Symmetric energy-momentum tensor in Maxwell, Yang-Mills, and Proca theories obtained using only Noether's theorem

Merced Montesinos*
Departamento de Física, Cinvestav, Av. Instituto Politécnico Nacional 2508, San Pedro Zacatenco, 07360, Gustavo A. Madero, Ciudad de México, México, e-mail: merced@fis.cinvestav.mx

Ernesto Flores
Facultad de Física e Inteligencia Artificial, Universidad Veracruzana, 91000, Xalapa, Veracruz, México

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The symmetric and gauge-invariant energy-momentum tensors for source-free Maxwell and Yang-Mills theories are obtained by means of translations in spacetime via a systematic implementation of Noether's theorem. For the source-free neutral Proca field, the same procedure yields also the symmetric energy-momentum tensor. In all cases, the key point to get the right expressions for the energy-momentum tensors is the appropriate handling of their equations of motion and the Bianchi identities. It must be stressed that these results are obtained without using Belinfante's symmetrization techniques which are usually employed to this end.

Keywords: Energy-momentum tensor; Noether's theorem; gauge field theory.

Los tensores de energía-momento invariantes de norma y simétricos para las teorías de Maxwell y Yang-Mills sin fuentes son obtenidos mediante traslaciones en el espacio-tiempo mediante una aplicación sistemática del teorema de Noether. Para el campo de Proca neutral y sin fuentes, el mismo procedimiento proporciona también el tensor de energía-momento simétrico. En todos los casos, el punto clave para obtener las expresiones correctas de los tensores de energía-momento es el manejo adecuado de las ecuaciones de movimiento y de las identidades de Bianchi. Debe ser enfatizado que estos resultados son obtenidos sin usar las técnicas de simetrización de Belinfante las cuales son usualmente empleadas para este fin.

Descriptores: Tensor de energía-momento; teorema de Noether; teoría de campo de norma.

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1. Introduction

One of the most beautiful and remarkable results in theoretical physics is that provided by Noether's theorem, which establishes a relationship between the symmetries of a given action and the conserved quantities for the dynamical system associated with this action principle [1]. However, the standard implementation of Noether's theorem to field theory leads, in the generic case, to a non-symmetric expression for the corresponding *canonical energy-momentum tensor* $\Theta^{\mu}{}_{\nu}$

$$\Theta^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^{\mu}_{\nu}\mathcal{L}\,, \tag{1}$$

where ϕ denotes the collection of independent fields involved in the Lagrangian density $\mathcal{L}$ and so in the action, and where all possible internal indices have not been explicitly written. Next, $\Theta^{\mu}{}_{\nu}$ is "improved" by Belinfante's method to obtain the *symmetric energy-momentum tensor* $T^{\mu\nu}$ [2]

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\gamma K^{\mu\nu\gamma}, \tag{2}$$

with $K^{\mu\nu\gamma} = -K^{\mu\gamma\nu}$ and so

$$\partial_\nu T^{\mu\nu} = \partial_\nu \Theta^{\mu\nu} + \partial_\nu\partial_\gamma K^{\mu\nu\gamma} = \partial_\nu \Theta^{\mu\nu}$$

(see Ref. 3 for a detailed description of Belinfante's method). A symmetric energy-momentum tensor, $T^{\mu\nu}$, is needed, for instance, when such matter fields are coupled to gravity in the context of general relativity [4]. For Maxwell theory, taking $\mathcal{L} = -(1/16\pi)F^{\mu\nu}F_{\mu\nu}$, Eq. (1) acquires the form

$$\Theta^{\mu}{}_{\nu} = -\frac{1}{4\pi}F^{\mu\alpha}\partial_\nu A_\alpha + \frac{1}{16\pi}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}\,, \tag{3}$$

while for Yang-Mills theory, taking $\mathcal{L} = -(1/16\pi)F^{\mu\nu}_{a}F^{a}_{\mu\nu}$, Eq. (1) becomes

$$\Theta^{\mu}{}_{\nu} = -\frac{1}{4\pi}F^{\mu\alpha}_{a}\partial_\nu A^{a}_{\alpha} + \frac{1}{16\pi}\delta^{\mu}_{\nu}F^{\alpha\beta}_{a}F^{a}_{\alpha\beta}\,, \tag{4}$$

which are neither symmetric nor gauge-invariant under their corresponding gauge transformations [3, 5, 6]. On the other hand, for the neutral Proca field, taking

$$\mathcal{L} = -\frac{1}{16\pi}\left(F^{\mu\nu}F_{\mu\nu} - 2m^2 A_\mu A^\mu\right),$$

Eq. (1) gives also the wrong expression [3]

$$\Theta^{\mu}{}_{\nu} = -\frac{1}{4\pi}F^{\mu\alpha}\partial_\nu A_\alpha + \frac{1}{16\pi}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{8\pi}m^2\delta^{\mu}_{\nu}A_\alpha A^\alpha\,. \tag{5}$$

As already mentioned, Eqs. (3), (4), and (5) can be "fixed" by using Belinfante's ideas [2, 3].

Let us call the combination of standard Noether's theorem and Belinfante's symmetrization techniques simply Noether-Belinfante's method. In spite of the success of Belifante's symmetrization techniques to fix the canonical energy-momentum tensor $\Theta^{\mu\nu}$ obtained by the standard Noether's theorem, it must be emphasized that Belifante's procedure is an *ad hoc* one which also has the ambiguity associated with the freedom of adding divergence terms to $\Theta^{\mu\nu}$. In spite of these properties, Belinfante's method has become to be very popular. Moreover, its permanent implementation in gauge theories to "fix" the energy-momentum tensor $\Theta^{\mu\nu}$ has created a paradigm consisting in the claim that Noether's theorem is *not* enough to determine the right form (via spacetime translations) for the energy-momentum tensor in gauge theories.

In this paper the issue of the incompleteness or correctness of Noether's theorem for gauge theories is analyzed, and our conclusion is that the paradigm is not correct. Our analysis includes Maxwell, Yang-Mills, and Proca fields, *i.e.*, Abelian, non-Abelian, and massive gauge fields are analyzed. More precisely, using spacetime translations, it is shown that the systematic implementation of Noether's theorem to the source-free Maxwell and Yang-Mills theories leads to symmetric and gauge-invariant expressions for their corresponding energy-momentum tensors, while for the Proca field the procedure yields also the symmetric and right energy-momentum tensor. In all three cases, the obtention of the right energy-momentum tensors is achieved by taking into account both the equations of motion and the Bianchi identities for the system under study. Therefore, the present results indicate that the action principle has *all* the information required to *uniquely* determine the right energy-momentum tensor and that there is *no* need to use Belifante's method because the expressions obtained in this paper, which are the right ones, follow only from a careful implementation of Noether's theorem. This is an unexpected result which goes against the already mentioned paradigm (for more details see Ref. 7).

Before beginning, some comments about our notation. Let $\mathcal{M}$ be the Minkowski spacetime where the Maxwell, Yang-Mills, and Proca fields exist and let $(x^\mu) = (ct, x, y, z)$ be Minkowskian coordinates in it; Greek indices $\mu, \nu \ldots$ take the values $0, 1, 2, 3$. The Minkowski metric is chosen to be *diagonal* $(\eta_{\mu\nu}) = (-1, +1, +1, +1)$. The symbol d^4x means $dx^0 \wedge dx \wedge dy \wedge dz$ and also $\partial_\mu = \partial/\partial x^\mu$. The detailed implementation of Noether's theorem is deliberate to stress our method.

2. Source-free Maxwell theory

The usual action principle for source-free Maxwell theory is

$$S[A_\mu] = \alpha \int_{\mathcal{R}} d^4x \; F_{\mu\nu}F^{\mu\nu}, \tag{6}$$

where $A{=}A_\mu(x)dx^\mu$, $(A_\mu){=}(\phi, \vec{A})$, is the potential 1-form and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor, and $\mathcal{R}$ is an arbitrary region of the four-dimensional Minkowski spacetime $\mathcal{M}$.

The first order variation of action (6) under the transformation of the variables $\tilde{\delta}A_\mu := \tilde{A}_\mu(x) - A_\mu(x)$ yields

$$\tilde{\delta}S = \int_{\mathcal{R}} d^4x \left[\frac{\delta S}{\delta A_\nu}\tilde{\delta}A_\nu\right] + \int_{\partial\mathcal{R}} \left(4\alpha F^{\mu\nu}\tilde{\delta}A_\nu\right) d\Sigma_\mu\,, \tag{7}$$

with

$$\frac{\delta S}{\delta A_\nu} = -(4\alpha\partial_\mu F^{\mu\nu})\,, \tag{8}$$

and so $\tilde{\delta}S = 0$ yields the equations of motion

$$\frac{\delta S}{\delta A_\nu} = -(4\alpha\partial_\mu F^{\mu\nu}) = 0\,, \tag{9}$$

provided that the boundary term in Eq. (7) vanishes.

The action (6) is fully invariant under the Poincaré group

$$\begin{aligned} S' &:= \alpha \int_{\mathcal{R}'} F'_{\mu\nu}F'^{\mu\nu}d^4x' \\ &= \alpha \int_{\mathcal{R}} F_{\mu\nu}F^{\mu\nu}d^4x \\ &= S. \end{aligned} \tag{10}$$

The word "fully" means that the symmetry is exact, *i.e.*, that the transformed action is equal to the original action without the presence of boundary terms. In order to apply Noether's theorem, the infinitesimal version of the Poincaré transformation is needed:

$$x'^\mu = x^\mu + \delta x^\mu\,, \quad \delta x^\mu = \varepsilon^\mu{}_\nu x^\nu + \varepsilon^\mu\,, \tag{11}$$

where $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ and ε^μ are the infinitesimal arbitrary constant parameters associated with the infinitesimal transformations under consideration. In addition, one has the transformation law for the 4-potential $A_\mu(x)$ and the 4-gradient $\partial/\partial x^\mu$ to first order in the parameters

$$\begin{aligned} A'_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu}A_\nu(x) = (\delta^\nu{}_\mu - \partial_\mu(\delta x^\nu))A_\nu \\ &= A_\mu(x) - (\partial_\mu\delta x^\nu)A_\nu\,, \\ \frac{\partial}{\partial x'^\mu} &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\delta^\nu{}_\mu - \partial_\mu(\delta x^\nu))\,\partial_\nu \\ &= \partial_\mu - (\partial_\mu\delta x^\nu)\partial_\nu, \end{aligned} \tag{12}$$

therefore, to first order in the parameters

$$\begin{aligned} F'_{\mu\nu}(x') :=& \partial'_\mu A'_\nu - \partial'_\nu A'_\mu\,, \\ =& F_{\mu\nu}(x) + (\partial_\nu\delta x^\alpha)F_{\alpha\mu} + (\partial_\mu\delta x^\alpha)F_{\nu\alpha}. \end{aligned} \tag{13}$$

On the other hand, one has $d^4x' = (1 + \partial_\mu\delta x^\mu)d^4x$. Note, however, that if one uses the explicit expression for δx^μ, then

$\partial_\mu \delta x^\mu = 0$ because of the antisymmetry of $\varepsilon_{\mu\nu}$. Nevertheless, $\partial_\mu \delta x^\mu = 0$ will not be used at this stage but rather at the end of the computations. Thus, to first order

$$\begin{aligned} S' &:= \alpha \int_{\mathcal{R}'} F'_{\mu\nu} F'^{\mu\nu} d^4x' \\ &= \alpha \int_{\mathcal{R}} F_{\mu\nu} F^{\mu\nu} (1 + \partial_\beta \delta x^\beta) d^4x \\ &+ \int_{\mathcal{R}} 4\alpha F^{\mu\nu} (\partial_\nu \delta x^\beta) F_{\beta\mu} d^4x \\ &= S[A_\mu] + \alpha \int_{\mathcal{R}} F_{\mu\nu} F^{\mu\nu} (\partial_\beta \delta x^\beta) d^4x \\ &+ \int_{\mathcal{R}} \partial_\nu \left(4\alpha F^{\mu\nu} \delta x^\beta F_{\beta\mu} \right) d^4x \\ &+ \int_{\mathcal{R}} \left[-4\alpha \delta x^\beta \partial_\nu (F^{\mu\nu} F_{\beta\mu}) \right] d^4x \,. \end{aligned} \tag{14}$$

To continue, it will be convenient to take into account the variation of the action (6) with respect to the 4-potential A_μ and given in Eq. (8) as well as the following definition:

$$B_{\mu\nu\beta} := \partial_\nu F_{\beta\mu} + \partial_\beta F_{\mu\nu} + \partial_\mu F_{\nu\beta}. \tag{15}$$

It is obvious that $B_{\mu\nu\beta} = 0$ is equivalent to the Bianchi identities

$$\partial_\nu F_{\beta\mu} + \partial_\beta F_{\mu\nu} + \partial_\mu F_{\nu\beta} = 0. \tag{16}$$

Equations (9) and (16) constitute the full set of Maxwell equations. However, Eqs. (9) and (16) will *not* be used at this stage but rather at the end of the computations, *i.e.*, we will be working "off-shell". Let $\mathcal{F}$ be the space formed by all configurations of the gauge potentials $A = A_\mu(x) dx^\mu$, *i.e.*, a point in $\mathcal{F}$ is a gauge potential which does not necessarily satisfy Eqs. (9) and (16).

Going back to Eq. (14), the integrand in the last term on the right-hand side of Eq. (14) can be rewritten by using Eq. (8) as

$$\begin{aligned} -4\alpha \delta x^\beta \partial_\nu (F^{\mu\nu} F_{\beta\mu}) &= -4\alpha \delta x^\beta (\partial_\nu F^{\mu\nu}) F_{\beta\mu} \\ &- 4\alpha \delta x^\beta F^{\mu\nu} \partial_\nu F_{\beta\mu} \\ &= -\delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} \\ &- 4\alpha \delta x^\beta F^{\mu\nu} \partial_\nu F_{\beta\mu} \,. \end{aligned} \tag{17}$$

Moreover, the last term on the right-hand side of Eq. (17) can be written by using the antisymmetry of $F_{\mu\nu}$ and Eq. (15) as

$$\begin{aligned} -4\alpha \delta x^\beta F^{\mu\nu} \partial_\nu F_{\beta\mu} &= -2\alpha \delta x^\beta F^{\mu\nu} \left(\partial_\nu F_{\beta\mu} - \partial_\mu F_{\beta\nu} \right) \\ &= -2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} \\ &+ 2\alpha \delta x^\beta F^{\mu\nu} \partial_\beta F_{\mu\nu} \\ &= -2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} \\ &+ \alpha \delta x^\beta \partial_\beta (F^{\mu\nu} F_{\mu\nu}) \\ &= -2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} \\ &+ \partial_\beta (\alpha \delta x^\beta F^{\mu\nu} F_{\mu\nu}) \\ &- \alpha F^{\mu\nu} F_{\mu\nu} (\partial_\beta \delta x^\beta). \end{aligned} \tag{18}$$

Therefore, inserting the results of Eqs. (17) and (18) back into Eq. (14)

$$\begin{aligned} S' = S[A_\mu] + \int_{\mathcal{R}} \Big[&\partial_\beta J^\beta - \delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} \\ &-2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} \Big] d^4x, \end{aligned} \tag{19}$$

where

$$\begin{aligned} J^\beta :=& 4\alpha F^{\mu\beta} \delta x^\gamma F_{\gamma\mu} + \alpha \delta x^\beta F^{\mu\nu} F_{\mu\nu} \\ =& T^\beta{}_\gamma \delta x^\gamma \,, \end{aligned} \tag{20}$$

is the Noether 4-current and

$$T^{\gamma\beta} := -4\alpha \left(F^{\gamma\mu} F^\beta{}_\mu - \frac{1}{4} \eta^{\gamma\beta} F^{\mu\nu} F_{\mu\nu} \right) , \tag{21}$$

is the *energy-momentum tensor* for the electromagnetic field.

Due to the fact that the action (6) is invariant under arbitrary transformations of the Poincaré group then it is, in particular, invariant under an infinitesimal transformation and so from Eq. (19)

$$\int_{\mathcal{R}} \left[\partial_\beta J^\beta - \delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} - 2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} \right] d^4x = 0 \tag{22}$$

for arbitrary spacetime regions $\mathcal{R}$. Therefore, the integrand in the last equation must identically vanish:

$$\partial_\beta J^\beta = \delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} + 2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta}. \tag{23}$$

Equation (23) is the cornerstone of the formalism of this paper. Note that Eq. (23) is *not* the so-called Noether's condition that is usually obtained in the standard implementation of Noether's theorem.

Now, let $\overline{\mathcal{F}}$ be the phase space formed by all those points of $\mathcal{F}$ which satisfy Eqs. (9) and (16). Therefore, for points of $\overline{\mathcal{F}}$, the right-hand side of Eq. (23) vanishes and the Noether 4-current J^β is conserved,

$$\partial_\beta J^\beta = 0. \tag{24}$$

Using the explicit form for δx^α, the Noether 4-current acquires the form

$$J^\beta = -\frac{1}{2}\varepsilon_{\gamma\phi}M^{\beta\gamma\phi} + \varepsilon_\gamma T^{\beta\gamma}\,, \tag{25}$$

with

$$M^{\beta\gamma\phi} := x^\gamma T^{\phi\beta} - x^\phi T^{\gamma\beta}\,, \tag{26}$$

the *angular momentum tensor* for the electromagnetic field [4]. Furthermore, from the continuity equation $\partial_\beta J^\beta = 0$,

$$\frac{1}{2}\varepsilon_{\gamma\phi}\left(\partial_\beta M^{\beta\gamma\phi}\right) + \varepsilon_\gamma\left(\partial_\beta T^{\gamma\beta}\right) = 0\,, \tag{27}$$

and from the fact that $\varepsilon_{\gamma\phi}$ and ε_γ are independent parameters it follows that each tensor is independently conserved

$$\partial_\beta M^{\beta\gamma\phi} = 0\,, \tag{28}$$

and

$$\partial_\beta T^{\gamma\beta} = 0\,. \tag{29}$$

Moreover, the energy-momentum tensor $T^{\gamma\beta}$ is symmetric. This can be seen from its definition in Eq. (21) or from the conservation of the energy-momentum and angular momentum tensors, *i.e.*, from Eqs. (28) and (29) it follows that $T^{\gamma\phi} = T^{\phi\gamma}$. Therefore, $T^{\gamma\beta}$ is *conserved*, *symmetric*, *gauge-invariant* under gauge transformations because it depends only on $F_{\mu\nu}$, and *traceless* because of $T^\mu{}_\mu = 0$. The Misner-Thorne-Wheeler's convention for the energy-momentum tensor is [4]

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)\,, \tag{30}$$

which corresponds to set $\alpha = -1/16\pi$ and $c = 1$ into the action (6). In an explicit form, the components of $T^{\mu\nu}$ read

$$\begin{aligned} T^{00} &= \frac{\vec{E}^2 + \vec{B}^2}{8\pi}\,,\\ T^{0i} &= \frac{(\vec{E}\times\vec{B})^i}{4\pi}\,,\\ T^{jk} &= \frac{1}{4\pi}\left[-(E^jE^k + B^jB^k) + \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\delta^{jk}\right]. \end{aligned} \tag{31}$$

3. Yang-Mills theory

The reader might wonder if the procedure applied to Abelian gauge fields holds also for non-Abelian gauge fields. The answer is in the affirmative. To see this, the Lagrangian action for the Yang-Mills fields is considered [8]:

$$S[A^a_\mu] = \alpha\int_{\mathcal{R}} d^4x\ F^a_{\mu\nu}F^{\mu\nu}_a, \tag{32}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - C^a{}_{bc}\,A^b_\mu A^c_\nu, \tag{33}$$

the strength of the Yang-Mills field $A = A^a_\mu dx^\mu \otimes T_a$ with T_a the generators of the Lie algebra of the gauge group.

Again, the first order change in the action (32) under the transformation $\tilde{\delta}A^a_\mu = \tilde{A}^a_\mu(x) - A^a_\mu(x)$ is:

$$\tilde{\delta}S = \int_{\mathcal{R}} d^4x\left[\frac{\delta S}{\delta A^a_\nu}\tilde{\delta}A^a_\nu\right] + \int_{\partial\mathcal{R}}\left(4\alpha F^{\mu\nu}_a\tilde{\delta}A^a_\nu\right)d\Sigma_\mu\,, \tag{34}$$

with[i]

$$\frac{\delta S}{\delta A^a_\nu} = -(4\alpha D_\mu F^{\mu\nu}_a), \tag{35}$$

and so $\tilde{\delta}S = 0$ gives the equations of motion

$$D_\mu F^{\mu\nu}_a = 0, \tag{36}$$

if the boundary term in Eq. (36) vanishes.

Again, the action (6) is fully invariant under the Poincaré group. In order to apply Noether's theorem, the infinitesimal version of the Poincaré transformation is needed and given by Eq. (11) together with the transformation for the Yang-Mills fields

$$\begin{aligned} A'^a_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu}A^a_\nu(x) = (\delta^\nu{}_\mu - \partial_\mu(\delta x^\nu))A^a_\nu\\ &= A^a_\mu(x) - (\partial_\mu\delta x^\nu)A^a_\nu. \end{aligned} \tag{37}$$

Therefore, to first order in δx^β

$$\begin{aligned} F'^a_{\mu\nu}(x') &= \partial'_\mu A'^a_\nu - \partial'_\nu A'^a_\mu - C^a{}_{bc}\,A'^b_\mu A'^c_\nu\\ &= F^a_{\mu\nu}(x) + (\partial_\nu\delta x^\alpha)F^a_{\alpha\mu} + (\partial_\mu\delta x^\alpha)F^a_{\nu\alpha}, \end{aligned} \tag{38}$$

and thus to first order

$$\begin{aligned} S' &:= \alpha\int_{\mathcal{R}'} F'^a_{\mu\nu}F'^{\mu\nu}_a d^4x'\\ &= \alpha\int_{\mathcal{R}} F^a_{\mu\nu}F^{\mu\nu}_a(1 + \partial_\beta\delta x^\beta)d^4x\\ &+ \int_{\mathcal{R}} 4\alpha F^{\mu\nu}_a(\partial_\nu\delta x^\beta)F^a_{\beta\mu}d^4x\\ &= S[A^a_\mu] + \alpha\int_{\mathcal{R}} F^a_{\mu\nu}F^{\mu\nu}_a(\partial_\beta\delta x^\beta)d^4x\\ &+ \int_{\mathcal{R}}\partial_\nu\left(4\alpha F^{\mu\nu}_a\delta x^\beta F^a_{\beta\mu}\right)d^4x\\ &+ \int_{\mathcal{R}}\left[-4\alpha\delta x^\beta\partial_\nu(F^{\mu\nu}_a F^a_{\beta\mu})\right]d^4x. \end{aligned} \tag{39}$$

As in the Abelian case, the object

$$B^a_{\mu\nu\beta} = D_\nu F^a_{\beta\mu} + D_\beta F^a_{\mu\nu} + D_\mu F^a_{\nu\beta}, \tag{40}$$

will be needed. The equation $B^a_{\mu\nu\beta} = 0$ is equivalent to the Bianchi identities

$$D_\nu F^a_{\beta\mu} + D_\beta F^a_{\mu\nu} + D_\mu F^a_{\nu\beta} = 0. \tag{41}$$

Equations (36) and (41) are the full set of Yang-Mills equations. As in the Abelian case, we will be working 'off-shell', *i.e.*, without using such equations at this stage but rather at the end of the computations.

We have in hand all the elements to continue. The integrand in the last term on the right-hand side of Eq. (39) can be rewritten using Eq. (35) as

$$\begin{aligned} -4\alpha\delta x^\beta \partial_\nu (F^{\mu\nu}_a F^a_{\beta\mu}) &= -4\alpha\delta x^\beta (D_\nu F^{\mu\nu}_a) F^a_{\beta\mu} \\ &\quad - 4\alpha\delta x^\beta F^{\mu\nu}_a D_\nu F^a_{\beta\mu} \\ &= -\delta x^\beta \frac{\delta S}{\delta A^a_\mu} F^a_{\beta\mu} \\ &\quad - 4\alpha\delta x^\beta F^{\mu\nu}_a D_\nu F^a_{\beta\mu}. \end{aligned} \tag{42}$$

Rewriting the last term on the right-hand side of Eq. (42), following the procedure used for the Abelian case,

$$\begin{aligned} -4\alpha\delta x^\beta F^{\mu\nu}_a D_\nu F^a_{\beta\mu} &= -2\alpha\delta x^\beta F^{\mu\nu}_a B^a_{\mu\nu\beta} \\ &\quad + \partial_\beta(\alpha\delta x^\beta F^{\mu\nu}_a F^a_{\mu\nu}) \\ &\quad - \alpha F^{\mu\nu}_a F^a_{\mu\nu}(\partial_\beta \delta x^\beta). \end{aligned} \tag{43}$$

Therefore, inserting the results of Eqs. (42) and (43) back into Eq. (39)

$$\begin{aligned} S' = S[A^a_\mu] + \int_{\mathcal{R}} \Big[&\partial_\beta J^\beta - \delta x^\beta \frac{\delta S}{\delta A^a_\mu} F^a_{\beta\mu} \\ &-2\alpha\delta x^\beta F^{\mu\nu}_a B^a_{\mu\nu\beta}\Big]\, d^4x, \end{aligned} \tag{44}$$

with

$$\begin{aligned} J^\beta &:= 4\alpha F^{\mu\beta}_a \delta x^\gamma F^a_{\gamma\mu} + \alpha\delta x^\beta F^{\mu\nu}_a F^a_{\mu\nu} \\ &= T^\beta{}_\gamma \delta x^\gamma\,, \end{aligned} \tag{45}$$

the Noether 4-current and

$$T^{\gamma\beta} := -4\alpha \left(F^{\gamma\mu}_a F^{a\beta}{}_\mu - \frac{1}{4}\eta^{\gamma\beta} F^{\mu\nu}_a F^a_{\mu\nu} \right), \tag{46}$$

is the *energy-momentum tensor* for the Yang-Mills fields.

Applying the same reasoning used in the Maxwell case, the relationship

$$\partial_\beta J^\beta = \delta x^\beta \frac{\delta S}{\delta A^a_\mu} F^a_{\beta\mu} + 2\alpha\delta x^\beta F^{\mu\nu}_a B^a_{\mu\nu\beta}, \tag{47}$$

is obtained. As in the Maxwell case, Eq. (47) is *not* the usual Noether's condition obtained by the standard Noether's theorem. As before, Eq. (47) plays a very important role here also. As before, if the equations of motion (36) and the Bianchi identities (41) hold, then the right hand side of last equation vanishes and the Noether 4-current J^β is conserved:

$$\partial_\beta J^\beta = 0. \tag{48}$$

Using the explicit form for δx^α, the Noether 4-current acquires the form

$$J^\beta = -\frac{1}{2}\varepsilon_{\gamma\phi} M^{\beta\gamma\phi} + \varepsilon_\gamma T^{\beta\gamma}, \tag{49}$$

with

$$M^{\beta\gamma\phi} := x^\gamma T^{\phi\beta} - x^\phi T^{\gamma\beta}, \tag{50}$$

the *angular momentum tensor* for the Yang-Mills fields.

Again, the same reasoning that follows Eq. (26) can be applied to conclude that $M^{\beta\gamma\phi}$ and $T^{\gamma\beta}$ are independently conserved and that $T^{\gamma\beta}$ is symmetric and gauge-invariant.

4. Source-free Proca theory

Now, the case of a non gauge-invariant theory will be discussed: the source-free Proca field [9]. It is interesting to know if the procedure of the present paper works also for this dynamical system.

The action principle for the source-free neutral Proca field is

$$S[A_\mu] = \alpha \int_{\mathcal{R}} d^4x \left[F_{\mu\nu} F^{\mu\nu} - 2m^2 A_\mu A^\mu \right], \tag{51}$$

where $A = A_\mu(x) dx^\mu$ is the potential 1-form and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

its strength.

The first order variation of the action (51) under the transformation $\tilde{\delta} A_\mu = \tilde{A}_\mu(x) - A_\mu(x)$ is

$$\tilde{\delta} S = \int_{\mathcal{R}} d^4x \left[\frac{\delta S}{\delta A_\nu} \right] \tilde{\delta} A_\nu + \int_{\partial\mathcal{R}} \left(4\alpha F^{\mu\nu} \tilde{\delta} A_\nu \right) d\Sigma_\mu, \tag{52}$$

with

$$\frac{\delta S}{\delta A_\nu} = -4\alpha \partial_\mu F^{\mu\nu} - 4\alpha m^2 A^\nu, \tag{53}$$

and so $\tilde{\delta} S = 0$ yields the equations of motion

$$\frac{\delta S}{\delta A_\nu} = -4\alpha \partial_\mu F^{\mu\nu} - 4\alpha m^2 A^\nu = 0, \tag{54}$$

provided that the boundary term in Eq. (52) vanishes.

From Eq. (53) it follows that

$$\partial_\nu \left(\frac{\delta S}{\delta A_\nu} \right) = -4\alpha m^2 \partial_\nu A^\nu, \tag{55}$$

because of the antisymmetry of $F_{\mu\nu}$. Eq. (55) will be used in the application of Noether's theorem.

Once again, applying an infinitesimal Poincaré transformation,

$$\begin{aligned}
S' :=&\alpha \int_{\mathcal{R}'} d^4x' \left[{F'}_{\mu\nu} F'^{\mu\nu} - 2m^2 {A'}_{\mu} A'^{\mu} \right] \\
=&\alpha \int_{\mathcal{R}} d^4x \left[F_{\mu\nu} F^{\mu\nu} - 2m^2 A_\mu A^\mu \right] (1 + \partial_\beta \delta x^\beta) \\
&+ \int_{\mathcal{R}} 4\alpha F^{\mu\nu} (\partial_\nu \delta x^\beta) F_{\beta\mu} d^4x \\
&+ \int_{\mathcal{R}} 4\alpha m^2 A^\mu A_\nu \left(\partial_\mu \delta x^\nu\right) d^4x \\
=&S[A_\mu] \\
&+ \alpha \int_{\mathcal{R}} d^4x \left[F_{\mu\nu} F^{\mu\nu} - 2m^2 A_\mu A^\mu \right] (\partial_\beta \delta x^\beta) \\
&+ \int_{\mathcal{R}} \partial_\nu \left[4\alpha F^{\mu\nu} \delta x^\beta F_{\beta\mu} \right. \\
&\left. + 4\alpha m^2 A^\nu A_\mu \delta x^\mu \right] d^4x \\
&+ \int_{\mathcal{R}} \left[-4\alpha \delta x^\beta \partial_\nu (F^{\mu\nu} F_{\beta\mu}) \right] d^4x \\
&- \int_{\mathcal{R}} 4\alpha m^2 \delta x^\nu \partial_\mu \left(A^\mu A_\nu\right) d^4x. \qquad (56)
\end{aligned}$$

By using Eq. (53), the next to last term on the right-hand side of Eq. (56) acquires the form

$$\begin{aligned}
-4\alpha \delta x^\beta \partial_\nu (F^{\mu\nu} F_{\beta\mu}) =& -4\alpha \delta x^\beta (\partial_\nu F^{\mu\nu}) F_{\beta\mu} \\
&- 4\alpha \delta x^\beta F^{\mu\nu} \partial_\nu F_{\beta\mu} \\
=& -\delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} \\
&- 4\alpha m^2 \delta x^\beta A^\mu F_{\beta\mu} \\
&- 4\alpha \delta x^\beta F^{\mu\nu} \partial_\nu F_{\beta\mu}. \qquad (57)
\end{aligned}$$

The last term in Eq. (57) has been already rewritten and it is given in Eq. (18). Therefore, using Eq. (18), Eq. (57) becomes

$$\begin{aligned}
-4\alpha \delta x^\beta \partial_\nu (F^{\mu\nu} F_{\beta\mu}) =& -\delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} \\
&- 4\alpha m^2 \delta x^\beta A^\mu F_{\beta\mu} \\
&- 2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} \\
&+ \partial_\beta (\alpha \delta x^\beta F^{\mu\nu} F_{\mu\nu}) \\
&- \alpha F^{\mu\nu} F_{\mu\nu} (\partial_\beta \delta x^\beta). \qquad (58)
\end{aligned}$$

In a similar way, the last line in Eq. (56) can be rewritten as

$$\begin{aligned}
-4\alpha m^2 \delta x^\nu \partial_\mu \left(A^\mu A_\nu\right) =& -4\alpha m^2 \delta x^\nu \left(\partial_\mu A^\mu\right) A_\nu \\
&- 4\alpha m^2 \delta x^\nu A^\mu \partial_\mu A_\nu \\
=& -4\alpha m^2 \delta x^\nu \left(\partial_\mu A^\mu\right) A_\nu \\
&- 4\alpha m^2 \delta x^\nu A^\mu \left[F_{\mu\nu} + \partial_\nu A_\mu\right] \\
=& \delta x^\nu A_\nu \partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right) \\
&- 4\alpha m^2 \delta x^\nu A^\mu F_{\mu\nu} \\
&- 2\alpha m^2 \delta x^\nu \partial_\nu \left(A_\mu A^\mu\right) \\
=& \delta x^\nu A_\nu \partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right) \\
&+ 4\alpha m^2 \delta x^\nu A^\mu F_{\nu\mu} \\
&+ \partial_\nu \left(-2\alpha m^2 \delta x^\nu A_\mu A^\mu\right) \\
&+ 2\alpha m^2 A_\mu A^\mu \left(\partial_\nu \delta x^\nu\right). \qquad (59)
\end{aligned}$$

On the right-hand side of the second equality in Eq. (59), the definition of $F_{\mu\nu}$ was used while in the third equality, Eq. (55) was used.

Inserting Eqs. (58) and (59) into Eq. (56),

$$\begin{aligned}
S' = S[A_\mu] + \int_{\mathcal{R}} \Bigg[& \partial_\beta J^\beta - \delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} \\
& -2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} + \delta x^\nu A_\nu \partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right) \Bigg] d^4x, \qquad (60)
\end{aligned}$$

where

$$\begin{aligned}
J^\beta :=& 4\alpha F^{\mu\beta} \delta x^\gamma F_{\gamma\mu} + \alpha \delta x^\beta F^{\mu\nu} F_{\mu\nu} \\
&- 2\alpha m^2 A_\mu A^\mu \delta x^\beta + 4\alpha m^2 A^\beta A_\mu \delta x^\mu \\
=& T^\beta{}_\gamma \delta x^\gamma \qquad (61)
\end{aligned}$$

is the Noether 4-current and

$$\begin{aligned}
T^{\gamma\beta} =& -4\alpha \left(F^{\gamma\mu} F^\beta{}_\mu - \frac{1}{4} \eta^{\gamma\beta} F^{\mu\nu} F_{\mu\nu} \right. \\
&\left. + \frac{1}{2} m^2 \eta^{\gamma\beta} A_\mu A^\mu - m^2 A^\gamma A^\beta \right), \qquad (62)
\end{aligned}$$

is the *energy-momentum tensor* for the Proca field.

Once again, due to the fact that the action principle is invariant under the Poincaré group, it follows that

$$\begin{aligned}
\int_{\mathcal{R}} \Bigg[& \partial_\beta J^\beta - \delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} \\
& -2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} + \delta x^\nu A_\nu \partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right) \Bigg] d^4x = 0, \qquad (63)
\end{aligned}$$

for arbitrary spacetime regions $\mathcal{R}$. Therefore, the integrand must identically vanish:

$$\partial_\beta J^\beta = \delta x^\beta \frac{\delta S}{\delta A_\mu} F_{\beta\mu} + 2\alpha \delta x^\beta F^{\mu\nu} B_{\mu\nu\beta} - \delta x^\nu A_\nu \partial_\mu \left(\frac{\delta S}{\delta A_\mu} \right), \tag{64}$$

which is the right Noether's condition.

Therefore, if the equations of motion (54) and the Bianchi identities hold, then the Noether 4-current J^β is identically conserved:

$$\partial_\beta J^\beta = 0. \tag{65}$$

Using the explicit expression for δx^μ, the 4-current acquires the form

$$J^\beta = -\frac{1}{2}\varepsilon_{\gamma\phi} M^{\beta\gamma\phi} + \varepsilon_\gamma T^{\beta\gamma}, \tag{66}$$

with

$$M^{\beta\gamma\phi} := x^\gamma T^{\phi\beta} - x^\phi T^{\gamma\beta}, \tag{67}$$

the *angular momentum tensor*.

5. Concluding Remarks

It has been shown that the symmetric and gauge-invariant expressions for the energy-momentum tensors of Maxwell and Yang-Mills fields can be obtained from a direct implementation of Noether's theorem under a correct handling of the terms involving the equations of motion and the Bianchi identities. The procedure also works for the Proca fields. The reader might then wonder about the cause of the failure of the standard Noether's approach, which leads to Eqs. (3), (4), and (5) instead of Eqs. (21), (46), and (62); respectively, or, equivalently, what is then the difference between the standard Noether's approach found in literature and the one of the present paper if after all both approaches deal with Noether's teorem? The answer is as follows. In the standard implementation of Noether's theorem to gauge theories only half of the full set of equations of motion are used, the Euler-Lagrange equations. However, when dealing with gauge theories, one has to keep in mind also the Bianchi identities which are *not* taken into account in the standard approach [3, 5, 6]. Nevertheless, as shown here, if they are both taken into account, Noether's theorem yields the right expressions for the energy-momentum tensors. Therefore, there is nothing mysterious or wrong in the implementation of Noether's theorem to gauge theories; what happens is just that the standard implementation of Noether's theorem is incomplete in the sense already explained, and this is why the canonical energy-momentum tensors so obtained need an "improvement" via, for instance, Belinfante's method. One could say that Noether-Belinfante's method is equivalent to the analysis performed here in the sense that both approaches agree in the final form for the energy-momentum tensor. This is so from an operational (and pragmatic) viewpoint. Nevertheless, there is a key conceptual difference between Noether-Belinfante's procedure and the one followed here. From the viewpoint of the present paper, Belinfante's method is *not* needed because the action has *all* the information required to *uniquely* determine the right expressions for the energy-momentum tensor $T^{\mu\nu}$ via translations in spacetime by using only Noether's theorem. Moreover, the formalism of the present paper, in contrast to Belinfante's method, has no ambiguities once the Lagrangian density has been chosen. In spite of these conceptual differences, our method agrees with Noether-Belinfante's in the computation of the correct form for the energy-momentum tensor. So, let us briefly say some words about the relationship between our method and Belinfante's. As we mentioned, a key element in our approach is the explicit incorporation and handling of the Bianchi identities. In our opinion, the Bianchi identities (or something equivalent to them) are "hidden" in Belinfante's method, which allows us to fix somehow the wrong canonical energy-momentum tensor. Of course that the precise relationship between the current formalism and the Noether-Belinfante's method must be explained, but that is beyond the scope of this paper.

Finally, it would be interesting to generalize the results of Refs. 10 and 11 to the case of gauge theories where the Lagrangian is singular [12], *i.e.*, $\det\left(\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j\right) = 0$. Such a generalization would involve building an action principle which would yield the original equations of motion for the gauge system simultaneously with its Jacobi variational equations. It is clear that the dynamical systems analyzed in this paper could be handled in the framework of such a generalization.

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$*$ Associate Member of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

i The covariant derivatives are defined as follows:

$$D_\mu \eta^a := \partial_\mu \eta^a + C^a{}_{bc} \eta^b A^c_\mu$$

and

$$D_\mu \lambda_a := \partial_\mu \lambda_a - C^b{}_{ac} \lambda_b A^c_\mu .$$

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