\right] dr = 0, \end{aligned} \tag{9}$$

and using the equations (7) we get

$$\begin{aligned} \frac{1}{i}\frac{dP}{dt} &= \int_0^\infty \frac{1}{2\mu_1}\left[u_1^*\frac{\partial^2 u_1}{\partial r^2} - \frac{\partial^2 u_1^*}{\partial r^2}u_1\right] dr \\ &+ \left[u_2^*\left(\frac{1}{i}\frac{\partial u_2}{\partial t}\right) - \left(\frac{1}{i}\frac{\partial u_2}{\partial t}\right)^* u_2\right] \\ &+ \int_0^\infty \frac{1}{2\mu_3}\left[u_3^*\frac{\partial^2 u_3}{\partial r^2} - \frac{\partial^2 u_3^*}{\partial r^2}u_3\right] dr = 0. \end{aligned} \tag{10}$$

With the help of the identity

$$u_i^*\frac{\partial^2 u_i}{\partial r^2} - \frac{\partial^2 u_i^*}{\partial r^2}u_i = \frac{\partial}{\partial r}\left(u_i^*\frac{\partial u_i}{\partial r} - \frac{\partial u_i^*}{\partial r}u_i\right)$$
$$i = 1, 3, \tag{11}$$

we can evaluate the integrals in Eq. (10) in terms of the integrand at $r = 0$ and ∞. As at $r = \infty$ there is no contribution to the scattered wave for any finite time (as will be seen in the following sections), we just have to consider the contribution at $r = 0$ which has a negative sign.

Thus, we obtain that [4]

$$
\begin{aligned}
-\frac{1}{i}\frac{dP}{dt} &= \frac{1}{2\mu_1}\left[(u_1^*)_{r=0}\left(\frac{\partial u_1}{\partial r}\right)_{r=0} - \left(\frac{\partial u_1^*}{\partial r}\right)_{r=0}(u_1)_{r=0}\right] \\
&+ \left[\left(\frac{1}{i}\frac{\partial u_2}{\partial t} + E_0 u_2\right)^* u_2 - u_2^*\left(\frac{1}{i}\frac{\partial u_2}{\partial t} + E_0 u_2\right)\right] \\
&+ \frac{1}{2\mu_3}\left[(u_3^*)_{r=0}\left(\frac{\partial u_3}{\partial r}\right)_{r=0} - \left(\frac{\partial u_3^*}{\partial r}\right)_{r=0}(u_3)_{r=0}\right] \\
&= 0 \qquad (12)
\end{aligned}
$$

where we have introduced the real constant

$$E_0 \equiv M - 1, \qquad (13)$$

(which obviously cancels in Eq. (12)) to indicate the difference between the mass of the compound particle and that of the two particles in channels 1 and 3. The E_0 is real but can be positive or negative and, for compactness, we shall only discuss the first case where the resonance energy E_0 is positive.

The expression (12) can be written as

$$\sum_{i=1}^{3}\left(y_{3+i}^* y_i - y_i^* y_{3+i}\right) = 0, \qquad (14)$$

where

$$
\begin{aligned}
y_1 &\equiv \frac{1}{2\mu_1}\left(\frac{\partial u_1}{\partial r}\right)_{r=0}, \qquad & y_2 &\equiv u_2, \\
y_3 &\equiv \frac{1}{2\mu_3}\left(\frac{\partial u_3}{\partial r}\right)_{r=0}, \qquad & y_4 &\equiv (u_1)_{r=0}, \\
y_5 &\equiv \frac{1}{i}\frac{\partial u_2}{\partial t} + E_0 u_2, \qquad & y_6 &\equiv (u_3)_{r=0}. \qquad (15)
\end{aligned}
$$

The bilinear form appears in many problems of mathematical physics [5], and it vanishes if there is a linear relation [5] between y_{3+i} and $y_i, i = 1, 2, 3$ of the form

$$y_{3+i} = \sum_{j=1}^{3} c_{ij} y_j \qquad (16)$$

with c_{ij} being a constant hermitian 3×3 matrix.

We can check this by just substituting y_{3+i} of Eq. (16) in Eq. (14), and taking out the factor $y_i, i = 1, 2, 3$ we arrive at the hermitian conjugate relation for the corresponding y_i^*.

We shall assume that all interactions of the two particle systems in channel 1 and 3 are only through the corresponding single particle indicated by the index 2 so that the matrix becomes

$$||c_{ij}|| = \begin{bmatrix} 0 & c_{12} & 0 \\ c_{21} & 0 & c_{23} \\ 0 & c_{32} & 0 \end{bmatrix} \qquad (17)$$

with $c_{21} = c_{12}^*$, $c_{32} = c_{23}^*$ and, for simplicity, we shall consider c_{12}, c_{32} as real numbers.

Thus the equations of motion of our problem become

$$i\frac{\partial u_1}{\partial t} = -\frac{1}{2\mu_1}\frac{\partial^2 u_1}{\partial r^2}, \quad i\frac{\partial u_3}{\partial t} = -\frac{1}{2\mu_3}\frac{\partial^2 u_3}{\partial r^2} \quad \text{for } r\neq 0 \quad (18)$$

$$
\begin{aligned}
(u_1)_{r=0} &= c_{12}u_2 \\
\frac{1}{i}\frac{\partial u_2}{\partial t} + E_0 u_2 &= c_{21}\frac{1}{2\mu_1}\left(\frac{\partial u_1}{\partial r}\right)_{r=0} \\
&+ c_{23}\frac{1}{2\mu_3}\left(\frac{\partial u_3}{\partial r}\right)_{r=0} \qquad (19) \\
(u_3)_{r=0} &= c_{32}u_2
\end{aligned}
$$

Note that Eqs. (19) would be changed if we had taken a different definition of the y_i in (15). Our choice guarantees that the relation between wave functions $(u_i)_{r=0}$ and derivatives $(\partial u_i/\partial r)_{r=0}, i = 1, 3$ in the stationary problem be consistent with Wigner's formalism as will be shown later (Eq. (27)).

3. The Laplace transform of our problem

The Laplace transform of a function u, of time t, and possibly of other variables will be indicated by a bar above it and defined as

$$\bar{u}(s) = \int_0^\infty e^{-st}u(t)dt \qquad (20)$$

with s being, in general, a complex number such that the integral can exist. From (20), we also see that

$$
\begin{aligned}
\int_0^\infty e^{-st}\frac{\partial u}{\partial t}dt &= \int_0^\infty \frac{\partial}{\partial t}(e^{-st}u)dt + s\bar{u} \\
&= -u(0) + s\bar{u} \qquad (21)
\end{aligned}
$$

assuming that s is in a region where $\exp(-st)$ tends to 0 when $t \to \infty$.

The physical problem, we are interested in, is where we have a plane wave in channel 1 *i.e.* $\exp(i\kappa\cdot\mathbf{r})$, and as the interaction can only occur at $l = 0$, there we shall need the radial part of the plane wave so that

$$u_1(r, 0) = \frac{\sin\kappa r}{\kappa} \qquad (22)$$

while the initial values for $u_2(t)$ and $u_3(r, t)$ will be 0.

Thus the Laplace transform of Eqs. (18), (19) becomes

$$
\begin{aligned}
&-i\frac{\sin\kappa r}{\kappa} + is\bar{u}_1 = -\frac{1}{2\mu_1}\frac{\partial^2\bar{u}_1}{\partial r^2}, \quad -is\bar{u}_3 = \frac{1}{2\mu_3}\frac{\partial^2\bar{u}_3}{\partial r^2} \\
&(\bar{u}_1)_{r=0} = c_{12}\bar{u}_2, \quad (\bar{u}_3)_{r=0} = c_{32}\bar{u}_2 \qquad (23) \\
&-is\bar{u}_2 + E_0\bar{u}_2 = c_{21}\frac{1}{2\mu_1}\left(\frac{\partial\bar{u}_1}{\partial r}\right)_{r=0} + c_{23}\frac{1}{2\mu_3}\left(\frac{\partial\bar{u}_3}{\partial r}\right)_{r=0}.
\end{aligned}
$$

The solutions of our problem would be provided by the inverse Laplace transform which implies an integration over s along the line parallel to the imaginary axis of the complex plane of s from $c - i\infty$ to $c + i\infty$, and to the right of all the poles of the problem.

It is convenient to change our variable s and contour, by introducing a new parameter k through the definition

$$s = -ik^2/2, \tag{24}$$

so that the contour can be modified as indicated in figure 1. Using the relation (21) the equations in (24) become

$$\frac{d^2\bar{u}_1}{dr^2}+k^2\mu_1\bar{u}_1=i\frac{2\mu_1 \sin\kappa r}{\kappa}, \quad \frac{d^2\bar{u}_3}{dr^2}+k^2\mu_3\bar{u}_3=0 \tag{25}$$

$$-\frac{k^2}{2}\bar{u}_2+E_0\bar{u}_2=c_{21}\frac{1}{2\mu_1}\left(\frac{\partial\bar{u}_1}{\partial r}\right)_{r=0} +c_{23}\frac{1}{2\mu_3}\left(\frac{\partial\bar{u}_3}{\partial r}\right)_{r=0} \tag{26}$$

$$(\bar{u}_1)_{r=0} = c_{12}\bar{u}_2, \quad (\bar{u}_3)_{r=0} = c_{32}\bar{u}_2,$$

where we can use the last Eqs. (26) to eliminate $\bar{u}_2$ in the first Eq. (26) to get

$$\begin{bmatrix} (\bar{u}_1)_{r=0} \\ (\bar{u}_3)_{r=0} \end{bmatrix} = \frac{1}{2(k_0^2-k^2)}\begin{bmatrix} c_{12}c_{21} & c_{12}c_{23} \\ c_{32}c_{21} & c_{32}c_{23} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\mu_1}\left(\frac{\partial\bar{u}_1}{\partial r}\right)_{r=0} \\ \frac{1}{\mu_3}\left(\frac{\partial\bar{u}_3}{\partial r}\right)_{r=0} \end{bmatrix}. \tag{27}$$

Note that this result is a Wigner R matrix formulation [6] for two channels and one resonant state. In Eq. (27) we have replaced E_0 by

$$E_0 = \frac{1}{2}k_0^2, \tag{28}$$

where k_0 is real for $E_0 > 0$, and imaginary for $E_0 < 0$, but for compactness, we shall restrict ourselves to the first case.

The Laplace transform problem is then fully determined by Eqs. (25) and boundary conditions (27). The former admit immediately the solutions

$$\bar{u}_1 = \frac{2i\sin\kappa r}{\kappa[k^2-(\kappa^2/\mu_1)]} + A_1\exp[i(k\sqrt{\mu_1}r)],$$
$$\bar{u}_3 = A_3\exp[i(k\sqrt{\mu_3}r)], \tag{29}$$

as the scattered waves must be purely outgoing. The coefficients A_1, A_3 will be obtained through Eq. (27) in the following section.

4. Determination of the coefficients A_1, A_3

At $r = 0$ we have from (29) that

$$(\bar{u}_1)_{r=0} = A_1; \quad (\bar{u}_3)_{r=0} = A_3$$
$$\frac{1}{\mu_1}\left(\frac{\partial\bar{u}_1}{\partial r}\right)_{r=0} = \frac{2i}{[k^2-(\kappa^2/\mu_1)]} + i\frac{kA_1}{\sqrt{\mu_1}};$$
$$\frac{1}{\mu_3}\left(\frac{\partial\bar{u}_3}{\partial r}\right)_{r=0} = \frac{ik}{\sqrt{\mu_3}}A_3. \tag{30}$$

Thus Eq. (27) becomes

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \frac{(-1)}{2(k^2-k_0^2)}\begin{bmatrix} c_{12}c_{21} & c_{12}c_{23} \\ c_{32}c_{21} & c_{32}c_{23} \end{bmatrix} \times \begin{bmatrix} \frac{2i}{[k^2-(\kappa^2/\mu_1)]} + \frac{ikA_1}{\sqrt{\mu_1}} \\ \frac{ik}{\sqrt{\mu_3}}A_3 \end{bmatrix}. \tag{31}$$

Multiplying both sides of Eq. (31) by $(k^2 - k_0^2)$, and passing to the left side the terms depending on A_1, A_3 we get

$$\begin{bmatrix} k^2-k_0^2+i(k/2\sqrt{\mu_1})c_{12}c_{21} & i(k/2\sqrt{\mu_3})c_{12}c_{23} \\ i(k/2\sqrt{\mu_1})c_{32}c_{21} & k^2-k_0^2+i(k/2\sqrt{\mu_3})c_{32}c_{23} \end{bmatrix}\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \frac{-i}{[k^2-(\kappa^2/\mu_1)]}\begin{bmatrix} c_{12}c_{21} \\ c_{32}c_{21} \end{bmatrix} \tag{32}$$

To get the explicit values of A_1, A_3 in terms of the parameters in Eq. (32) we need to apply to both sides of this equation, the inverse of the matrix that appears on the left hand side. To find this inverse we note that for a 2×2 matrix we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = (ad-bc)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{33}$$

For the matrix in Eq. (32) the determinant is

$$ad - bc = (k^2-k_0^2)\left[k^2-k_0^2+i\frac{k}{2}\left(\frac{c_{12}c_{21}}{\sqrt{\mu_1}}+\frac{c_{32}c_{23}}{\sqrt{\mu_3}}\right)\right]. \tag{34}$$

Remembering that after Eq. (17) we assume all c_{ij} real so $c_{ij} = c_{ji}$, and introducing the notation

$$\Gamma_1 \equiv \frac{1}{2}c_{12}^2, \quad \Gamma_3 \equiv \frac{1}{2}c_{23}^2, \quad \sqrt{\Gamma_1\Gamma_3} \equiv \frac{1}{2}c_{12}c_{23}$$
$$\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_3}} \equiv \frac{1}{\sqrt{\mu}}, \quad \frac{\Gamma}{\sqrt{\mu}} \equiv \frac{\Gamma_1}{\sqrt{\mu_1}} + \frac{\Gamma_3}{\sqrt{\mu_3}}, \tag{35}$$

we obtain

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = (-i)\left[k^2 - \left(\frac{\kappa^2}{\mu_1}\right)\right]^{-1} (k^2 - k_0^2)^{-1}\left\{(k^2 - k_0^2) + i\left(\frac{k}{2\sqrt{\mu}}\right)\Gamma\right\}$$
$$\times \begin{bmatrix} k^2 - k_0^2 + i(k/2\sqrt{\mu_3})\Gamma_3 & -i(k/2\sqrt{\mu_3})\sqrt{\Gamma_1\Gamma_3} \\ -i(k/2\sqrt{\mu_1})\sqrt{\Gamma_1\Gamma_3} & k^2 - k_0^2 + i(k/2\sqrt{\mu_1})\Gamma_1 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \sqrt{\Gamma_1\Gamma_3}. \end{bmatrix} \quad (36)$$

Carrying out the operations applying the 2×2 matrix to the two component vectors we get

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \frac{(-i)}{[k^2 - (\kappa^2/\mu_1)][(k^2 - k_0^2) + ik(\Gamma/\sqrt{\mu})]} \times \begin{bmatrix} \Gamma_1 \\ \sqrt{\Gamma_1\Gamma_3} \end{bmatrix} \quad (37)$$

where a factor $(k^2 - k_0^2)$ appearers in the numerator which is canceled with the same factor in the denominator given by (34).

The $(\Gamma_1/\sqrt{\mu_1})$, $(\Gamma_3/\sqrt{\mu_3})$ can be considered as the widths of the resonant state in channels 1 and 3, while $(\Gamma/\sqrt{\mu})$, defined in Eq. (35), is the total width [6].

The poles of the scattering amplitudes A_1, A_3 are give by the zeros of the denominator in Eq. (37). Two of them are $k = \pm(\kappa\sqrt{\mu_1})$ and the other two are roots $\lambda_\pm$ of the equation

$$k^2 + ik(\Gamma/\sqrt{\mu}) - k_0^2 = 0,$$
$$\lambda_\pm = -i\left(\frac{\Gamma}{2\sqrt{\mu}}\right) \pm \sqrt{k_0^2 - \frac{\Gamma^2}{4\mu}}. \quad (38)$$

The scattering amplitudes in channel 1 and 3 can then be written as

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \frac{(-i)}{[k - (\kappa/\sqrt{\mu_1})][k + (\kappa/\sqrt{\mu_1})](k - \lambda_+)(k - \lambda_-)} \times \begin{bmatrix} \Gamma_1 \\ \sqrt{\Gamma_1\Gamma_3} \end{bmatrix}. \quad (39)$$

The Laplace transform solution of our two channel problems $\bar{u}_i(r,k)$ has now been fully determined through Eqs. (29) and (39). Then we proceed to express the solution as a function of time using an inverse Laplace transform.

5. Transient effects in the two channel interaction

As $\bar{u}_i(r,s)$ is defined in terms of $u_i(r,t)$ by the expression (20), the inverse operations will be

$$u_i(r,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{u}_i(r,s)ds, \quad (40)$$

and using the relation (24) for s in terms of the variable k it becomes

$$u_i(r,t) = -\frac{1}{2\pi i}\int_{C'} e^{-i\frac{k^2 t}{2}}\bar{u}_i(r,k)kdk, \quad (41)$$

where the contour C' is shown in Fig. 1.

In the second quadrant of the complex k plane $k{=}-k_x{+}ik_y$ where k_x, k_y are real and positive. Thus

$$\exp\left[-i\frac{k^2 t}{2}\right] = \exp\left[-i(k_x^2 - k_y^2)t/2\right] \times \exp(-k_x k_y t) \quad (42)$$

and when $|k| \to \infty$, the expression in (42) tends to zero. We can then close the contour C' by the dashed arc of a circle indicated in Fig. 1 and, if k_0 is real (*i.e.* $M > 1$) we can deform it to the contour C in Fig. 1, where we bypass the poles $k = \pm(\kappa/\sqrt{\mu_1})$ at the real axis by the indicated circles. Thus our $u_i(r,t)$ becomes

$$u_i(r,t) = -\frac{1}{2\pi}P\int_{-\infty}^{\infty}\exp(-ik^2t/2)k\bar{u}_i(r,k)dk, \quad (43)$$

where P stands for the principal value of the integral in order, to take into account the semicircles around the poles $k = \pm(\kappa/\sqrt{\mu_1})$. From Eq.(29) the radial scattered $\bar{u}_i(r,k)$ is given by

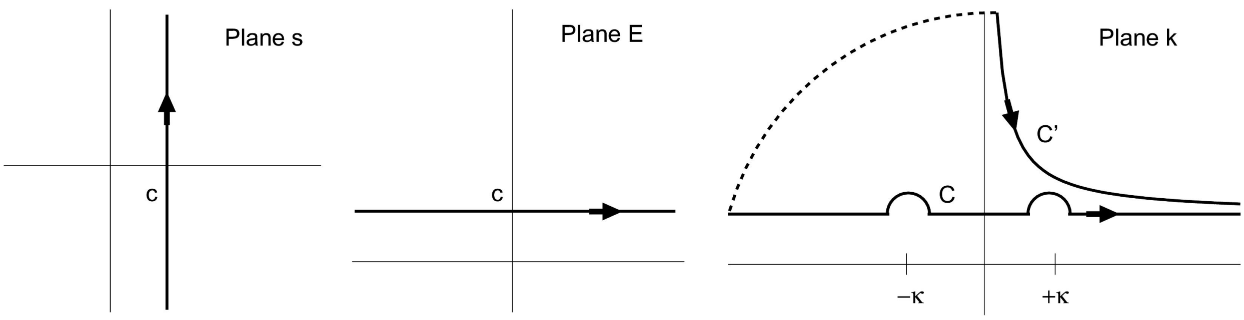

FIGURE 1. Contours in the complex plane for the inverse of the Laplace transform.

$$\bar{u}_i(r,k) = A_i \exp\left[ik(\sqrt{\mu_i}r)\right] \quad i = 1,3 \tag{44}$$

with the A_i of Eq. (39). We see that it is convenient, first, to use the theory of residues to write

$$F(k) \equiv \frac{k}{[k-(\kappa/\sqrt{\mu_1})][k+(\kappa/\sqrt{\mu_1})](k-\lambda_+)(k-\lambda_-)} = \sum_{\alpha=1}^{4} \frac{R_\alpha}{(k-q_\alpha)} \tag{45}$$

where

$$q_1 = (\kappa/\sqrt{\mu_1}), \quad q_2 = -(\kappa/\sqrt{\mu_1}), \quad q_3 = \lambda_+, \quad q_4 = \lambda_-,$$

$$R_\alpha = \lim_{k\to q_\alpha}\left[(k-q_\alpha)F(k)\right]. \tag{46}$$

We, thus, obtain from Eqs. (29), (39) and (45) that the scattered part of the wave function in channel 1, which we continue to denote by $u_1(r,t)$, is

$$u_1(r,t) = \sum_{\alpha=1}^{4} R_\alpha \Gamma_1 \frac{i}{2\pi} P \int_{-\infty}^{\infty} \frac{\exp[ik(\sqrt{\mu_1}r) - \frac{1}{2}ik^2 t]}{(k-q_\alpha)} dk = \sum_{\alpha=1}^{4} R_\alpha \Gamma_1 M(\sqrt{\mu_1}r, q_\alpha, t) \tag{47}$$

where the function $M(\rho,q,t)$ is given by Ref. [1]

$$M(\rho,q,t) = \frac{1}{2}\exp\left[i\left(q\rho - \frac{1}{2}q^2 t\right)\right] \times \text{erfc}\left[(1-i)(4t)^{-1/2}(\rho - qt)\right] \tag{48}$$

with

$$\text{erfc}(z) = 2\pi^{-1/2}\int_z^{\infty} \exp(-z^2)dz = 1 - erf(z) \tag{49}$$

From Eqs. (44) and (37) we have that for the other channel

$$u_3(r,t) = \sum_{\alpha=1}^{4} R_\alpha \sqrt{\Gamma_1\Gamma_3} M(\sqrt{\mu_3}r, q_\alpha, t) \tag{50}$$

We have thus the explicit analytic expressions for the transient scattered [1] amplitudes as functions of the $M(\rho,q,t)$, whose properties are known in Refs. [1, 2]. We proceed then to discuss their physical implications, and later to compare them with results derived from other procedures in reference [2] .

6. Physical implications

The discussion of transient effects in the probability density *i.e.* $|u_i(r,t)|^2, i = 1,3$ for the two channel problem is interesting in itself, but in the introduction we mentioned that we would like to use the width in the channel with index 3 as a kind of model for the time dependent shutter effect in the channel with index 1. Thus we shall only concern ourselves with the behavior of $|u_1(r,t)|^2$, in which the effect of the other channel only appears through $(\Gamma_3/\sqrt{\mu_3})$ both in the total width $\Gamma/\sqrt{\mu}$ of (35) and the poles $\lambda_\pm$ of (38). Thus we first write explicitly $u_1(r,t)$ as

$$u_1(r,t) = \frac{\Gamma_1}{2\mu_1}\left\{\frac{M[\sqrt{\mu_1}r, (\kappa/\sqrt{\mu_1}), t]}{(\kappa^2/\mu_1) + i(\kappa/\sqrt{\mu_1})(\Gamma/\sqrt{\mu}) - k_0^2} + \frac{M[\sqrt{\mu_1}r, -(\kappa/\sqrt{\mu_1}), t]}{(\kappa^2/\mu_1) - i(\kappa/\sqrt{\mu_1})(\Gamma/\sqrt{\mu}) - k_0^2} - \frac{2\lambda_+ M(\sqrt{\mu_1}r, \lambda_+, t)}{[(\kappa^2/\mu_1) - \lambda_+^2]\sqrt{4k_0^2 - (\Gamma^2/\mu)}} + \frac{2\lambda_- M(\sqrt{\mu_1}r, \lambda_-, t)}{[(\kappa^2/\mu_1) - \lambda_-^2]\sqrt{4k_0^2 - (\Gamma^2/\mu)}}\right\} \tag{51}$$

In the Eq. (51) the function $M(\sqrt{\mu_1}r, q, t)$ is given by Eq. (48), where κ is the wave number of our incident plane wave, $\Gamma/\sqrt{\mu}$ is given by (35) and $k_0^2 = 2(M-1)$.

In Fig. 2 we give a numerical example of $|u_1(r,t)|^2$ as function of time. We indicate in Fig. 2 specific values of $\mu_1, (\Gamma/\sqrt{\mu_1}), k_0$ and the point r of observation and keep $(\Gamma_3/\sqrt{\mu_3})$ as a parameter. By changing the value of this parameter in the interval $0 \le (\Gamma_3/\sqrt{\mu_3}) \le \infty$, we evaluate in Fig. 2 the probability density $|u_1(r,t)|^2$ as function of time and compare it in the next section, with the diffraction in time effect when a shutter is opened as a function of time.

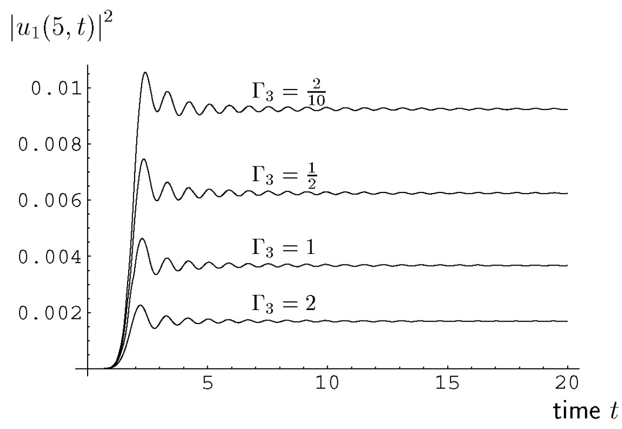

FIGURE 2. Scattered probability density in channel 1 as function of the width $(\Gamma_3/\sqrt{\mu_3})$ in channel 3, for the fixed values of the parameters given below: $M = 6$, $m = 1$, $k_0 = \sqrt{2(M-1)} = \sqrt{10}$, $\mu_1 = \Gamma_1 = 1$, $\mu_3 = 1$, $\kappa = 4$, $r = 10$. The values above the curves are those of the width $(\Gamma_3/\sqrt{\mu_3})$.

We can get though some idea of the effect of channel 3 on 1 by going to the limit $t \to \infty$, where we know that the final stationary system is established.

We know that when $t \to \infty$ we have [1]

$$M(\sqrt{\mu_1}r, \kappa, t) \to \exp\left\{i\left[q\sqrt{\mu_1}r - \frac{1}{2}q^2 t\right]\right\},$$
$$\text{if} \quad \left(-\frac{\pi}{4}\right) \le argq \le \left(\frac{3\pi}{4}\right). \quad (52)$$

$$M(\sqrt{\mu_1}r, \kappa, t) \to 0,$$
$$\text{if} \quad \left(\frac{3\pi}{4}\right) \le argq \le \left(\frac{7\pi}{4}\right). \quad (53)$$

Thus in Eq. (51) we are left only with the first term in the curly bracket and

$$|u_1(r,t)|^2 = \left(\frac{\Gamma_1}{2\mu_1}\right)^2 \frac{1}{(\kappa^2 - k_0^2)^2 + \kappa^2(\Gamma/\sqrt{\mu})^2} \quad (54)$$

As from Eq. (35) we have

$$(\Gamma/\sqrt{\mu}) = (\Gamma_1/\sqrt{\mu}) + (\Gamma_3/\sqrt{\mu}),$$

we see that if $(\Gamma_3/\sqrt{\mu_3}) \to 0$ then it becomes the standard one level formula in a single channel [6], while for $(\Gamma_3/\sqrt{\mu_3}) \to \infty$ it vanishes. Thus in a certain way the other channel acts as a fully open shutter when $(\Gamma_3/\sqrt{\mu_3}) \to 0$ and a closed me when $(\Gamma_3/\sqrt{\mu_3}) \to \infty$.

7. The time dependent shutter problem as viewed by Kleber and Scheitler

In the present paper; we considered the transient effects in a two channel interaction so as to get, in the first channel, some of the behavior of a time dependent shutter.

We mentioned, in the abstract, the difficulties in the mathematical analysis of an arbitrary time dependent opening of a shutter. There is one case though, discussed in the papers of reference [2], in which this can be carried out, which corresponds to the one dimensional time dependent Schroedinger equation.

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{V}{t}\delta(x)\right]\psi(x,t) = 0, \quad (55)$$

where they use units in which $\hbar = m = 1, V$ is a positive constant and the initial condition is

$$\psi(x,0) = \begin{cases} \exp(ikr), & \text{if } x < 0; \\ 0, & \text{if } x > 0. \end{cases} \quad (56)$$

This problem is solved in the full interval $0 \le t \le \infty$ in the papers of reference 2, in which for $t = 0$ the shutter at the origin $x = 0$ is fully closed while at $t = \infty$ is completely open.

Obviously, this general solution does not correspond to our previous analysis in which the width in channel 3 *i.e.* $(\Gamma_3/\sqrt{\mu_3})$ is adjusted to different values to influence the probability density coming from channel 1.

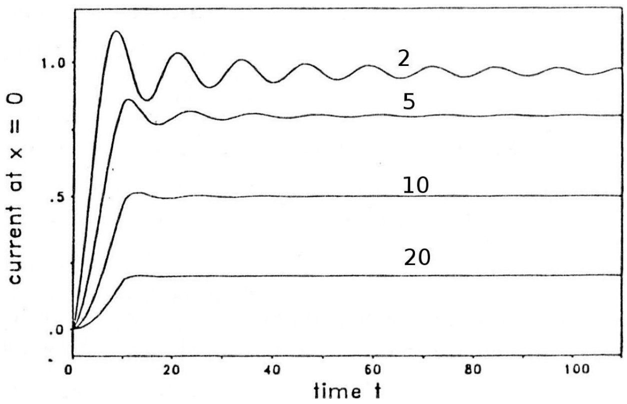

FIGURE 3. Tunneling [in units $\hbar\kappa/m$] as function of time [in units $(m/\hbar\kappa^2)$] for different values of the parameter V and a fixed $T = 10$. The values of V are indicated above the curves. (From G. Scheitler Ph. D. thesis in reference[2], p. 19, Fig. 2.7).

Fortunately in his Ph. D. thesis [2], Scheitler also discussed a potential whose strength can be stopped after a time T *i.e.* where the coefficient of $\delta(x)$ in Eq. (55) is

$$\begin{aligned} &(V/t) \quad \text{for } 0 \le t \le T \\ &(V/T) \quad \text{for } T \le t \le \infty. \end{aligned} \quad (57)$$

In this case the problem was also solved analytically by Scheitler [2] and the current for $x = 0$ is plotted in Fig. 3 as a function of the time t at the different values of T at which the potential strength becomes the constant (V/T), which are indicated above the curves in Fig. 3.

8. Conclusion

Transient phenomena in quantum mechanics are more easily discussed if the perturbation is applied suddenly, say at time $t = 0$. Thus for time $t < 0$, we have a fixed time independent potential (let us cal it V_-), while for $t > 0$ we have another one (let us call it V_+). If we start with the eigenstates of the potential V_- they will show transient time dependent effects at $t > 0$ as shown for example in reference[1] for the problem of “Diffraction in time” when we suddenly remove a shutter.

Physically a perturbation can not be applied suddenly so the situation described in the previous paragraph is an idealization.

Thus, in fact all perturbations imply some time dependent potentials which lead to the difficulties mentioned in the paper and, of particular relevance, as the problem is not then invariant under time translations the energy is no longer an integral of motion.

In the present paper, we discuss a different way in which we can make changes in our problem by enlarging the space in which it is described. Thus, we can change the behavior in part of the problem we are interested in (channel 1 in the present paper) by controlling the parameter $(\Gamma_3/\sqrt{\mu_3})$ that we have introduced in the enlarged part of the space (channel 3).

Notice that while probability density discussed in Fig. 2 and current in Fig. 3 are closely related as is remarked in Ref. [1], the parameters in the two pictures are independent. The figures show though that the transient effects in the two channel time independent interactions can lead to a behavior in the first channel with resemblance to some aspects for a time dependent shutter in a single channel.

While our discussion deals with a particular problem, the ideas presented in this paper are, in principle, valid for more general situations.

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