On the stabilization of bubble solitons

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We analyze the instability of bubble soliton solutions for the cubic quintic nonlinear Schrödinger equation. This equation, for instance, can be obtained studying the nonlinear excitations of the DNA model. We have found that under specific restrictions concerning the main parameter of the model and soliton velocities, these solutions are stable.

Keywords: Solitons; NSE; stability.

Se analiza la estabilidad de las soluciones solitónicas tipo burbuja para la ecuación no-lineal cúbica-quinta de Schrödinger. Esta ecuación puede ser obtenida, por ejemplo, estudiando las excitaciones no-lineales del modelo del ADN. Hemos encontrado que esas soluciones son estables bajo restricciones específicas concernientes al parámetro principal del modelo y las velocidades solitónicas.

Descriptores: Solitones; NSF; estabilidad.

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1. Introduction

The role of solitons in contemporary physics is generally recognized, and the most characteristic property of solitons is that they are objects which in many aspects resemble particles. One of the crucial issues of nonlinear waves, and especially of what are known as soliton solutions, is the question of their stability. Recent experiments on Bose Einstein condensations have revealed the striking interest of solitons named dark and kink solitons [1, 2]. It is well know that the generalized nonlinear Schrödinger equation (NLSE) with general nonlinearity can support multi stable temporal or spatial solitons in several dimensions. These solutions resemble the well known kink, dark or drop solitons [3]. It has also been suggested also by several authors that nonlinear dynamics has a word to say concerning the energy transport along proteins or indeed DNA. One of the most widely known theories in this context is probably that of Davydov‘s soliton solution of the nonlinear Schrödinger equation [4]. The internal excitation of molecules and their motion around equilibrium positions are inseparably linked. Localized opening or so named bubbles of double stranded DNA is essential in a number of cellular processes such as the initiation of gene transcription and DNA replication. Recent studies of dynamics of these twist induced bubbles (or dark solitons) in a random DNA sequence show that small bubbles (less than several tens of base pairs) are delocalized along the DNA, whereas larger bubbles become localized in various regions [5]. Thus, the study of the stability of these structures, *i.e.* bubbles or dark solitons, is an important task for research. In recent years, many authors have studied structural stability in the framework of various evolution equations for different types of perturbations.

In this contribution, we analyze the stabilization of the bubble (dark) soliton solution that was obtained using the Hirota method for the so called cubic-quintic nonlinear Schrödinger equation (CQNSE) in paper [6]:

$$i\varphi_t + \varphi_{xx} - \left(3\left|\varphi\right|^2 - (2A+1)\right)\left(\left|\varphi\right|^2 - 1\right)\varphi = 0, \quad (1)$$

with A being the parameter of the model. This form can be obtained from the standard equation

$$i\psi_t + \psi_{xx} - \varkappa_1\psi + \varkappa_2\left|\psi\right|^2\psi - \varkappa_3\left|\psi\right|^4\psi = 0,$$

by making transformations of parameters and scale variables. As it is well known, form (1) is convenient for obtaining soliton solutions [3]. In the following section we will give a brief and comprehensive exposition of how the Boussinesq-like equation may be derived from Eq. (1). Section 2 is devoted to the construction of the soliton solutions. The results obtained here will play a crucial role in the rest of the paper. In Sec. 3, we analyze the stabilization of special dark soliton solutions obtained via the Boussinesq equation. The conclusions appear in the final part of this work.

2. The Boussinesq equation

The main idea for finding many soliton solutions exploited in the mentioned paper was to map the Eq. (1) into another completely integrable nonlinear equation, by choosing certain parameter values and a suitable vacuum, which for this case is:

$$\varphi_0 = \sqrt{\frac{2A+1}{3}} = \sqrt{a}.$$

The linear waves around this vacuum have the Bogoliubov dispersion relation

$$\omega^2 = k^2 \left(k^2 + \frac{4}{3}(A-1)(2A+1) \right) \quad (2)$$

which will be useful in further calculations. Next, analyzing the nonlinear oscillations in the neighborhood of the stable vacuum φ_0, the perturbed solution is considered in Eq. (1)

$$\varphi = \varphi_o - \phi\,(x,t)\,, \quad (3)$$

with $\phi\,(x,t)$ being the new unknown function to be solved. Then we introduce two new functions:

$$\Phi\,(x,t) = \phi\,(x,t) + \phi\,(x,t)^*$$
$$\Theta\,(x,t) = i\left(\phi\,(x,t) - \phi\,(x,t)^*\right), \quad (4)$$

where the asterisk above the unknown function indicates its complex conjugate. Equation (1) can be transformed by using the Eqs. (3) and (4) to the next system

$$\Theta_\tau = \Phi_{\xi\xi} - 6ab^2\Phi + 6a^{\frac{3}{2}}\Phi^2$$
$$\Phi_\tau = -\Theta_{\xi\xi},$$

which that finally can be transformed to the Boussinesq-like equation (Bq)

$$U_{\tau\tau} - U_{\xi\xi} + 6(U^2)_{\xi\xi} + U_{\xi\xi\xi\xi} = 0 \quad (5)$$

with

$$\tau = 6ab\;t, \qquad \xi = \sqrt{6ab}\;x,$$
$$\Phi = \frac{6}{\sqrt{a}}\,U, \qquad b = a - 1 = \frac{2}{3}\,(A-1)\,. \quad (6)$$

3. Soliton solutions

The cubic quintic nonlinear Schr¨odinger equation can support bubble or dark soliton which present interesting behaviours for the case in which the system is emerging from a state of three degenerated vacua [6]. It is important to notice that when the vacuum degeneracy is slightly destroyed by making A = 1 + 3=2 ε; bubbles or grey solitons can profusely appear around the quasi-stable vacuum φ_o. Analyzing the behavior of grey solitons, one finds that they are able to interact elastically without loss of energy, besides, they condensate forming slowtravelling bubbles The general method for finding soliton solutions used here is the direct Hirota method [7]. The general soliton solutions of Eq. (5) can be represented as

$$U = \frac{\partial^2}{\partial \xi^2} \ln f(\xi,\tau), \quad (7)$$

and then it is possible to write Eq. (7) in the following bilinear form:

$$-\left(f_\tau\right)^2 + f f_{\tau\tau} + \left(f_\xi\right)^2 - f f_{\xi\xi} + 3\left(f_{\xi\xi}\right)^2$$
$$-4 f_\xi f_{\xi\xi\xi} + f f_{\xi\xi\xi\xi} = 0 \quad (8)$$

Further, the preceding equation could be rewritten in a condensed form as

$$\left(D_\xi D_\tau \;+ D_\xi^4\right) f \cdot f = 0 \quad (9)$$

where the Hirota operators D_ξ^m obey this non-standard operation

$$D_\xi^m D_\tau^n \; a \cdot b$$
$$= \left(\partial_\xi - \partial_{\xi'}\right)^m \left(\partial_\xi - \partial_{\xi'}\right)^n a\,(\xi,\tau)\, b(\xi^{'},\tau^{'})|_{\xi'=\xi,\tau'=\tau}$$

Clearly the bilinear form of the Bq-equation belongs to the class

$$Q(D_x, D_y, \ldots) f.f = 0, \quad Q(0) = 0$$

For this class of equations, in contrast to U, soliton solutions are indeed simple in terms of f in contrast to U. This method has turned out to be quite efficient. After some calculations, the multi-soliton solutions can be presented as

$$f(\xi,\tau) = \sum_{\mu=0.1} \exp^{/} \left[\sum_{i=1}^{N} \mu_i \eta_i + \sum_{1 \le i < j}^{N} \mu_i \mu_j A_{ij} \right] \quad (10)$$

$$\eta_i = p_i \xi - \varepsilon_i \Omega_i \tau - \eta_i^0, \;\; \varepsilon_i = +1, \;\; \varepsilon_i = -1 \quad (11)$$

$$\Omega_i = p_i\,(1 - p_i)^{1/2}\,, \quad (12)$$

with $\exp^{'}$ meaning $\exp^{'}[.] = \varepsilon \cdot \exp[.]$, and, parameter ε by definition can take the two arbitrary values, ± 1. This means that, for constructing the specific solution, one could take only one of the available values of the parameter ε. The velocity of each soliton, antisoliton is denoted by $v_i = \Omega_i/p_i$. Here the p_i and η_i are two real constants relating to the amplitude and phase, respectively, of the ith soliton, and the coefficients A_{ij} fulfil

$$\exp\left[A_{ij}\right] = \left| \frac{\left(\varepsilon_i v_i - \varepsilon_j v_j\right)^2 - 3\left(p_i - p_j\right)^2}{\left(\varepsilon_i v_i - \varepsilon_j v_j\right)^2 - 3\left(p_i + p_j\right)^2} \right| = |a_{ij}| \quad (13)$$

It is easy to check also that the velocity of the ith soliton determines the manner in which it can travel along the unidimensional medium. While the usual soliton solution of the normal Boussinesq (Bq.) equation travels faster because its amplitude is greater than the other, our soliton will behave in a opposite way. In other words, the small soliton should travel faster than the tall one. This occurs because the velocity of each soliton takes the form $v_i = \sqrt{1 - p_i^2}$.

Indeed, let us take, for purposes of studying the stabilization, the one soliton solution of this conglomerate (10). By taking into consideration only the real part, $Re\;\varphi\,(x,t) = \Gamma\,(x,t)$ of the complex solution $\varphi(x,t)$ in (3), the end of calculations one obtains

$$\Gamma(x,t) = \sqrt{a} - \Phi, \text{ with}$$
$$\Phi = \frac{3b}{4\sqrt{a}} p^2$$
$$\times \left(\text{sech}\left[\frac{p}{2}\sqrt{6ab}\left(x \pm \sqrt{6ab} V t \right) + Ln\alpha \right] \right)^2, \quad (14)$$

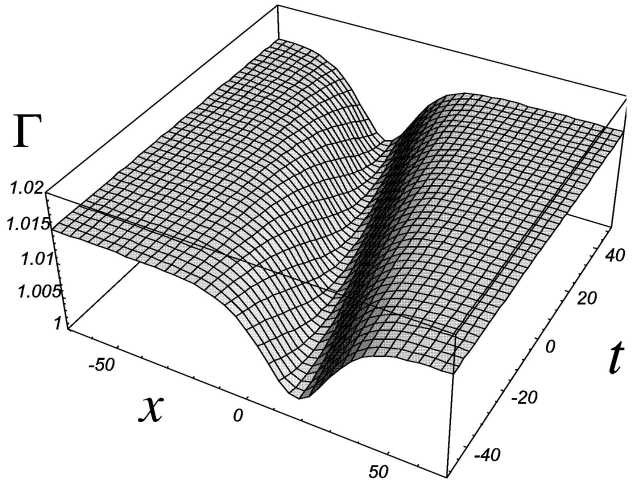

FIGURE 1. Bubble-or gray-like soliton that is obtained from the cubic quintic Schrodinger equation with the aid of the Boussinesq Eq. (5).

where α is the coefficient of the exponent function in the next expression

$$f\,(\xi,\tau)=1+\alpha\;e^{p(\xi+V\tau)}$$

and determines the solutions of the Eq. (5).

Next, for solution (14) we analyze the conditions under which this soliton is or is not stable depending on its velocity. The Bq equation was obtained considering the strict restriction when the value of the main parameter is $A=1+(3/2)\varepsilon$, with $\varepsilon \ll 1$. This parameter value destroys the vacuum degeneration (when $A=1$, the potential energy has three degenerated vacua), enabling us to have two degenerated local minima with a global one between them at the center of the potential. This value of the parameter A meets the condition of small amplitude values for the nonlinear oscillations $|\phi\,(x,t)| \ll 1$ around the stable vacuum field φ_0. The one grey soliton solution for a selected parameter value A is shown in Fig. 1.

4. Stabilization

It is known that the stability criterion for dark solitons should be defined through the renormalized momentum. We shall use the renormalized momentum introduced by Jones and Roberts [8]. Let us recall briefly the main points in the derivation of the equations for determining the criteria of instability of dark solitons; here we follow the general method developed by Pelinovsky *et al.* in Ref. 9. Since the integral of motion for the case of nonzero boundary conditions is divergent, it is then necessary to introduce the renormalized invariants.

Let us overview some results of the work [9] which will be useful in this study. The general idea in this approach is shortly outlined here. The analysis of the stability of dark soliton solutions can be carried out in the framework of the perturbation theory if the soliton parameters vary slow with time. The analysis was done in the framework of the perturbation theory of solitons. It is supposed that the amplitude of instability-induced perturbations remains small for long time interval, and the parameters of the dark soliton vary slowly in such a manner that it is possible then to introduce a small parameter ε that will characterize small perturbations of an unstable dark soliton. Then it is convenient to look for the solutions of Eq. (1) as the asymptotic expansion

$$\varphi=[\varphi_s(z;v,q)+\varepsilon\psi_1(z;v,\varphi_0,X,T) +\varepsilon^2\psi_2(z;v,\varphi_0,X,T)+O(\varepsilon^2)]e^{iS(X,T)} \quad (15)$$

with

$$z=x-\frac{1}{\varepsilon}X_s(T),\quad X_s(T)=\int_0^T v(T)dT, X=\varepsilon x, T=\varepsilon t.$$

Where $v(T)$ is the slow varying soliton velocity, $S(X,T)$ is the local phase of the background wave near the soliton, X and T are the common slow space and time variables, while X_s is the coordinate of the soliton "center". After analysing Eq. (1) and taking into consideration Eq. (15), the "asymptotic differential" equation for the soliton velocity is obtained. Further analysis of the linear approximation leads us to obtain an eigenvalue of the asymptotic equation, which gives a general criterium that the dark soliton instability occurs provided that

$$\frac{\partial M_r}{\partial v}|_{v=v_0}<0,$$

where M_r stands for the renormalized momentum and v_0 is the unperturbed soliton velocity. Soliton instability is weak near the instability threshold when the velocity v of the unstable dark soliton is close to a critical value v_c defined by the instability threshold equation

$$\frac{\partial M_r}{\partial v}|_{v=v_{cr}}=0.$$

We use the renormalized momentum

$$M_r=\frac{i}{2}\int_{-\infty}^{\infty}\left[\left(\varphi-\sqrt{a}\right)\frac{\partial\varphi^*}{\partial z}-\left(\varphi^*-\sqrt{a}\right)\frac{\partial\varphi}{\partial z}\right]dz. \quad (16)$$

For a travelling wave, we use the standard definition of wave function $\varphi(\xi,\tau)=\varphi(\xi-V\tau)=\varphi(\eta)$, with $V=v_o$ being the unperturbed soliton velocity.The equations which are needed to calculate the slope of the renormalized momentum (16) are

$$M_r=-\frac{\sqrt{6ab}}{2}\int_{-\infty}^{\infty}\left(\Phi\frac{d\Theta}{d\eta}-\Theta\frac{d\Phi}{d\eta}\right)d\eta$$

$$\Theta=-\frac{1}{V}\frac{d}{d\eta}\Phi+\frac{6ab}{V}\int\Phi d\eta-\frac{6a^{\frac{3}{2}}}{V}\int\Phi^2 d\eta$$

where the function Φ is taken from Eq. (14). Functions Φ and Σ vanish when $\eta\to\pm\infty$ and the parameter p is related to the velocity as follows:

$$V^2=6ab-4p^2$$

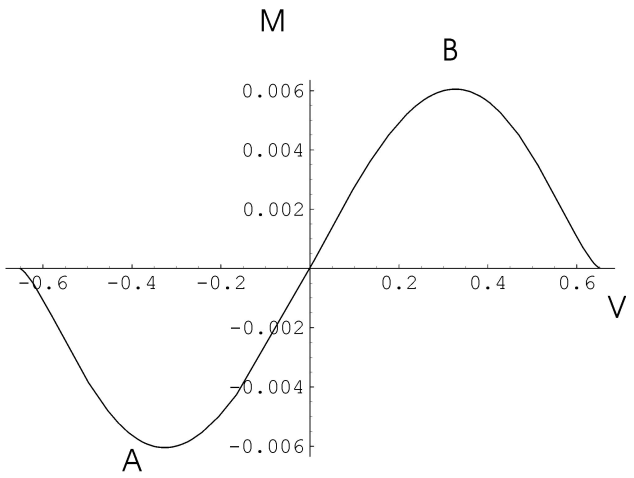

FIGURE 2. The segment AB of the curve for the renormalized momentum is the region where the bubbles are stable. Outside of this region, the soliton-like solution could destroy forming radiation for example.

Analyzing the results, we obtain the functional dependence of M_r on the speed V of dark soliton

$$M_r(V) = \frac{1}{8}\sqrt{6b}Va^4\left(6ab - V^2\right)^{\frac{3}{2}}.$$

From what is above, it is not hard to check that gray or dark solitons are stable in the region for the speed values AB of the curve presented in Fig. 2. To the lef and to the right of these points, gray solitons decompose into other structures. Indeed, according to the analysis of paper [9], the soliton, after reaching the critical velocity, destroys itself by emitting radiation and finally decaying.

The velocity that corresponds to the critical points of maxima and minima is determined by the equation

$$\frac{\partial M_r}{\partial v}|_{v=v_{cr}} = 0, \tag{17}$$

which gives the following results for the maximum and minimum of the curve AB:

$$V_{cr}^2 = \frac{2A+1}{3}\left(A-1\right).$$

This critical velocity is half the velocity of sound:

$$v_{cr} = \frac{1}{2}v_s \tag{18}$$

which is obtained from the dispersion relation (2) by using

$$v_s^2 = \lim_{k\to 0}\frac{\omega^2}{k^2} = \frac{4}{3}\left(A-1\right)\left(2A+1\right).$$

As it well established, the soliton amplitude $\lambda(A)$ is given by

$$\lambda(A) = \frac{\sqrt{3}}{2}\sqrt{\frac{A-1}{1+2A}}\left({v_s}^2 - V^2\right). \tag{19}$$

From equation (19), we can infer that when the soliton´s velocity is near the velocity of sound, its amplitude dissapears and the soliton ceases to exist. The existence of gray solitons then is closely related to the value of the main parameter A > 1.

On the other hand, the width of solitons travelling with speeds close to that of sound, are wider than those travelling with small velocities. In fact the width is given by

$$\Delta = \frac{7.05}{\sqrt{v_s^2 - V^2}}. \tag{20}$$

5. Conclusions

The analysis of stability implies that small amplitude bubbles or gray solitons in this particular case, in order to be true solitons, have to travel with smaller velocity than the critical one, which is one half of the velocity of sound according to expression (18). So, there is a critical velocity v_{cr}; such that the gray solitons are stable at $v \leq v_{cr}$ and unstable at $v > v_{cr}$: While the velocity is approaching the critical value, the width (20) is growing while the amplitude is decreasing. In contrast, when the depth approaches the vacuum state, its width narrows. We observe that only the gray solitons that are slow in comparison to sound, can be considered stable solitons. Obviously, solitons whose velocities surpass the critical value of one half of the velocity of sound, are all unstable according to the Eq. (18). Further, in order to analyze the radiation of dark solitons when they are passing the critical barrier in their velocity we could use the important results obtained in Ref. 9.

Let us now make some asseverations about the situation when we have two solitons with two different velocities. The gray soliton which moves with a greater velocity is shallower than the slower one.

According to the general formula for their analytical expressions, in order to have two stable bubble solitons, the velocity of these structures should satisfy the following relation:

$$v_i^2 + v_j^2 - \frac{1}{2}\delta_i\delta_j v_i v_j - \frac{3}{2} > 0 \text{ or} \tag{21}$$

$$v_i^2 + v_j^2 - \frac{1}{2}\delta_i\delta_j v_i v_j - \frac{3}{2} < 0$$

The signs of $\delta_1\delta_2$ determine how the interacting solitons move.

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