

A survey of embedded solitons

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At the end of the nineties a brand-new type of soliton was discovered: *the embedded solitons*. Initially they were found in optical systems, and afterwards they were also found in hydrodynamic models, liquid crystal theory and discrete systems. These peculiar solitary waves are interesting because they exist under conditions in which, until recently, the propagation of solitons was thought to be impossible. At first these nonlinear waves were believed to be necessarily isolated and unstable, but later on it was found that they can be stable and may exist in families. This paper explains what these *embedded solitons* are, in which models they have been found, and what variants exist (stable, unstable, continuous, discrete, etc.).

Keywords: Embedded solitons; solitary waves; nonlinear waves; liquid crystals; discrete systems.

Al final de los noventa se descubrió un nuevo tipo de solitones: *los solitones embebidos*. Inicialmente estas peculiares ondas se encontraron en sistemas ópticos, y posteriormente también se hallaron en modelos hidrodinámicos, en la teoría de cristales líquidos, y en sistemas discretos. Estas ondas solitarias son de interés porque existen en condiciones bajo las cuales, hasta hace poco, se consideraba que era imposible la propagación de solitones. En un principio se creyó que estas ondas no lineales forzosamente eran soluciones aisladas e inestables, pero más tarde se encontró que pueden ser estables y existir en familias. En este artículo se explica qué son estos *solitones embebidos*, en qué modelos han sido hallados, y qué variantes existen (estables, inestables, continuos, discretos, etc.).

Descriptores: Solitones embebidos; ondas solitarias; ondas no lineales; cristales líquidos; sistemas discretos.

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1. Introduction

Solitons are solitary waves which are able to propagate in nonlinear media, *i.e.* in systems which are described by nonlinear equations [*partial differential equations* (PDEs), in the case of continuous systems, and *differential-difference equations* (DDEs), in the case of discrete systems]. The solitons were discovered by Zabusky and Kruskal in 1965 [1], while they were studying the Korteweg-de Vries (KdV) equation, and the history of this discovery can be found in several texts (see, for example, the book of Newell [2]). In the beginning (in the sixties and early seventies), the term *soliton* was only applied to very stable waves, which recover their initial shapes and velocities after interacting with similar waves, and whose behavior was governed by nonlinear PDEs integrable by inverse scattering [3]. However, later on the meaning of this term (soliton) became less restrictive, and now it is usually applied to any solitary wave capable of propagating in a nonlinear system.

There are several categories of solitons: bright, dark, topological, non-topological, Bragg solitons, vector and vortex solitons, spatiotemporal solitons (optical bullets), lattice solitons, etc.. Most of these categories have been known since the seventies. However, a brand-new category was discovered very recently at the end of the nineties: *the embedded solitons*. The discovery of these solitary waves was a surprise because they exist under conditions in which, prior to 1997, the propagation of solitons was thought to be impossible. In the present survey we shall see what these *embedded solitons* are, in which models they appear, and the different types of embedded solitons known to date (*i.e.*, stable, unstable, continuous, discrete, etc.).

This paper is structured as follows: Sec. 2 will explain the relationship between standard solitons (*i.e.*, not embedded) and the small amplitude periodic waves which are able to propagate in a nonlinear system. Section 3 will present the first systems in which isolated embedded solitons were found. Section 4 introduces the few systems which have continuous families of embedded solitons, and explains what *double embedded solitons* are. In Sec. 5, it will be shown that embedded solitons also exist in discrete systems (*i.e.*, *embedded lattice solitons* exist). Finally, Sec. 6 contains some closing remarks.

2. Standard solitons and linear waves

In any nonlinear system in which the propagation of solitons is possible, the propagation of small-amplitude periodic waves which satisfy the linearized version of the nonlinear equations which control the system is also possible. However, for a soliton to exist, it is absolutely necessary that no resonance occur between the soliton and these linear waves. Otherwise, energy will be transferred from the soliton to the linear waves due to a resonant process, and the soliton will weaken continuously. As we shall see next, this no-resonance condition has different forms, depending on the *real* or *complex* nature of the soliton.

Let us examine the *real* case first. As an example of real solitons we consider the solitons of the KdV equation [4]:

$$\frac{\partial w}{\partial t} + 6w \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0. \quad (1)$$

In this case the solitons have the following form [5]:

$$w(x, t) = 2\kappa^2 \operatorname{sech}^2 \kappa (x - 4\kappa^2 t), \quad (2)$$

where κ is an arbitrary real constant. We can see that the velocity, $4\kappa^2$, of these solitons is always positive. On the other hand, if we substitute the function:

$$w(x, t) = \sin(kx - \omega t) \tag{3}$$

in the linear part of the KdV equation, we will find that these linear waves must obey the dispersion relation:

$$\omega(k) = -k^3, \tag{4}$$

thus implying that the phase velocity of these waves is always negative:

$$\frac{\omega}{k} = -k^2. \tag{5}$$

Consequently, in this case the solitons travel to the right, while the linear waves travel to the left. Due to this fact, the KdV solitons do not resonate with the linear waves.

Now let us consider the *complex* solitons. In this case the most famous equation with complex solitons is the nonlinear Schrödinger (NLS) equation [6-9]:

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0. \tag{6}$$

This equation occurs in many fields, but it is particularly important in telecommunications, as it describes the propagation of light pulses along optical fibers [6]. In this case $u(z, t)$ is a complex function, z and t are real variables, and the fundamental solitons of this equation have the form [7]:

$$u(z, t) = A \operatorname{sech}(At) \exp\left(i \frac{A^2}{2} z\right), \tag{7}$$

where A is an arbitrary real constant. This expression shows that the NLS solitons have an oscillatory component whose wavenumber, $A^2/2$, is always positive. Concerning the linear waves, they are also complex in this case, and if we substitute

$$u(x, t) = \exp[i(kz - \omega t)] \tag{8}$$

in the linear part of the NLS equation, it follows that these waves must satisfy the linear dispersion relation:

$$k(\omega) = -\frac{1}{2}\omega^2. \tag{9}$$

This expression shows that all the linear waves which are able to propagate in an NLS system have *negative* wavenumbers, contrary to the solitons whose wavenumbers are *positive*. Due to this difference, the NLS solitons do not resonate with linear waves.

The results mentioned above led to the conjecture that *real solitons* cannot have *velocities* which are contained in the range of velocities allowed to linear waves, and *complex solitons* cannot have *wavenumbers* which are permitted to these waves. Confidence in this conjecture increased when it was found that any soliton-like initial condition which evolves according to the equation [10-12]

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} - i\varepsilon_3 \frac{\partial^3 u}{\partial t^3} + \gamma |u|^2 u = 0, \tag{10}$$

inevitably begins to radiate, and the frequency of the emitted radiation is defined by the equation [13]

$$\frac{A^2}{2} = -\frac{1}{2}\omega^2 + \varepsilon_3 \omega^3, \tag{11}$$

where A is the amplitude of the initial wave. Since $A^2/2$ is precisely the wavenumber corresponding to a soliton of amplitude A , Eq. (11) is exactly the resonance condition between the soliton-like initial condition and the linear waves which satisfy the linear part of Eq. (10).

In a similar way, it was also found that any localized initial condition (similar to a soliton) which evolves according to the equation [14-17]

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + \varepsilon_4 \frac{\partial^4 u}{\partial t^4} + \gamma |u|^2 u = 0, \tag{12}$$

necessarily begins to radiate as a consequence of a resonance with linear waves. The frequency of the radiation is defined in this case by the equation [13]:

$$\frac{A^2}{2} = -\frac{1}{2}\omega^2 + \varepsilon_4 \omega^4, \tag{13}$$

which is the resonance condition between the soliton-like initial condition (of amplitude A), and the small-amplitude linear waves which can propagate in a system governed by Eq. (12).

The two above mentioned results, concerning Eqs. (10) and (12), strengthened the assumption that a complex soliton cannot have a wavenumber which is contained in the range of the (inverse) dispersion relation $k(\omega)$, and this conjecture prevailed until 1997.

3. Isolated embedded solitons

During the eighties and nineties, several variants of the NLS equation were thoroughly studied. Among them were equations which describe the propagation of ultrashort (sub-picosecond) pulses in optical fibers [10,13-17], and equations which describe the propagation of intense optical pulses under conditions where the Kerr-type dependence of the refractive index on the intensity of light is no longer accurate enough [18-28]. In the first case (ultrashort pulses) it is necessary to consider extensions of the NLS equation which include higher-order derivatives, as in (10) and (12). On the other hand, in the case of intense pulses (under non-Kerr conditions), there are extensions of the NLS equation which contain saturable nonlinearities of the form [18-22]:

$$\frac{|u|^2}{1 + \gamma |u|^2}, \tag{14}$$

and equations which consider polynomial nonlinearities of the form [23-29]:

$$\gamma_1 |u|^2 - \gamma_2 |u|^4. \tag{15}$$

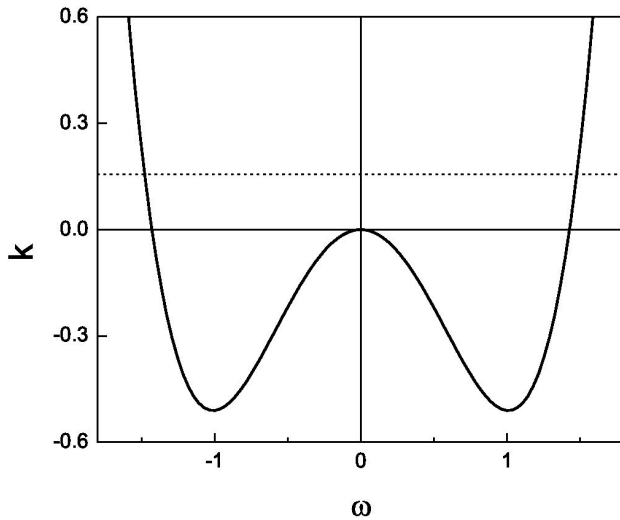


FIGURE 1. Graph of the dispersion relation (21) for $\varepsilon_2 = 1$ and $\varepsilon_4 = 24/49$. The horizontal dotted line indicates the value of the soliton's wavenumber k_{sol} given by (20).

To describe the propagation of very short, intense pulses in non-Kerr materials, the following equation has been studied [30,31]:

$$i \frac{\partial u}{\partial z} + \varepsilon_2 \frac{\partial^2 u}{\partial t^2} + \varepsilon_4 \frac{\partial^4 u}{\partial t^4} + \gamma_1 |u|^2 u - \gamma_2 |u|^4 u = 0. \quad (16)$$

This equation, however, only applies when the frequency of the carrier wave is close to the frequency at which the third-order dispersion vanishes. If this restriction is removed, the following equation should be used instead:

$$i \frac{\partial u}{\partial z} + \varepsilon_2 \frac{\partial^2 u}{\partial t^2} - i \varepsilon_3 \frac{\partial^3 u}{\partial t^3} + \varepsilon_4 \frac{\partial^4 u}{\partial t^4} + \gamma_1 |u|^2 u - \gamma_2 |u|^4 u = 0. \quad (17)$$

In 1997 it was found that equation (16) has soliton-like solutions of the following form [30]:

$$u(z, t) = \left(\frac{\gamma_1}{2\gamma_2} \right)^{\frac{1}{2}} \operatorname{sech} \left(\frac{t}{\sqrt{14\varepsilon_4/\varepsilon_2}} \right) \exp \left(i \frac{5}{32} \frac{\gamma_1^2}{\gamma_2} z \right), \quad (18)$$

if the coefficients ε_i and γ_i satisfy the condition:

$$\frac{\varepsilon_4}{\varepsilon_2^2} = \frac{24}{49} \frac{\gamma_2}{\gamma_1^2}. \quad (19)$$

Note that the soliton (18) has the following wavenumber:

$$k_{sol} = \frac{5}{32} \frac{\gamma_1^2}{\gamma_2} = \frac{15}{196} \frac{\varepsilon_2^2}{\varepsilon_4}. \quad (20)$$

On the other hand, the small-amplitude periodic waves (linear waves) which satisfy the linear part of Eq. (16) must obey the dispersion relation:

$$k(\omega) = \varepsilon_4 \omega^4 - \varepsilon_2 \omega^2. \quad (21)$$

If $\varepsilon_4 > 0$, the range of this function contains all the positive values of k , and consequently the soliton's wavenumber (which is positive if $\varepsilon_4 > 0$) is contained in the range

of wavenumbers permitted for linear waves, as can be seen in Fig. 1. With this discovery, the widely accepted conjecture stating that a soliton could not have a wavenumber contained in the linear spectrum of the system was shown to be false. In the next few years, other systems with this unusual type of soliton were discovered and will be presented below.

In 1998 Champneys, Malomed and Friedman studied the system of NLS-like equations [32]:

$$i \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial t} + D \frac{\partial^2 u}{\partial t^2} + (\sigma |u|^2 + |v|^2) u + v = 0, \quad (22)$$

$$i \frac{\partial v}{\partial x} - i \frac{\partial v}{\partial t} + D \frac{\partial^2 v}{\partial t^2} + (|u|^2 + \sigma |v|^2) v + u = 0, \quad (23)$$

which are useful in describing two different systems:

- (i) the propagation of two pulses of circularly polarized light travelling along a birefringent twisted fiber, and
- (ii) the propagation of two optical pulses travelling in opposite directions along a nonlinear optical fiber with a grating (in this case we must switch x and t).

The linear analysis carried out by Champneys *et al.* suggested the existence of isolated solitary waves, and their numerical simulations confirmed the existence of these waves. When this paper was published, it was not realized that these solitary waves might be similar to the soliton solutions for Eq. (16). However, if the linear coupling terms appearing in (22)-(23) are disregarded, and the periodic waves

$$u(x, t) = \exp[i(kx - \omega t)], \quad (24)$$

$$v(x, t) = \exp[i(-kx - \omega t)], \quad (25)$$

are substituted in the linear parts of (22)-(23), it is easily found that these waves must satisfy, respectively, the linear dispersion relations:

$$k(\omega) = \omega - D\omega^2, \quad (26)$$

$$k(\omega) = \omega + D\omega^2. \quad (27)$$

Since the ranges of these functions include positive and negative wavenumbers, it could be suspected that the isolated solitary waves found numerically in [32] might have wavenumbers immersed in the linear spectrums of the system. Later, on Champneys and Malomed found that this was precisely the case [33], and so the numerical solutions of the system (22)-(23) found in [32] were shown to be similar to the soliton solutions for Eq. (16).

A new interesting system with explicit soliton-like solutions was found by Yang, Malomed and Kaup (YMK) in 1999. YMK studied the propagation of optical pulses in a nonlinear medium (a Kerr medium), taking into account the interplay of the fundamental (FH) and the second-harmonic (SH) fields. Such a system is described by the equations [34,35]:

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + \gamma_1 (|u|^2 + 2|v|^2) u + u^* v = 0, \quad (28)$$

$$i \frac{\partial v}{\partial z} - \frac{\delta}{2} \frac{\partial^2 v}{\partial t^2} + 2\gamma_2 (|v|^2 + 2|u|^2) v + \frac{1}{2} u^2 + qv = 0, \quad (29)$$

where u is the FH, v is the SH, z is the propagation distance, t is the reduced time, γ_1 and γ_2 are the Kerr coefficients (which are different because the FH and SH fields have different frequencies), q is the group-velocity mismatch (originated by the frequency difference of the FH and SH fields), and δ is the relative dispersion of the SH.

In the particular case when SH is much weaker than FH, the system (28)-(29) reduces to:

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + \gamma_1 |u|^2 u + u^* v = 0, \quad (30)$$

$$i \frac{\partial v}{\partial z} - \frac{\delta}{2} \frac{\partial^2 v}{\partial t^2} + 4\gamma_2 |u|^2 v + \frac{1}{2} u^2 + \kappa v = 0. \quad (31)$$

This system has the soliton solutions:

$$u(z, t) = A e^{ikz} \operatorname{sech}(\sqrt{2kt}), \quad (32)$$

$$v(z, t) = B e^{2ikz} \operatorname{sech}^2(\sqrt{2kt}), \quad (33)$$

where

$$k = \frac{1}{2(1 + 2\delta)} \left[q - \frac{3\delta}{2(4\gamma_2 + 3\delta\gamma_1)} \right], \quad (34)$$

$$A = -3\delta k / 2\gamma_2, \quad (35)$$

$$B = 2k(1 + 3\delta\gamma_1 / 4\gamma_2). \quad (36)$$

If we now substitute the periodic waves

$$u(z, t) = \exp[i(kz - \omega t)], \quad (37)$$

$$v(z, t) = \exp[2i(kz - \omega t)], \quad (38)$$

in the linear parts of (30) and (31), we find the following linear dispersion relations corresponding, respectively, to $u(z, t)$ and $v(z, t)$:

$$k = -\frac{1}{2}\omega^2, \quad (39)$$

$$2k = 2\delta\omega^2 + q. \quad (40)$$

The range of the linear dispersion relation (39) contains only negative wavenumbers, and since the wavenumber k of the soliton (32) is necessarily positive (because of the root $\sqrt{2k}$), the u -soliton is a normal soliton (with a wavenumber lying outside the linear spectrum). Something different occurs in the case of the v -soliton. In this case, the range of the linear dispersion relation (40) contains *positive* wavenumbers if $\delta > 0$, and consequently the wavenumber of the v -soliton (which is $2k$) might be contained within the linear spectrum. If $q > 0$, this will occur for $2k > q$, and if $q < 0$ this will happen for $2k > 0$. These two inequalities imply that the wavenumber of the v -soliton will be contained within the linear spectrum if δ is positive, γ_1 and γ_2 are negative, and q is contained in the interval:

$$\frac{3\delta}{2(4\gamma_2 + 3\delta\gamma_1)} < q < -\frac{3}{4(4\gamma_2 + 3\delta\gamma_1)}. \quad (41)$$

Consequently, there are four features that make the system (30)-(31) a very interesting one:

- (i) it describes a physically relevant system (a second-harmonic generating system),
- (ii) it has *explicit* analytical soliton solutions,
- (iii) the profiles of the FH and SH solitons are different, the FH-soliton is given by a hyperbolic secant (which is usual in optical solitons), but the profile of the SH-soliton is a *squared* hyperbolic secant (which is usual in hydrodynamics, but unusual in optics), and
- (iv) if $\delta > 0$, $\gamma_1, \gamma_2 > 0$, and (41) is satisfied, the wavenumber of the v -soliton is contained in the linear spectrum of the system.

Yang, Malomed and Kaup recognized that the v -solitons were particularly interesting, and decided to baptize these peculiar waves with a new name: *embedded solitons*, thus indicating that the wavenumber of these solitons is “embedded” in the linear spectrum of the system.

In the same year (1999), Champneys and Malomed extended the model (22)-(23), including second-derivative (wave) terms, as shown below [33]:

$$i \frac{\partial u}{\partial t} + i \frac{\partial u}{\partial x} + (2k)^{-1} \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} \right) + \left(\sigma |u|^2 + |v|^2 \right) u + v = 0, \quad (42)$$

$$i \frac{\partial v}{\partial t} - i \frac{\partial v}{\partial x} + (2k)^{-1} \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} \right) + \left(\sigma |v|^2 + |u|^2 \right) v + u = 0. \quad (43)$$

Proceeding as in Ref. 32, Champneys and Malomed found that this system has *moving* embedded solitons, and they found that these solitons are isolated and unstable. No explicit analytical expressions for these embedded solitons are known in this case.

By the year 2000, it was clear that embedded solitons may exist in different optical systems. However, it was not clear why these solitons can exist, because at first sight it seemed that these solitary waves should resonate with linear waves, losing energy in the process. Qualitative explanations for the existence of embedded solitons appeared in 2001 [36], and a quantitative explanation was presented in 2003 [37], in a paper dealing with Eq. (17). In Ref. 37, it was proved that Eq. (17) has bright and dark soliton solutions, and the bright solitons have the form:

$$u(z, t) = A \operatorname{sech} \left(\frac{t - az}{w} \right) \exp [i(qz + rt)], \quad (44)$$

where the soliton parameters are defined by the equations:

$$A = \left(\frac{6}{5\gamma_2} \right)^{\frac{1}{2}} \left[\gamma_1 - \left(2\varepsilon_2 + \frac{3\varepsilon_3}{4\varepsilon_4} \right) \left(\frac{\gamma_2}{24\varepsilon_4} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad (45)$$

$$w = \left(\frac{24 \varepsilon_4}{\gamma_2} \right)^{1/4} \frac{1}{A}, \quad (46)$$

$$r = \frac{\varepsilon_3}{4 \varepsilon_4}, \quad (47)$$

$$a = 2 \varepsilon_2 r + 8 \varepsilon_4 r^3, \quad (48)$$

$$q = -\varepsilon_2 r^2 - 3 \varepsilon_4 r^4 + (\varepsilon_2 + 6 \varepsilon_4 r^2) \left(\frac{\gamma_2}{24 \varepsilon_4} \right)^{1/2} \times A^2 + \frac{\gamma_2}{24} A^4. \quad (49)$$

In Ref. 37, it was shown that these solitons are embedded if ε_i and γ_i are positive, and satisfy the condition $A > 0$. To explain why these solitons do not resonate with the linear waves (in spite of having a wavenumber contained in the linear spectrum of the system), it is necessary to calculate the double Fourier transform (in space and time) of Eq. (17). To calculate the Fourier transforms of the nonlinear terms, $u(z, t)$ is considered to have the form given in (44), without imposing any conditions on the values of the parameters A , a , w , q and r . In this way it is found that the double Fourier transform of $u(z, t)$ has the following form:

$$\tilde{u}(k, \omega) = \frac{\pi w A \left[\operatorname{sech} \left(\frac{\pi w (r + \omega)}{2} \right) \right] \sum_{n=0}^4 (c_n \omega^n) \delta(k - q - a(r + \omega))}{-(q + ar) - a\omega - \varepsilon_2 \omega^2 + \varepsilon_3 \omega^3 + \varepsilon_4 \omega^4}, \quad (50)$$

where the coefficients c_n are functions of γ_1 , γ_2 , A , w and r . Expressions such as this usually imply that resonances occur at the values of the frequencies where the denominator becomes zero, since at these frequencies the Fourier transform diverges. In the present case, the denominator becomes zero when ω is any of the roots of the equation:

$$q + ar = \varepsilon_4 \omega^4 + \varepsilon_3 \omega^3 - \varepsilon_2 \omega^2 - a\omega, \quad (51)$$

which is precisely the resonance condition between the soliton (whose intrinsic wavenumber is $q + ar$) and the linear waves. The mystery with embedded solitons is that *this resonance does not occur*. And the solution of the mystery lies in the expression (50). We can see on the r.h.s. of this equation that both the numerator and the denominator, contain fourth-order polynomials in ω . Consequently, if we could make these two polynomials coincide, they would cancel each other out, and the denominator in Eq. (50) would disappear, thus eliminating the divergence of $\tilde{u}(k, \omega)$, and eliminating the resonance between the soliton and the linear waves.

To find out if the two polynomials occurring in (50) can indeed cancel each other out, we can equate the coefficients of similar powers of ω appearing in these polynomials. In this way we obtain a system of five simultaneous equations for the parameters A , w , a , q , and r , whose solution (as the reader may have already guessed) is obtained when these parameters take the values defined by the Eqs. (45)-(49). Therefore, the values of the parameters A , w , a , q , and r presented in (45)-(49) are precisely the values required to cancel the two polynomials occurring in (50). When these polynomials vanish, the resonances also disappear. That is why the embedded solitons of Eq. (17) do not resonate with linear waves.

It is worth observing that the coefficients c_n of the fourth-order polynomial which appears in the numerator of Eq. (50) depend on the nonlinear coefficients γ_1 and γ_2 (and also on the pulse parameters A , w , and r), but they do not depend

explicitly on the dispersive coefficients ε_2 , ε_3 and ε_4 . Hence, the existence of this polynomial is a consequence of the nonlinear terms which appear in Eq. (17). On the contrary, the polynomial in the denominator of (50) depends on the dispersive coefficients (and the pulse parameters q , a and r), but it does not depend explicitly on the nonlinear coefficients. Hence, the existence of this polynomial is a consequence of the dispersive terms of Eq. (17). Therefore, the mutual cancellation of these two polynomials is the result of a delicate balance between nonlinearity and dispersion.

In the systems mentioned in the preceding paragraphs the embedded solitons were found to be *isolated solutions*. That is, in each case the embedded soliton has a unique amplitude, a unique width, and a unique wavenumber. The values of these parameters are determined by the coefficients of the equation considered. Moreover, these solitons are *semi-stable* solutions. In other words, when one of these solitons is perturbed, and the perturbation increases its energy, the perturbed soliton tends to stabilize, approaching the exact soliton configuration. On the other hand, when the perturbation diminishes the soliton's energy, it starts resonating with the linear waves, and loses energy in the process. In this case the perturbed soliton does not attain a new equilibrium state. For some time it was believed that all embedded solitons should (necessarily) be isolated and semi-stable. However, as we shall see in the next section, the embedded solitons can exist in families, and in this case they can be stable solutions.

4. Continuous families of embedded solitons

The first example which showed that embedded solitons can exist in *continuous families* was found by Champneys and Malomed in 1999 [38], while studying the interaction of three spatial solitons propagating in a planar waveguide (with a quadratic nonlinearity), which has a set of parallel scores

which act as a Bragg grating. The process considered was the following: two light pulses (with carrier waves of equal frequencies) are introduced in the planar waveguide, making opposite angles with the Z axis (which is the direction of the parallel scores). Due to the nonlinearity, a second-harmonic wave is generated, which propagates in the direction of the parallel scores (*i.e.* along the Z direction) [39]. A diagram of the system can be seen in Fig. 2. The evolution of the three waves is then described by the following equations:

$$i \frac{\partial u}{\partial z} + i \frac{\partial u}{\partial x} + v + uv^* = 0, \quad (52)$$

$$i \frac{\partial v}{\partial z} - i \frac{\partial v}{\partial x} + u + uv^* = 0, \quad (53)$$

$$2i \frac{\partial w}{\partial z} + D \frac{\partial^2 w}{\partial x^2} - qw + uv = 0, \quad (54)$$

where u and v are the two components of the fundamental harmonic (FH), w is the second-harmonic (SH) wave, q is a phase mismatch, and D is an effective diffraction coefficient. To find out if this system may have soliton solutions, Champneys and Malomed analyzed the linearized problem, proposing solutions which asymptotically (at the tails of the solitons) behave in the form:

$$u(z, x) \sim \exp(ikz) \exp(\lambda\xi), \quad (55)$$

$$v(z, x) \sim \exp(ikz) \exp(\lambda\xi), \quad (56)$$

$$w(z, x) \sim \exp(2ikz) \exp(2\lambda\xi), \quad (57)$$

where $\xi = x - cz$. Substituting these expressions in the linearized equations, and calculating the eigenvalues, it was found that the linear analysis indicates that embedded solitons may exist, isolated and in families (depending on the value of q), and the existence of some of these embedded solitons was corroborated numerically.

Two years later (in 2001), an *explicit* family of embedded solitons was found by Yang while studying a complicated extension of the Korteweg-de Vries equation [40]:

$$u_t + 6uu_x + u_{xxx} + u_{xxxx} + 10uu_{xx} + 20u_x u_{xx} + 30u^2 u_x = \epsilon F(u), \quad (58)$$

where:

$$F(u) = - (auu_{xxx} + bu_x u_{xx} + cu^2 u_x) \quad (59)$$

is a perturbative term ($\epsilon \ll 1$), and a, b, c are real constants. In this case, however, the *embedding* is different, because now the solitons are real. In this case, the quantity that is embedded in the linear spectrum of the system is not a *wavenumber* (or a *frequency*), but the *velocity* of the solitons. The form of the solitons of Eq. (58) is:

$$u(x, t) = \frac{1}{2} k^2 \operatorname{sech}^2 \left[\frac{k}{2} (x - Ct) \right], \quad (60)$$

where $C = k^2 + k^4$ is the velocity of the soliton, and $k > 0$ is an arbitrary parameter. In Ref. 40, Yang proved that all the

solitons of this family are embedded if the coefficients occurring in (59) satisfy the condition $(a, b, c) \propto (1, -1, 3)$. This was the first continuous family of *real* and *explicit* embedded solitons reported in the literature.

In the complex case, *explicit continuous families* of embedded solitons also exist. An example of such a family was found in 2003, in a model which describes the propagation of light in liquid crystals. In a first approximation, the propagation of a light pulse in a liquid-crystal waveguide is similar to the propagation of light in a solid optical fiber. The pulse has a tendency to disperse because different frequencies travel at different velocities, but this dispersion effect can be balanced by the nonlinear dependence of the refractive index on light intensity. Because of this, in a first approximation the NLS equation is useful in describing the behavior of a light pulse in a liquid-crystal waveguide. In liquid crystals, however, the nonlinear effect is much stronger than in a solid optical fiber, and therefore a correction is needed. In Ref. 41, it was shown that the NLS approximation can be improved by using the following equation:

$$\frac{\partial u}{\partial z} - \epsilon \frac{\partial^3 u}{\partial t^3} - \gamma |u|^2 \frac{\partial u}{\partial t} = 0. \quad (61)$$

Since this equation reduces to the modified Korteweg-de Vries (mKdV) equation when u is real, it is evident that *real* solitons exist. A not so evident result is that Eq. (61) also has a *continuous family* of *complex* solitons of the form:

$$u(z, t) = A \operatorname{sech} \left(\frac{t - az}{w} \right) \exp [i(qz + rt)], \quad (62)$$

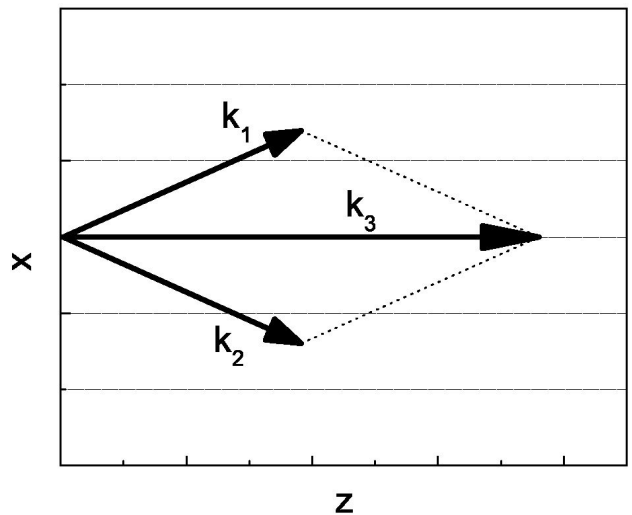


FIGURE 2. Diagram of the system described by Eqs. (52)-(54). \mathbf{k}_1 and \mathbf{k}_2 are the wavenumbers corresponding to the two FH (fundamental harmonic) waves, and \mathbf{k}_3 is the wavenumber of the SH (second harmonic) wave \mathbf{k}_3 . The horizontal lines indicate the direction of the parallel scores which form the Bragg grating.

where the parameters A , a , w , q y r must satisfy the following three conditions:

$$A^2 w^2 = \frac{6\varepsilon}{\gamma}, \quad (63)$$

$$a = 3\varepsilon r^2 - \frac{1}{6}\gamma A^2, \quad (64)$$

$$q = \frac{1}{2}\gamma A^2 r - \varepsilon r^3. \quad (65)$$

Due to the presence of the third derivative in (61), the range of the dispersion relation $k(\omega)$ contains the entire real axis, and so the intrinsic wavenumber ($q + ar$) of any soliton of the family (62) is necessarily immersed in the linear spectrum. Consequently, *all the solitons of this family are embedded solitons* (according to their wavenumbers). Moreover, if $a\varepsilon > 0$, the *velocity* for the soliton is contained in the range of velocities permitted to linear waves. Therefore, in this case (when $a\varepsilon > 0$), both the *wavenumber* and the *velocity* of the soliton are contained in the corresponding linear spectrums. These are the *double embedded solitons*. When $a\varepsilon < 0$ only the soliton's wavenumber is contained in the linear spectrum, and we speak in this case of *single embedded solitons*.

The numerical results presented in [41] showed that the solitons of Eq. (61) are *stable solutions*. This is an interesting result, since it is the only example known to date of *stable embedded solitons* with an *explicit* analytical expression.

The following extension of Eq. (61):

$$\frac{\partial u}{\partial z} + \frac{\partial^3 u}{\partial t^3} + 6|u|^2 \frac{\partial u}{\partial t} = i\alpha |u|^2 u - \gamma \frac{\partial |u|^2}{\partial t} u \quad (66)$$

was also studied by Yang [42], who found that this equation also has a *family of stable embedded solitons*. However, in this case no analytical solutions were found.

5. Embedded lattice solitons

By 2003 it was known that the embedded solitons can appear in very different contexts, such as liquid crystals, hydrodynamic models, and several types of optical systems. However, all these systems share at least one common characteristic: they are *continuous* systems. When this common denominator is recognized, a question immediately arises: *can embedded solitons exist in discrete systems?* In other words: *do embedded lattice solitons exist?* The answer to this question was found recently [43], when it was shown that explicit embedded lattice solitons are solutions of a discrete version of Eq. (16) of the following form:

$$i \frac{\partial r_n}{\partial t} + \varepsilon_2 \Delta_2 r_n + \varepsilon_4 \Delta_4 r_n + \frac{1}{2} \gamma_1 |r_n|^2 (r_{n+1} + r_{n-1}) - \frac{2}{3} \gamma_2 |r_n|^4 [r_{n+2} + 4\alpha (r_{n+1} + r_{n-1}) + r_{n-2}] = 0,$$

where $r_n(t)$ is a complex-valued function of time defined at the lattice sites, the coefficients ε_2 , ε_4 , γ_1 , γ_2 and α are real,

and Δ_2 and Δ_4 are the finite-difference operators defined as follows:

$$\Delta_2 r_n \equiv \frac{r_{n+1} - 2r_n + r_{n-1}}{(\Delta x)^2}, \quad (67)$$

$$\Delta_4 r_n \equiv \frac{r_{n+2} - 4r_{n+1} + 6r_n - 4r_{n-1} + r_{n-2}}{(\Delta x)^4}, \quad (68)$$

where Δx is the lattice spacing. In Ref. 43 it was shown that Eq. (67) has bright and dark lattice solitons, if the coefficients of the equation satisfy certain conditions. In particular, the bright solitons have the form

$$r_n = A e^{-iCt} \operatorname{sech}(Bn\Delta x), \quad (69)$$

where the constants A , B and C are defined by the following set of algebraic equations:

$$A^2 = \sqrt{\frac{3\varepsilon_4 \sinh^2(2B\Delta x)}{2\gamma_2 (\Delta x)^2}}, \quad (70)$$

$$\begin{aligned} & \cosh^6(B\Delta x) - \left(\frac{3\gamma_1^2 (\Delta x)^4}{32\gamma_2 \varepsilon_4} + 1 \right) \cosh^4(B\Delta x) \\ & + \left(\frac{\varepsilon_2 (\Delta x)^2}{4\varepsilon_4} - 1 \right) \cosh^3(B\Delta x) \\ & + \frac{1}{4} \cosh^2(B\Delta x) - \frac{1}{2} \left(\frac{\varepsilon_2 (\Delta x)^2}{4\varepsilon_4} - 1 \right) \cosh(B\Delta x) \\ & + \frac{1}{4} \left(\frac{\varepsilon_2}{4\varepsilon_4} (\Delta x)^2 - 1 \right)^2 = 0, \end{aligned} \quad (71)$$

$$\begin{aligned} C &= -\frac{2\varepsilon_2}{(\Delta x)^2} [\cosh(B\Delta x) - 1] \\ & - \frac{4\varepsilon_4}{(\Delta x)^4} [\cosh(B\Delta x) - 1]^2, \end{aligned} \quad (72)$$

and the coefficient α must satisfy the following condition:

$$\alpha = \frac{1 - 2 \cosh^2(B\Delta x)}{16 \cosh^3(B\Delta x)}. \quad (73)$$

To find out if the soliton (69) is an embedded lattice soliton, it is necessary to determine whether its internal frequency C is contained within the range of frequencies permitted for linear waves (*i.e.* within the range of the dispersion relation). In this case we pay attention to the *frequencies*, instead of the *wavenumbers*, because in Eq. (67) the evolution variable is the *time*, whereas in equations such as (16) the evolution variable was the propagation distance along the nonlinear medium. Reference 43 shows that C falls within the range of the linear dispersion relation if the following inequalities are satisfied:

$$1 - \cosh(B\Delta x) < \frac{\varepsilon_2 (\Delta x)^2}{2\varepsilon_4} < 3 - \cosh(B\Delta x). \quad (74)$$

Depending on the values of the coefficients ε_n , γ_n and α , these two inequalities may or may not be fulfilled. When they are satisfied, the soliton (68) is an *embedded lattice soliton*. If any of these conditions is not satisfied, we will have a standard (*i.e.* not embedded) lattice soliton.

6. Conclusions

Embedded solitons (ES) are particularly interesting nonlinear waves because they exist under conditions under which, until recently, it was believed that the propagation of solitons was impossible. As we have seen in this brief survey, this field is rather new, but it has grown faster than expected. At first (1997), it was thought that the existence of these peculiar waves was a very rare and isolated phenomenon. However, soon enough these new solitons were found in several nonlinear models. Most of these models are related to the propagation of light in nonlinear media, and in every case the existence of the ES is the result of the interplay between nonlinearity and dispersion. To understand why nonlinear optics is such a fruitful source of models with ES, we must observe that in materials such as silica glass or liquid crystals the nonlinear dependence of the refractive index on light intensity leads naturally to partial differential equations with several nonlinear terms, which may balance the effect of the dispersive terms. Moreover, there are many different systems where light pulses (or light rays) may interact, and there are many parameters that can be controlled: light intensity, direction of light, number of rays, light frequency, dispersive characteristics of the systems, periodicity of the media, etc.. When the propagation of a single light pulse is considered, usually only one equation is needed. Equations (16), (17), (61) and (66) are examples of these types of models. On the other hand, when we want to describe the interaction of two (or more) pulses, a system of equations is usually needed. Systems (22)-(23), (28)-(29), (42)-(43) and (52)-(54) are examples of these models. It is worth observing that these systems involve two interesting ingredients: second-harmonic generation and resonant gratings. It remains an open question

whether new ES could be found in systems involving third-harmonic generation and/or other types of periodic media.

It is important to observe that there are different types of ES, and we can classify them in different ways. There are isolated ES and continuous families of ES. There are stable and unstable ES. There are continuous and discrete ES. Finally, we have ES described by explicit analytical expressions, and ES for which no analytical expressions are known.

The discovery of the ES has shown that the destructive resonances that frequently hinder the propagation of solitary waves in nonlinear systems, can be cancelled (in certain cases) by higher-order nonlinear terms. This idea suggests that new embedded solitons might be found in highly nonlinear systems not yet studied.

Even in the systems with embedded solitons already known, there are several aspects which merit a further study. The emission of radiation by *perturbed* embedded solitons is one of these issues. The *stability* of the embedded solitons which appear in families is another.

The rigorous mathematical analysis of the systems with embedded solitons is another field which is just beginning to grow [44]- [46], and where surely there is plenty to do.

Note added in proff: while the proofs of this article were being revised, the first example of stable embedded lattice solitons appeared in the paper B.A. Malomed *et al.*, *Chaos* **16** (2006) 013112.

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